A NEW CONCEPT OF CONVERGENCE SPACE

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This paper is dedicated to my colleagues Jirí Adamék and Walter Tholen on the occasion of their sixtieth birthdays

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Abstract: The notion of b-convergence is presented for studying preuniform convergence spaces in the sense of Preuß (1993) and set-convergence spaces introduced by Wyler in 1989 from a common point of view.

The well-known supertopologies as defined by Doitchinov in 1964 and also the filtermerotopies in the sense of Katétov (1965) can be integrated as well. Even the grill-defined presupernear operators, introduced by the author (1999) are contained in this new broader concept.

Moreover, we discuss all the properties for describing categories in the realm of Convenient Topology, especially the properties of being cartesian closed or extensional.

1. Introduction

In this paper we present a new type of convergence, which generalizes the “classical” ones of set-convergence in the sense of Wyler [11] and of preuniform convergence in the sense of Preuß [9] by bringing them both together.

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Thus, a comprehensive theory of convergence space is being established, which enables us to simultaneously express generalized “topological” and “uniform” aspects.

Hence the branches of Convenient Topology and of non-symmetric Convenient Topology are both involved and can be discussed in connection with generalized Cauchy spaces or filter spaces, respectively [6].

As a basic concept we consider uniform filters converging to bounded subsets, thus defining by suitable axioms the so-called $b$-convergences. Morphisms between the corresponding spaces are then defined in an obvious way, i.e., they are bounded maps which preserve uniform filters and so-called $b$-continuous functions. The resulting category $b$-$\text{CONV}$ is “topological” in the sense that it is fibre-small, initially complete and moreover has the terminal separator property. So in general, subspaces and products, or quotients and sums as well are simultaneously formed by supplying the corresponding sets with the initial (respectively final) $b$-convergence with respect to the given data (see e.g., [9]). Moreover, we show that pointed $b$-convergence leads us to a strong topological universe in which the constructs $\text{TOP}$ and $\text{UNIF}$ can both be embedded in particularly nice fashion.

2. Basic concepts

As usual, $PX$ denotes the power set of a set $X$, and we use $\mathcal{B}^X \subseteq PX$ to denote a collection of bounded subsets of $X$, also known as B-sets. Moreover, $\text{FIL}(X \times X)$ denotes the set of all uniform filters on $X$.

2.1 Definition. We call a pair $(\mathcal{B}^X, \tau)$ consisting of a B-set $\mathcal{B}^X$ and a function $\tau : \mathcal{B}^X \rightarrow P(\text{FIL}(X \times X))$ a $b$-convergence space and $\tau$ a $b$-convergence (on $\mathcal{B}^X$), if the following axioms are satisfied:

(bC1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;
(bC2) $\emptyset \in \mathcal{B}^X$;
(bC3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(bC4) $x \in X$ implies $\hat{x} \times \hat{x} \in \tau(\{x\})$;
(bC5) $\tau(\emptyset) = \{P(X \times X)\}$;
(bC6) $B \in \mathcal{B}^X$, $U \in \tau(B)$ and $U \subseteq \mathcal{U} \in \text{FIL}(X \times X)$ imply $\mathcal{U} \in \tau(B)$.

(Here $\hat{x}$ denotes the filter generated by the set $\{x\}$.) In general, for filters $\mathcal{F}$ and $\mathcal{G}$ their cross product is defined by $\mathcal{F} \times \mathcal{G} := \{ T \subseteq X \mid \exists F \in \mathcal{F} \exists G \in \mathcal{G} \exists T' \subseteq F \times G \}$. 
If $\mathcal{U} \in \tau(B)$ for some $B \in \mathcal{B}^X$, we say the uniform filter $\mathcal{U}$ $b$-converges to $B$.

Given two $b$-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$, a function $f : X \rightarrow Y$ is called $b$-continuous iff it is bounded, which means

(c1) $\{ f[B] \mid B \in \mathcal{B}^X \} \subseteq \mathcal{B}^Y$,

and in addition we have that $f$ preserves uniform filters in the sense that

(c2) $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau_X(B)$ imply $(f \times f)(\mathcal{U}) \in \tau_Y(f[B])$, where

$$(f \times f)(\mathcal{U}) := \{ V \subseteq Y \times Y \mid (f \times f)^{-1}[V] \in \mathcal{U} \}.$$ 

Moreover, we denote the corresponding category by $\textbf{b-CONV}$, and mention here its interesting property of being topological (see Th. 4.1).

2.2 Examples. (i) Consider a set-convergence space $(X, \mathcal{M}^X, q)$, where $X$ is a set, $\mathcal{M}^X$ is a $B$-set $q \subseteq \text{FIL}(X) \times \mathcal{M}^X$ relates filters on $X$ with bounded sets subject to the following conditions (O. Wyler):

(SC1) $\dot{A} A A$ for any $A \in \mathcal{M}^X$, where $A := \{ B \subseteq X \mid B \supseteq A \}$;

(SC2) $\mathcal{F} \in \text{FIL}(X)$ implies $\mathcal{F} \emptyset$ iff $\mathcal{F} = PX$;

(SC3) $A \in \mathcal{M}^X$, $\mathcal{F}_1 q A$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \in \text{FIL}(X)$ imply $\mathcal{F}_2 q A$.

These data induce a function $\tau_q$ from $\mathcal{M}^X$ into $P(\text{FIL}(X \times X))$ by setting for each $A \in \mathcal{M}^X$

$$\tau_q(A) := \{ \mathcal{U} \in \text{FIL}(X \times X) \mid \exists \mathcal{F} \in \text{FIL}(X). \mathcal{F} q A \text{ and } \dot{A} \times \mathcal{F} \subseteq \mathcal{U} \}.$$ 

(ii) A special case arises for a surrounding system (neighborhood system) $(\mathcal{M}^X, \Theta)$ on a set $X$, where $\mathcal{M}^X$ is a $B$-set and $\Theta : \mathcal{M}^X \rightarrow \text{FIL}(X)$ is a function into the set of all filters on $X$ (including the zero-filter $PX$) satisfying the following properties [3]:

(SS1) $\Theta(\emptyset) = PX$;

(SS2) $A \in \mathcal{M}^X$ and $U \in \Theta(A)$ imply $U \supseteq A$.

Then in analogy we may set for each $A \in \mathcal{M}^X$:

$$\tau_\Theta(A) := \{ \mathcal{U} \in \text{FIL}(X) \mid \dot{A} \times \Theta(A) \subseteq \mathcal{U} \}.$$ 

(iii) Let $(X, J_X)$ be a preuniform convergence space, where $J_X \subseteq \text{FIL}(X \times X)$ satisfies the following two conditions (Preuß):

(PUC1) $x \in X$ implies $\dot{x} \times \dot{x} \in J_X$;

(PUC2) $\mathcal{U}_1 \subseteq J_X$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2 \in \text{FIL}(X \times X)$ imply $\mathcal{U}_2 \subseteq J_X$.

In this case we consider $PX$ as a $B$-set and define a function from $PX$ into $P(\text{FIL}(X \times X))$ by setting $\tau_X(B) := J_X$ for each nonempty $B \subseteq X$,

and $\tau_X(\emptyset) := \{ P(X \times X) \}$.

(iv) At last, consider a presupernear space $(\mathcal{B}^X, N)$, where $\mathcal{B}^X$ is a $B$-set (on a set $X$) and $N : \mathcal{B}^X \rightarrow P(P(X))$ is a function satisfying the following conditions [7]:


(SN1) \( N_1 \ll N_2 \in N(B) \) implies \( N_1 \in N(B) \);
(SN2) \( N(\emptyset) = \{\emptyset\} \) and \( B^X \notin N(B) \) for each \( B \in B^X \);
(SN3) \( x \in X \) implies \( \{\{x\}\} \in N(\{x\}) \).
Moreover, let \((B^X, N)\) be grill-defined, which means that in addition we have

(G) for each \( N \in N(B) \) there exists a grill \( G \in \mathcal{GRL}(X) \) with \( N \subseteq G \in N(B) \).
Recall that \( G \subseteq PX \) is called a grill on the set \( X \) (G. Choquet), provided that

(G1) \( \emptyset \notin G \);
(G2) \( G_1 \cup G_2 \in G \) iff \( G_1 \in G \) or \( G_2 \in G \).

Then we set for each \( B \in B^X \):

\[
\tau_N(B) := \{ U \in \mathcal{FIL}(X \times X) \mid \exists G \in \mathcal{GRL}(X), G \in N(B) \text{ and sec } G \times \text{sec } G \subseteq U \},
\]
Recall that sec \( G := \{ T \subseteq X \mid \forall G \in G. G \cap T \neq \emptyset \} \).

3. Categorical concepts

Sets of bounded subsets of a set \( X \) are described axiomatically by the postulates (bc1) through (bc3) (see also Def. 2.1).

Having B-sets \( B^X \) and \( B^Y \), respectively, a function \( f : X \to Y \) is called bounded, if it preserves bounded sets (see again Def. 2.1).

The category \( \mathbf{BOUND} \) with pairs \((X, B^X)\) consisting of a set \( X \) and a corresponding B-set \( B^X \) as objects and bounded maps as morphisms is a topological universe, which means it is cartesian closed and extensional and hence has universal one-point extensions. If no confusion is possible, we consider the B-sets \( B^X \) as objects of \( \mathbf{BOUND} \).

We recall the defining conditions for a concrete category \( \mathbf{C} \) to be called topological:

(CT1) “Existence of initial structures”: For any set \( X \), any family \((X_i, T_i)_I\) of \( \mathbf{C} \)-objects indexed by a class \( I \), and any family \((f_i : X \to X_i)_I\) of maps indexed by \( I \), there exists a unique \( \mathbf{C} \)-structure \( T \) on \( X \) that is initial with respect to \((X, f_i, (X_i, T_i), I)\). I.e., for any \( \mathbf{C} \)-object \((Y, S)\) a function \( g : Y \to X \) is a \( \mathbf{C} \)-morphism from \((Y, S)\) to \((X, T)\) iff for every \( i \in I \) the composite map \( f_i \circ g : Y \to X_i \) is a \( \mathbf{C} \)-morphism from \((Y, S)\) to \((X_i, T_i)\).

(CT2) “Fibre-smallness”: For any set \( X \) the \( \mathbf{C} \)-fibre, i.e., the class of all \( \mathbf{C} \)-structures on \( X \), is a set.
TV (CT3) “Terminal separator property”: For any set $X$ with cardinality 1 there exists precisely one $\mathcal{C}$-structure. Moreover, a topological category (construct) is cartesian closed (i.e., has natural function space structures), provided that for any pair $(A, B)$ of $\mathcal{C}$-objects the set $\text{Mor}(A, B)$ of all $\mathcal{C}$-morphisms from $A$ to $B$ can be equipped with the structure of a $\mathcal{C}$-object, denoted by $\text{Pow}(A, B)$ and called power-object or natural function space, such that the following are satisfied:

1. The evaluation map $e : A \times \text{Pow}(A, B) \to B$ defined by $e(a, f) := f(a)$ for each pair $(a, f) \in A \times \text{Pow}(A, B)$ is a $\mathcal{C}$-morphism;
2. For each $\mathcal{C}$-object $C$ and each $\mathcal{C}$-morphism $f : A \times C \to B$ the map $\hat{f} : C \to \text{Pow}(A, B)$ defined by $\hat{f}(a)(c) := f(a, c)$ is a $\mathcal{C}$-morphism.

For a topological category $\mathcal{C}$ with universal one-point extensions the last expression means that every $\mathcal{C}$-object $A$ can be embedded via the addition of a single point $\infty$ into an object $A^* := A \cup \{\infty\}$ such that the following holds:

- For every $\mathcal{C}$-morphism $f : U \to A$ from a subspace $U$ of a $\mathcal{C}$-object $B$ into $A$ the unique function $f^* : B \to A^*$ defined by
  $$f^*(b) := \begin{cases} f(b), & \text{if } b \in U, \\ \infty & \text{otherwise} \end{cases}$$
  is a $\mathcal{C}$-morphism.

For basic literature concerning these definitions the reader is referred to the book of Preuß [9].

4. Convenient properties in the realm of $b$-$\text{CONV}$

The aim of Convenient Topology (see [9]) consists in the study of “strong topological universes”, in which “convergence” structures are available. Furthermore, such a strong topological universe should be easily described by means of suitable axioms and should not be too large.

Thus, the construct $\text{PUCONV}$ of preuniform convergence spaces in the sense of Preuss is a good candidate for this purpose in Convenient Topology.

As already pointed out there also exist convergence structures (e.g., set-convergences, supertopologies and grill-defined presupernear operators), which cannot be subsumed by the above-mentioned construct $\text{PUCONV}$. This motivated our broader concept of b-convergence.
Now we will examine whether $b\text{-}CONV$ or some of its interesting subcategories satisfies the proposed axioms for being a topological universe, or strong topological universe, respectively.

4.1 Theorem. $b\text{-}CONV$ is a topological category.

Proof. For a B-set $B \in \mathbb{B}^X$ and a class $I$ let $(\mathbb{B}^{X_i}, \tau_i)_I$ be a family of $b$-convergence spaces and $(f_i : X \longrightarrow X_i)_I$ a family of bounded maps from $\mathbb{B}^X$ to $\mathbb{B}^{X_i}$. We set

$$
\tau_{in}(B) = \begin{cases} 
\{P(X \times X)\}, & \text{if } B = \emptyset; \\
\{\mathcal{U} \in \text{FIL}(X \times X) \mid \forall i \in I. (f_i \times f_i)(\mathcal{U}_i) \in \tau_i(f_i[B])\}, & \text{if } B \neq \emptyset.
\end{cases}
$$

Then $\tau_{in}$ is the initial $b$-convergence on $\mathbb{B}^X$ with respect to the given data.

To (bc4): For $x \in X$ we have $(f_i \times f_i)(\dot{x} \times \dot{x}) = ((f_i(x), f_i(x))) \in \tau_i(\{f_i(x)\})$. All the remaining axioms are easy to verify. By definition, the functions $f_i$ are $b$-continuous. Now let $(\mathbb{B}^Y, \Gamma)$ be a $b$-convergence space and $g : Y \longrightarrow X$ be a map such that $f_i \circ g$ is $b$-continuous from $(\mathbb{B}^Y, \Gamma)$ to $(\mathbb{B}^{X_i}, \tau_i)$ for every $i \in I$. Consider $\mathcal{U} \in \Gamma(B)$ and $B \in \mathbb{B}^Y \setminus \{\emptyset\}$. Then we have $(f_i \times f_i)((g \times g)(\mathcal{U})) = ((f_i \circ g) \times (f_i \circ g)(\mathcal{U})) \in \tau_i(f_i[B])$ for each $i \in I$. Hence, $(g \times g)(\mathcal{U}) \in \tau_{in}(g[B])$ follows, which shows that $f$ is $b$-continuous from $(\mathbb{B}^Y, \Gamma)$ to $(\mathbb{B}^X, \tau_{in})$.

The other two axioms of being a topological category are obviously satisfied. ♦

4.2 Definition. A $b$-convergence $\tau$ on $\mathbb{B}^X$ and the corresponding pair $(\mathbb{B}^X, \tau)$ are called isof orm, provided

(if) $\emptyset \neq B_2 \subseteq B_1 \in \mathbb{B}^X$ implies $\tau(B_2) \subseteq \tau(B_1)$.

4.3 Remark. Isotone set-convergence spaces $(X, M^X, q)$, where $q$ satisfies in addition

(SC4) $A_2 \subseteq A_1 \in M^X$ and $\mathcal{F} q A_2$ imply $\mathcal{F} q A_1$,

lead us to isof orm $b$-convergences $\tau_q$. Grill-defined pseudosupernear operators $N$ determine isof orm $b$-convergences $\tau_N$ as well, where $N$ satisfies axioms (SN1) through (SN4).

$ib\text{-}CONV$ denotes the full subcategory of $b\text{-}CONV$ spanned by the isof orm $b$-convergences.

4.4 Remark. Cartesian closedness of topological constructs implies that quotient maps are finitely productive, but not necessarily productive (i.e., not closed under the construction of arbitrary products). We will call a topological construct satisfying this latter property strong.
4.5 Lemma. Let $X$ be a set, $(\mathcal{B}^X_i, \tau_i)_I$ be a family of isoform $b$-convergence spaces and $(f_i : X_i \rightarrow X)_I$ be a family of maps. Then we set
\[
\tau(B) := \begin{cases} 
P(X \times X), & \text{if } B = \emptyset; \\
\{ \mathcal{U} \in \mathcal{FIL}(X \times X) \mid \exists i \in I \exists \mathcal{U}_i \in \tau_i(f_i^{-1}[B]). (f_i \times f_i)(\mathcal{U}_i) \subseteq \mathcal{U} \} \cup \{ \hat{x} \times \hat{x} \mid x \in X \}, & \text{if } B \neq \emptyset. 
\end{cases}
\]
Consequently $\tau_X$ is the final $ib\text{-CONV}$-structure on $\mathcal{B}^X$ with respect to the given data.
Proof. This is evident. \(\square\)

4.6 Remark. If $(f_i : X_i \rightarrow X)_I$ is an epi-sink in $Set$ (i.e., $X = \bigcup \{ f_i[X_i] \mid i \in I \}$), then for every $B \in \mathcal{B}^X \setminus \{ \emptyset \}$ we have
\[
\tau_{\text{fin}}(B) = \{ \mathcal{U} \in \mathcal{FIL}(X \times X) \mid \exists i \in I \exists \mathcal{U}_i \in \tau_i(f_i^{-1}[B]). (f_i \times f_i)(\mathcal{U}_i) \subseteq \mathcal{U} \}. 
\]

4.7 Theorem. $ib\text{-CONV}$ is a strong topological construct.
Proof. Let $((\mathcal{B}^X_i, \tau_{X_i}) \xrightarrow{f_i} (\mathcal{B}^Y_i, \tau_{Y_i}))_I$ be a non-empty family of quotient maps in $ib\text{-CONV}$ indexed by a set $I$, and consider the corresponding product diagram in $ib\text{-CONV}$
\[
\begin{array}{ccc}
(\mathcal{B}^X, \tau_X) & \xrightarrow{\prod f_i} & (\mathcal{B}^Y, \tau_Y) \\
p_{X_i} & \downarrow & p_{Y_i} \\
(\mathcal{B}^X_i, \tau_{X_i}) & \xrightarrow{f_i} & (\mathcal{B}^Y_i, \tau_{Y_i})
\end{array}
\]
where $(\mathcal{B}^X, \tau_X) := \prod_{i \in I} (\mathcal{B}^X_i, \tau_{X_i})$ and $(\mathcal{B}^Y, \tau_Y) := \prod_{i \in I} (\mathcal{B}^Y_i, \tau_{Y_i})$.
Since all $f_i$ are surjective, $\prod f_i$ is surjective as well. For each $i \in I$, and every $B_i \in \mathcal{B}^Y_i \setminus \{ \emptyset \}$ we have
\[
\tau_{Y_i}(B_i) := \{ \mathcal{U} \in \mathcal{FIL}(Y_i \times Y_i) \mid \exists \mathcal{U}_i \in \tau_{X_i}(f_i^{-1}[B_i]). (f_i \times f_i)(\mathcal{U}_i) \subseteq \mathcal{U} \}
\]
because $f_i$ is a quotient map. For every $B \in \mathcal{B}^Y \setminus \{ \emptyset \}$ define
\[
\tau_{Y}(B) := \{ \mathcal{V} \in \mathcal{FIL}(Y \times Y) \mid \exists \mathcal{W} \in \tau_X((\prod f_i)^{-1}[B]). (\prod f_i \times \prod f_i)(\mathcal{W}) \subseteq \mathcal{V} \}
\]
which implies
\[
\tau_{Y_i}(B) = \tau_{Y}(B) := \{ \mathcal{U} \in \mathcal{FIL}(Y \times Y) \mid (p_{Y_i} \times p_{Y_i})(\mathcal{U}) \in \tau_{Y_i}(p_i[B]) \}. 
\]
This means that $\prod f_i$ is a quotient map. \(\square\)

4.8 Theorem. $ib\text{-CONV}$ is extensional.
Proof. For an isoform $b$-convergence space $(\mathcal{B}^X, \tau)$ we put $X^* := X \cup \{ \infty \}$ and $\mathcal{B}^* := \mathcal{B}^X \cup \{ \{ \infty \} \}$, and we define a $b$-convergence $\tau^*$ on $\mathcal{B}^*$ by setting:
4.10 Definition. We call a saturated and equiform b-convergence space $(\mathcal{B}^X, \tau)$ a preuniform convergence as follows:

$$\tau^*(B) = \begin{cases} \{P(X^* \times X^*)\}, & \text{if } B = \emptyset; \\ \{\infty \times \infty\}, & \text{if } B = \{\infty\}; \\ \{U \in \mathcal{FIL}(X^* \times X^*) | \exists U \in \tau(B). U^* \supseteq \bar{U}\}, & \text{if } B \in \mathcal{B}^X \setminus \{\emptyset\} \end{cases}$$

with $\bar{U} := \{\bar{U} | U \in U\}$ and $\bar{U} := U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$.

4.11 Theorem. The category $\text{puconver}$ with preuniform convergence spaces as objects and uniformly continuous maps between them.
as morphisms is isomorphic to the category $\text{pub-CONV}$ of preuniform $b$-convergence spaces and $b$-continuous functions.

**Proof.** This is immediately clear. $\diamond$

4.12 Remark. Special cases of preuniform convergences are the so-called principal ones; i.e., a preuniform convergence space $(X, J_X)$ is called a principal preuniform convergence space, provided there is a filter $\mathcal{U}$ on $X \times X$ such that $J_X := [\mathcal{U}]$, where

$$[\mathcal{U}] := \{ \mathcal{V} \in \text{FIL}(X \times X) \mid \mathcal{V} \supseteq \mathcal{U} \}.$$

Now it is easy to verify how diagonal filters, especially semiuniformities, quasiuniformities or uniformities, can be described by means of their corresponding preuniform $b$-convergences, namely as the pointed ones.

4.13 Definition. A $b$-convergence $\tau$ on $B^X$ and the corresponding pair $(B^X, \tau)$ are called pointed, provided

(p) $B \in B^X \setminus \{ \emptyset \}$ implies $\tau(B) = \bigcup \{ \tau(\{x\}) \mid x \in B \}$.

4.14 Remark. Hence, pointed $b$-convergences are of interest when studying spaces dealing with “uniform aspects”. Moreover, we note that pointed $b$-convergence spaces are necessarily isomorphic.

The considerations above allow us to consider further specializations, for instance by setting $B^X := \emptyset \cup \{ \{x\} \mid x \in X \}$. So in case of having a $b$-convergence in $B^X$, this leads us to a corresponding “generalized” convergence relation, and vice versa, so that the category $\text{GCONV}$ of generalized convergence spaces and related maps can be considered as such an isomorphic one [9] with respect to Ex. 2.2(i), defining topological spaces in special cases. Moreover, we note that the corresponding $b$-convergence is necessarily pointed.

4.15 Theorem. The category $\text{pb-CONV}$, whose objects are the pointed $b$-convergence spaces, is bicoreflective in $\text{ib-CONV}$.

**Proof.** Given an isomorphic $b$-convergence space $(B^X, \tau)$, for each $B \in B^X$ we set

$$\tau^p(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{ \mathcal{U} \in \text{FIL}(X \times X) \mid \exists x \in B. \mathcal{U} \in \tau(\{x\}) \} & \text{otherwise.} \end{cases}$$

Hence $(B^X, \tau^p)$ is a pointed $b$-convergence space. Evidently, the axioms (bC1) through (bC6) are satisfied. Now consider $B \in B^X \setminus \{ \emptyset \}$ and $\mathcal{U} \in \tau^p(B)$. Then we have $\mathcal{U} \in \tau(\{x\})$ for some $x \in B$, hence $\mathcal{U} \in \tau^p(\{x\})$, which shows $\mathcal{U} \in \bigcup \{ \tau^p(\{x\}) \mid x \in B \}$.

Conversely, $\mathcal{U} \in \bigcup \{ \tau^p(\{x\}) \mid x \in B \}$ implies $\mathcal{U} \in \tau^p(\{x'\})$ for some $x' \in B$. Therefore we obtain $\mathcal{U} \in \tau(\{x\})$, which shows $\mathcal{U} \in \tau^p(B)$. 

Moreover, the identity map \(1_X\) is \(b\)-continuous on \((\mathcal{B}^X, \tau^p)\), i.e., \(\mathcal{U} \in \tau^p(B)\), and without restriction \(B \neq \emptyset\) implies \(\mathcal{U} \in \tau(\{x\})\) for some \(x \in B\). Since \(\tau\) is isoform, \(\mathcal{U} \in \tau(B)\) results, which concludes this verification.

Now let \((\mathcal{B}^Y, \Gamma)\) be a pointed \(b\)-convergence space and \(f: (\mathcal{B}^Y, \Gamma) \rightarrow (\mathcal{B}^X, \tau)\) a \(b\)-continuous function. We have to show that \(f: (\mathcal{B}^Y, \Gamma) \rightarrow (\mathcal{B}^X, \tau^p)\) is again \(b\)-continuous. Without restriction let \(\emptyset \neq B \in \mathcal{B}^Y\). Then \(\mathcal{U} \in \Gamma(B)\) implies the existence of some \(y \in B\) such that \(\mathcal{U} \in \Gamma(\{y\})\), since \(\Gamma\) is pointed. Hence \((f \times f)(\mathcal{U}) \in \tau^p(\{f(y)\})\) by hypothesis. Consequently, we get \((f \times f)(\mathcal{U}) \in \tau(\{f(y)\}) \subseteq \tau(f[B])\). \(\Box\)

4.16 Remark. Since \(pb\text{-}\textsc{CONV}\) is bicoreflective in \(ib\text{-}\textsc{CONV}\) (see Th. 4.15), it is again a topological category. Thus quotients and sums in \(pb\text{-}\textsc{CONV}\) are formed as in \(b\text{-}\textsc{CONV}\), whereas subspaces and products arise from the corresponding structures in \(ib\text{-}\textsc{CONV}\) by applying the corresponding bicoreflection.

4.17 Corollary. For an isoform \(b\)-convergence space \((\mathcal{B}^X, \tau)\) the \(b\)-convergence \(\tau^*\) is pointed iff \(\tau\) is pointed.

Proof. \(B \in \mathcal{B}^* \setminus \{\emptyset\}\) implies \(B \in \mathcal{B}^X\) or \(B = \{\infty\}\). In the latter case \(\tau^*(B) = \tau^*(\{\infty\}) = \{\infty \times \infty\} = \bigcup \{\tau^*(\{x\})\mid x \in \{\infty\} = B\}\). In the first case consider \(\mathcal{U}^* \in \tau^*(B)\), hence there exists \(\mathcal{U} \in \tau(B)\) with \(\mathcal{U}^* \supseteq \mathcal{U}\). Because \(\tau\) is pointed, \(\mathcal{U} \in \tau(\{x\})\) follows, which shows that \(\mathcal{U}^* \in \tau^*(\{x\})\). Conversely, let \(\tau^*\) be pointed. \(B \in \mathcal{B}^X \setminus \{\emptyset\}\) and \(\mathcal{U} \in \tau^*(B)\) imply \(\mathcal{U} \in \tau^*(B)\), hence by hypothesis \(\mathcal{U} \in \tau(\{x\})\) follows. Now choose \(\mathfrak{U} \in \tau(\{x\})\) with \(\mathfrak{U} \supseteq \mathcal{U}\). Then \(\mathfrak{U} \subseteq \mathcal{U}\), because \(V \in \mathcal{U}\) implies \(V = U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)\) for some \(U \in \mathcal{U}\), which shows that \(\mathcal{U} \in \tau(\{x\})\).

4.18 Theorem. \(pb\text{-}\textsc{CONV}\) is extensional.

Proof. Use the results obtained above.

In the case of non-symmetric Convenient Topology, we will further check whether the category \(pb\text{-}\textsc{CONV}\) can serve as a topological universe.

4.19 Theorem. For two \(pb\)-convergence spaces \((\mathcal{B}^X, \tau_X)\) and \((\mathcal{B}^Y, \tau_Y)\) consider the set \([\mathcal{B}^X, \mathcal{B}^Y]_{pb}\) of \(b\)-continuous functions \(f: X \rightarrow Y\) from \((\mathcal{B}^X, \tau_X)\) to \((\mathcal{B}^Y, \tau_Y)\). We define a \(b\)-convergence on the corresponding \(B\)-set \(\mathcal{B}^{Y \times X}\) (see also the beginning of Sec. 3) by setting for each \(B^* \in \mathcal{B}^{Y \times X} \setminus \{\emptyset\}\):
Proof. By construction it only remains to prove the axioms (bc4), (bc6) and (pb), respectively.

To (bc4): We have \( \tilde{f} \times \hat{f} \in \tau(\{f\}) \), since by hypothesis \( \tilde{f} \times \hat{f}(\mathcal{U}) = \tau_Y(\{f\}(B)) = \tau_Y(f[B]) \).

To (bc6): The inclusion \( \mathcal{U}^*(\mathcal{U}) \subseteq \mathcal{U}^*(\mathcal{U}) \) is valid for every \( \mathcal{U}^*, \mathcal{U}^* \in \mathcal{F}(\mathcal{B}^X, \mathcal{B}^Y) \) with \( \mathcal{U}^* \subseteq \mathcal{U}^* \) and each \( \mathcal{U} \in \tau_X(B) \) where \( B \in \mathcal{B}^X \).

To (pb): For \( B^* \in \mathcal{B}^{Y^X} \setminus \{\emptyset\} \) consider \( \mathcal{U}^* \in \tau(B^*) \) and choose \( f \in B^* \). Then \( B \in \mathcal{B}^X \setminus \{\emptyset\} \) and \( \mathcal{U} \in \tau_X(B) \) imply \( \mathcal{U} \in \tau_X(\{x\}) \) for some \( x \in B \), since \( \tau_X \) is pointed. By hypothesis \( \mathcal{U}^*(\mathcal{U}) \in \tau_Y(\{x\}) \) follows, hence \( \mathcal{U}^*(\mathcal{U}) \in \tau_Y(\{f(x)\}) = \tau_Y(\{f\}(\{x\})) \), because \( \tau_Y \) is pointed as well, which yields \( \mathcal{U}^* \in \tau(\{f\}) \).

The evaluation map

\[ e : (\mathcal{B}^X, \tau_X) \times ([\mathcal{B}^X, \mathcal{B}^Y]_{pb}, \tau) \rightarrow (\mathcal{B}^Y, \tau_Y) \]

is b-continuous, since the following equation

\[ (e \times e)(\mathcal{U} \times \mathcal{U}) = \mathcal{U}^*(\mathcal{U}) \]

holds for each \( \mathcal{U} \in \tau_X(B) \) and \( \mathcal{U}^* \in \tau(B^*) \), where \( B \in \mathcal{B}^X \setminus \{\emptyset\} \) and \( B^* \in \mathcal{B}^{Y^X} \setminus \{\emptyset\} \).

Moreover, finite products in \( \text{pb-CONV} \) can be described as follows:

\[ (\tau_X \times \tau_Y)(B) := \{ \mathcal{U} \in \mathcal{F}(X \times X, Y \times Y) \mid \exists \mathcal{U}_X \in \tau_X(p_X[B]) \exists \mathcal{U}_Y \in \tau_Y(p_Y[B]) \} \supseteq \mathcal{U} \times \mathcal{U} \]

where \( B \neq \emptyset \) and \( p_X, p_Y \) denote the corresponding projections.

Now, in fact, let \( \mathcal{U} \in (\tau_X \times \tau)(B) \) for some \( B \in \mathcal{B}^{X \times Y^X} \setminus \{\emptyset\} \) hence \( \mathcal{U} \in (\tau_X \times \tau)(\{x, f\}) \) for some \( (x, f) \in B \), since \( \tau_X \times \tau \) is pointed.
\[ \mathcal{U} \supseteq \mathcal{U}_X \times \mathcal{U}^* \text{ for some } \mathcal{U}_X \in \tau_X(p_X[\{(x, f)\}]) \text{ and } \mathcal{U}^* \in \tau(p_{X^*}[\{(x, f)\}]), \text{ hence } (e \times e)(\mathcal{U}_X \times \mathcal{U}^*) = \mathcal{U}^*(\mathcal{U}_X) \in \tau_Y(\{f\}{x}) \] by definition of \( \tau \).

Consequently, we obtain 
\[ \tau_Y(\{f\}{x}) = \tau_Y(\{f(x)\}) = \tau_Y(\{e(x, f)\}) \subseteq \tau_Y(e[B]) \]
since \( \tau_Y \) in particular is isoform.

Now, let \((B^Z, \tau_Z)\) be a pointed \( b \)-convergence space and let \( f : (B^X, \tau_X) \times (B^Z, \tau_Z) \rightarrow (B^Y, \tau_Y) \) be a \( b \)-continuous function. Then the map \( \hat{f} : (B^Z, \tau_Z) \rightarrow ([B^X, B^Y]_{pb}, \tau) \) defined by \( \hat{f}(z)(x) := f(x, z) \) for every \( x \in X, z \in Z \), is again \( b \)-continuous (see (2) in Sec. 3).

Suppose \( \mathfrak{U} \in \tau_Z(\hat{B}) \) for some \( \hat{B} \in B^Z \setminus \{\emptyset\} \). Then \( \hat{B} \in \tau_Z(z) \) for some \( z \in \hat{B} \). We aim to show that \((\hat{f} \times \hat{f})(\mathfrak{U}) \in \tau(\{\hat{f}(z)\}) \) holds, because this implies \((\hat{f} \times \hat{f})(\mathfrak{U}) \in \tau(\hat{f}[B]) \), since \( \tau \) in particular is isoform.

So let \( B \in B^X \setminus \{\emptyset\} \) and \( \mathfrak{U} \in \tau_X(B) \), hence \( \mathfrak{U} \in \tau_X(\{x\}) \) for some \( x \in B \). Consequently \((f \times f)(\mathfrak{U}) \) holds, which shows that \((f \times f)(\mathfrak{U}) \in \tau(\{f(z)\}{x}) \) holds, which implies \((f \times f)(\mathfrak{U}) \in \tau(\{f(z)\}) \). This concludes the proof. ♦

4.20 Corollary. \( pb\text{-CONV} \) is cartesian closed.

4.21 Theorem. \( pb\text{-CONV} \) is a topological universe.

Proof. Taking into account Rem. 4.16, Th. 4.19 and Th. 4.18, the claim easily follows.

![Figure 1. Relationships of the categories mentioned in this paper](image-url)
4.22 Remark. Since $pb\text{-}CONV$ is bicoreflective in $ib\text{-}CONV$ (see Th. 4.15) and closed under formation of products in $ib\text{-}CONV$, $pb\text{-}CONV$ is again strong.

4.23 Theorem. $pb\text{-}CONV$ is a strong topological universe.

References