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FOUR TOPOLOGIES EXAMINED BY SOME CARDINAL FUNCTIONS

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Abstract: Real linear spaces equipped with core, directional, Klee and finite topologies are examined by cardinal functions such as density, cellularity, extent, character, weight and tightness. There are also stated the cardinality of the family of open and regularly open sets. Moreover, it is shown that there exists an open set G in the directional topology such that its interior in the finite topology is empty and the complement of G in every finite dimensional subspace is nowhere dense in the Euclidean topology.

1. Introduction and notation

We continue the research on four topologies undertaken in [7] and [8]. All these topologies are defined in real linear spaces. In this paper a linear space is meant as a real linear space of dimension at least 1. Usually it is denoted by X and its dimension (i.e., the cardinality of its Hamel

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base) by $\dim X$.

Topologies we investigate are denoted by $\tau_0(X)$, $\tau_1(X)$, $\tau_2(X)$ and $\tau_3(X)$, or shortly by τ_0, τ_1, τ_2 and τ_3 if it is clear which a space X is taken into consideration. Such sets as the interior, the closure and the boundary of a set A in the topology τ_i, $i = 0, 1, 2, 3$, are denoted by $\mathrm{Int}_i A$, $\mathrm{Cl}_i A$, $\mathrm{Fr}_i A$, resp. Analogous convention concerns, e.g., open sets, compact sets, boundary sets, so we talk, e.g., about i-open sets, i-compact sets, i-boundary sets with $i \in \{0, 1, 2, 3\}$. We put the index j if the property at hand holds true for every one of three topologies τ_1, τ_2 and τ_3. If the property holds also for the topology τ_0, we put the general index j into parentheses, so we have, e.g., (j)-open sets.

The topology τ_0 has been introduced by Klee and Kakutani in [9] and they named it a finite topology. It is defined as the strongest topology such that in any finite dimensional space it induces the Euclidean topology. Lelong in [17] showed that this topology is also the strongest of all topologies defined on X such that for every $y \in X$ the function f_y, where $f_y(x, r) = x + ry$, is continuous on $X \times \mathbb{R}$, where $\mathbb{R}$ denotes the real line equipped with Euclidean topology (in the next we always take $\mathbb{R}$ with this topology). In [17] there are discussed generalizations of such topologies, namely the topologies in linear spaces over fields satisfying special conditions (the real space is a particular case of these spaces). These topologies are used to make insight into subharmonic functions in linear spaces and into so-called φ-topologies which are determined by a family φ of functions.

Probably the most known topology among four topologies discussed in this paper is the topology τ_1, in [10] Klee called it a *core topology*. It is the strongest topology such that it induces the Euclidean topology on any line. The topology τ_1 may be defined in various ways and we later give some of them. This topology was investigated in [10], [11] and [12], as well as in [14], [6], [15], [19] and in [13]. Moreover, in [13] there is presented its application in optimization. Generalizations of the core topology are given, e.g., in [20], [21], [22], [3], [4], [5] and [18]. These generalizations are mostly obtained in two ways, or the Euclidean topology is not induced on all lines, or there is induced a topology which is stronger than the Euclidean one. The combination of these both ways is also dealt with. An interesting property of the core topology is that it is the strongest topology in a real space such that the addition and the multiplication are separately continuous. Moreover, the topology τ_1 has this property that any directionally continuous function is continuous in this topology (in this paper a function is called directionally continuous if it is defined on X, assumes values in $\mathbb{R}$, and its restriction to any line is continuous in the Euclidean topology).

The topology determined by the family of all directionally continuous functions (i.e., the weakest topology such that any directionally continuous function is continuous in this topology) in [7] is called a *directional topology*. We denote it by τ_2. In [7, p. 60] it is shown that if $\dim X \geq 2$, then τ_2 is essentially weaker then the core topology.

The topology τ_3, in [7] called a Klee topology, has been first defined in [12] for finite dimensional spaces. In [7] there is given its extension to arbitrary real linear spaces, and this generalization is made via the topology τ_0. The definition of this topology will be given later. In [7] there is proved that in spaces of dimension at least 2 the Klee topology is essentially weaker than the directional topology. Roughly saying, the Klee topology and the core topology approximate the directional topology. We hope that the exploration of these topologies lets more completely recognise the nature of the space of directionally continuous functions.

There are already known some properties of these topologies, e.g., their relation to the separation axioms [12], [6], [8], the structure of the (j)-compact sets [19], [8], the structure of (j)-connected components of open sets [8], the Baire property of 1-open sets [6]. There is also solved the problem of the classification to sequential spaces and Frechet spaces [8].

The standard research of any topology includes the determination of values of basic cardinal functions. In this paper we deal with following cardinal functions (their names are followed by their denotations): the density – d, the cellularity, or Souslin number – c, the hereditary cellularity – hc, character – χ, the pseudocharacter – ψ, the weight – w, the π-character – π-χ, the π-weight – π-w, the number of open sets (i.e., the cardinality of collection of all open sets) – o, the number of regular open sets – ro (in [2] a regular set is called an open domain), the extent – e, the tightness – t. Some particular statements concerning the weight and the density of topologies τ_0 and τ_1 are stated in [12] and [6]. Here we complete them by results for topologies τ_2 and τ_3, and related to the dimension of space at hand. By $f_i(X)$ we denote the value of the cardinal function f for the topological space (X, τ_i). If $f_1(X) = f_2(X) = f_3(X)$, then, analogously as above, we shortly write $f_j(X)$. If $f_0(X) = f_j(X)$, we write $f_{(j)}(X)$.

Before we will give the definition of the Klee topology we establish some terminology and denotations used in this paper.

The sets of natural, rational, real and nonnegative numbers are denoted by $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}_+$, respectively. The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$, and that of $\mathbb{R}$ by $\mathfrak{c}$. $\aleph_0$ denotes also the initial number of this cardinality. Analogous convention concerns $\mathfrak{c}$. Other ordinal numbers are denoted by Greek letters.

The zero element of X is written as 0. The closed line segment between different points $a, b \in X$ is designated as $\langle a, b\rangle = \{\lambda a + (1-\lambda)b : : 0 \le \lambda \le 1\}$, analogous denotations are used for (semi-)open intervals, e.g. $\langle a, b) = \langle a, b\rangle \setminus \{b\}$, $(a, b) = \langle a, b\rangle \setminus \{a, b\}$. For any sets $S \subset \mathbb{R}$ and $A, B \subset X$ and for any $s \in S$ and $x \in X$ we write $SA = \{sa : s \in S, a \in \in A\}$, $sA = \{s\}A$, $A + B = \{a + b : a \in A, b \in B\}$, $x + B = \{x\} + B$. For $x \in X$, $r \in \mathbb{R}$, and for any family $\mathcal{B}$ of subsets in X we set $x + \mathcal{B} = = \{x + B : B \in \mathcal{B}\}$, $r\mathcal{B} = \{rB : b \in \mathcal{B}\}$.

We say that subspaces L and M of X are complementary each to another in X if $L + M = X$ and $L \cap M = \{0\}$. Then we write $\operatorname{codim} L = \dim M$.

A cone generated by the set $A \subset X$ and with its vertex at the point $y \in X$ is the set $\operatorname{Con}(A, y) = \{y + rx : \ r \ge 0, \ x \in A\}$. If $y = 0$, then we put $\operatorname{Con} A = \operatorname{Con}(A, 0)$.

We write $\sum\limits_{t \in T} a_t$ when almost all summing elements a_t are equal to 0.

The linear space spanned by the set $A \subset X$ is defined to be the set

$$\operatorname{Lin} A = \left\{ \sum_{t \in T} \alpha_t u_t : \alpha_t \in \mathbb{R}, u_t \in A \right\}.$$

The family of all functions defined on a set A and assuming values in a set B is denoted by B^A. The restriction of the function f to the set A contained in the domain of f is denoted by $f|A$, and $f^{-1}(B) = = \{a : f(a) \in B\}$ is the inverse-image of the function f assuming values in the set B. The superposition $f \circ g$ of functions f and g is defined by $f \circ g(x) = f(g(x))$. A linear map f such that $f \circ f = f$ is called a projection. If $\wp$ is a projection in X, $\wp(X) = L$ and $\wp^{-1}(0) = M$, we say that the projection $\wp$ maps onto L and parallelly to M.

Sequences are denoted as (x_n), (y_n) etc. We write $(x_n) \subset A$ if $x_n \in A$ for all $n \in \mathbb{N}$.

If f_k's, $(k = 1, 2, \ldots, n)$ are real functions then $\sup\{f_k : k = = 1, 2, \ldots, n\}$ is the function φ defined by formula $\varphi(x) = \sup\{f_k(x) : : k = 1, 2, \ldots, n\}$.

The Euclidean norm of an element $x \in \mathbb{R}^n$ is denoted by $\|x\|$, and $K(x, r)$ stands for the open ball centered at x and of radius r. $\rho_A(x)$ is the distance of the point x to the set A, i.e., $\rho_A(x) = \inf\{\|x - a\| : a \in A\}$, and $\rho(A, B) = \inf\{\|x - a\| : a \in A, b \in B\}$ is the distance between sets A and B, where $A, B \subset \mathbb{R}^n$.

Let A be any subset in X. The *core* of A (with respect to X) denoted by $\mathrm{Cor}_X A$, or shortly $\mathrm{Cor}A$, is defined to be the subset of A such that $a \in \mathrm{Cor}A$ if and only if for every $x \in X \setminus \{a\}$ there exists an element y in the segment (a, x) such that $\langle a, y\rangle \subset A$. Following [6] we call a set A a *core set* if $A = \mathrm{Cor}A$. The family of all core sets is a topology, and it is nothing else than the core topology.

The definition of the topology τ_3 will be here given in terms of a Klee pair. A pair (U, F) of subsets $U, F \subset X$ is called a *Klee pair* for a point $x \in X$ if

1° U is 0-open in X,

2° $F \subset U$,

3° $\{x\} \cup F$ is 0-closed in X,

4° $x \in \mathrm{Cor}\,(\{x\} \cup F)$.

The *Klee topology* in X is the topology, the base of which is the family consisted of all open sets in $\tau_0(X)$ and all sets of the form $\{x\} \cup U$, where $x \in X \setminus U$, U is open in $\tau_0(X)$ and there exists a subset F of X such that (U, F) is the Klee pair for x.

Topological notions are as they are defined in [2], however we allow some exceptions. They affect, e.g., the notions of a neighbourhood. By the neighbourhood of a point x we mean a set A such that x belongs to its interior.

The family $\mathcal{B}$ of open sets in a topology is called a π-base for this topology if for every open set G there exists a set $B \in \mathcal{B}$ such that $B \subset \subset G$. If we also demand G to include the point x, then $\mathcal{B}$ is called a π-base for the topology at the point x.

If $L \subset Y$ and Y is a space equipped with the topology η, then the topology induced in L by the topology η in Y is denoted by $\eta|L$. As in [16, p. 270], for a set $A \subset Y$ and an ordinal number α the derived set of an order α is denoted by $A^{(\alpha)}$. First ordinal number α such that the set $A^{(\alpha)}$ is perfect is called the *rank* of A and denoted by $\delta(A)$.

From [7] and [8] let us here recall following facts:
Fact 1.1. *The inclusions* $\tau_0 \subset \tau_3 \subset \tau_2 \subset \tau_1$ *take place in any* X *and they turn into equalities only if* $\dim X = 1$.
Fact 1.2. *If* B *is a Hamel base of* X *and*

$$K = \left\{ x = \sum_{b \in B} \alpha_b b : |\alpha_b| < \frac{1}{3} \right\},$$

then for every $b \in B$ *the set* $U_b = b + K$ *is an* 0*-open neighbourhood of* b, *and* $U_{b_1} \cap U_{b_2} = \emptyset$ *for* $b_1 \neq b_2$.

2. Density, cellularity and extent

Lemma 2.1. *Let* B *be a Hamel base of* X. B *is* 0*-closed set and every its element is* 0*-isolated of* B.
Proof. For every finite dimension subspace L of X the set $L \cap B$ is finite, so it is 0-closed. Therefore B is 0-closed. By Fact 1.2, the family $\{K + b : b \in B\}$ consists of 0-open and pairwise disjoint sets. Hence every $b \in B$ is the 0-isolated point of B. $\Diamond$
Theorem 2.1. *There hold the equalities*

$$d_{(j)}(X) = c_{(j)}(X) = \sup\{\aleph_0, \dim X\}.$$

Proof. According to Cor. 1 in [6, p. 244], the space (X, τ_1) is separable if $\dim X \leq \aleph_0$. Hence $(X, \tau_{(j)})$ is separable if $\dim X \leq \aleph_0$. In the next let $\dim X > \aleph_0$. Let $\{b_t : t \in T\}$ be a Hamel base of X and let $S(X)$ be the set of elements $\sum\limits_{t \in T} \alpha_t b_t$, where for all $t \in T$ the coefficients α_t are rational numbers. We will show that $S(X)$ is 1-dense. Let $x \in X \setminus \{0\}$. Therefore there exists a finite set $T_x \subset \mathcal{T}$ such that $x \in L = \mathrm{Lin}\{b_t : t \in T_x\}$. Hence $x \in \mathrm{Cl}_1(S(X) \cap L) = L$. Since L is 1-close, so $L \subset \mathrm{Cl}_1(S(X))$ and, finally, $x \in \mathrm{Cl}_1(S(X))$. It says that $S(X)$ is 1-dense, in consequence it is (j)-dense. This proves that $d_{(j)}(X) \leq \sup\{\aleph_0, \dim X\}$.

Now we are going to find an inequality involving $c_{(j)}(X)$ and $\sup\{\aleph_0, \dim X\}$. In this aim let's notice that in $\mathbb{R}^n$ there exists a countable family of open (and therefore (j)-open) and pairwise disjoint sets, so $c_{(j)}(X) \geq \aleph_0$. By Fact 1.2 there exits a family $\mathcal{F}$ of 0-open and pairwise disjoint sets such that $\mathrm{card}\mathcal{F} = \dim X$. It implies that $c_{(j)}(X) \geq$ $\geq \sup\{\aleph_0, \dim X\}$.

The relation $c_{(j)}(X) \leq d_{(j)}(X)$ given in [2, p. 86] completes the proof. $\Diamond$

Theorem 2.2. *There hold* $hc_0(X) = e_0(X) = \sup\{\aleph_0, \dim X\}$.

Proof. It's clear that $e_0(X) \geq \aleph_0$ (it's enough to see it for $\mathbb{N}x$, where $x \neq 0$). Taking into account Fact 1.2 we have $e_0(X) \geq \sup(\aleph_0, \dim X)$.

We will show that $hc_0(X) \leq \sup(\aleph_0, \dim X)$. First we consider the case $\dim X \geq \aleph_0$. Let $B = \{b_t : t \in T\}$ be a Hamel base of X, and $\mathcal{T}$ be the family of all finite subsets of T. Obviously, $\operatorname{card}\mathcal{T} = \operatorname{card} T$. Let $\mathcal{L} = \{\operatorname{Lin}\{b_t : t \in S\} : S \in \mathcal{T}\}$. Let's suppose that there exists a set $A \subset X$ such that $\operatorname{card} A > \dim X$ and every its element is isolated. Because $\operatorname{card} A > \dim X = \operatorname{card}\mathcal{T}$, so there exists $L_0 \in \mathcal{L}$ such that $\operatorname{card}(A \cap L_0) = \operatorname{card} A > \aleph_0$.

It is easy to state that $hc_0(\mathbb{R}^n) = \aleph_0$ for every $n \in \mathbb{N}$. Indeed, if this equality would not hold then there should exist an uncountable set $C \subset \mathbb{R}^n$ such that every its point is isolated and $r(x) = \rho_{C\setminus\{x\}}(x) > 0$ for each $x \in C$. Taking $C_n = \left\{x : r(x) > \frac{1}{n}\right\}$, $n \in \mathbb{N}$, we see that there exists $n_0 \in \mathbb{N}$ such that $\operatorname{card} C_{n_0} = \operatorname{card} C$. Hence the open balls $K\left(x_1, \frac{1}{2n_0}\right)$ and $K\left(x_2, \frac{1}{2n_0}\right)$ are disjoint for different x_1, $x_2 \in C_{n_0}$. Therefore there exists the uncountable family of balls which are disjoint each with other. In $\mathbb{R}^n$ it is impossible. This contradiction proves that $hc_0(\mathbb{R}^n) \leq \aleph_0$.

In consequence, there does not exist a set A, the existence of which was assumed above. This way it is proved that $hc_0(X) \leq \dim X$ if $\dim X \geq \aleph_0$. In conclusion, we have $hc_0(X) \leq \sup(\aleph_0, \dim X)$ and, in view of the inequality $e_0(X) \leq hc_0(X)$, it proves the thesis. $\Diamond$

Theorem 2.3. *For every space of dimension at least* 2 *there holds* $e_j(X) = \operatorname{card} X$.

Proof. It's obvious that in $\mathbb{R}^2$ any circle is j-closed and it is composed of j-isolated points only. Hence $e_j(X) \geq \mathfrak{c}$. Let $B = \{b_t : t \in T\}$ be a Hamel base of the space X. Lemma 2.1 states that the base B is 0-closed in X and consisting exclusively of 0-isolated elements. Now, by Fact 1.1, B is (j)-closed. Because of $\operatorname{card} X = \sup\{\mathfrak{c}, \dim X\}$, the proof is finished. $\Diamond$

The inequality $hc_j(X) \geq e_j(X)$ and Th. 2.3 imply

Corollary 2.1. *For every space* X *there holds* $hc_j(X) = e_j(X) = \operatorname{card} X$.

Theorem 2.4. *For every* X *there holds* $ro_{(j)}(X) = \sup\{\mathfrak{c}, 2^{\dim X}\}$.

Proof. Let a set D be a (j)-dense in X and $\operatorname{card} D = d_{(j)}(X)$. Since $\operatorname{Cl}_{(j)}G = \operatorname{Cl}_{(j)}(D \cap G)$ for every (j)-open set G, so $ro_{(j)}(X) \leq 2^{d_{(j)}(X)} =$

$= 2^{\sup\{\aleph_0, \dim X\}} = \sup\{\mathfrak{c}, 2^{\dim X}\}$.

Since the topology $\tau_0|X$ is Euclidean for X if $\dim X < \aleph_0$, so $ro_0(X) = \mathfrak{c}$. By Fact 1.1, it follows that $ro_{(j)}(X) \geq \mathfrak{c}$.

In the next $\dim X \geq \aleph_0$ and let B be a Hamel base of X. First we notice that the set K defined in Fact 1.2 is 0-regularly open. Therefore for every set $B_0 \subset B$ the set $\bigcup\limits_{b \in B_0} (b + K)$ is 0-regularly open. In consequence, $ro_0(X) \geq 2^{\dim X}$ and it implies that $ro_{(j)}(X) \geq 2^{\dim X}$. Therefore $ro_{(j)}(X) \geq \sup\left\{\mathfrak{c}, 2^{\dim X}\right\}$ and this completes the proof. ◊

3. Character, weight

Theorem 3.1. *For every space X there holds $\psi_{(j)}(X) = \aleph_0$.*

Proof. It's obvious that $\psi_{(j)}(x) = \psi_{(j)}(0)$ for every $x \in X$. Let B be a Hamel base of the space X. Then, for K defined as in Fact 1.2, we have $\bigcap\limits_{n=1}^{\infty} \frac{1}{n} K = \{0\}$. It implies that $\psi_0(X) = \aleph_0$. The equality $\psi_j(X) = \aleph_0$ follows from Fact 1.1. ◊

From Thms. 2.1 and 2.4 it follows

Corollary 3.1. *For every X and for $i = 0, 2, 3$ there holds*

$$w_i(X) \leq \sup\left\{\mathfrak{c}, 2^{\dim X}\right\}.$$

Proof. By [7] τ_0 is hereditary normal. By [8] τ_2 and τ_3 are totally regular. Hence all these topologies are regular, so for $i = 0, 2, 3$ the family of all i-regular open sets is the base of the topology τ_i. Therefore, by Th. 2.4, we have $w_i(X) \leq \sup\{\mathfrak{c}, 2^{\dim X}\}$. ◊

Now we will deal with the character and π-character of topological spaces (X, τ_i), where $i = 0, 1, 2, 3$.

Since any translation, i.e. the transformation $f_y : X \to X$ defined by the formula $f_y(x) = x + y$, where $y \in X$, is the homeomorphism mapping the space $\left(X, \tau_{(j)}\right)$ onto itself, so the family $\mathcal{B}$ is a (π-)base for the topology $\tau_{(j)}$ at the point 0 iff for each $x \in X$ the family $x + \mathcal{B}$ is a (π-)base for the topology $\tau_{(j)}$ at the point x. Therefore (π-)character of the space is equal to (π-)character at 0. Thanks to this property we will deal with (π-)character at 0 only and in the next we will not mention it.

Lemma 3.1. *Let L and M be complementary subspaces, $\wp$ be a projection onto L and parallel to M. Then $\wp(G)$ is (j)-open in L for every (j)-open set G.*

Proof. Let $x \in M$. Then $G \cap (x+L) \in \tau_{(j)}(x+L)$ and $(x+L) \cap M = \{x\}$. Consequently, $\wp(G \cap (x+L)) = (G \cap (x+L)) - x \in \tau_{(j)}(L)$. Hence $\wp(G) = \bigcup_{x \in M} \wp(G \cap (x+L)) \in \tau_{(j)}(L)$. $\Diamond$

Fact 3.1. *Let f be one of following cardinal functions $\chi_{(j)}$, π-$\chi_{(j)}$, $w_{(j)}$, π-$w_{(j)}$. Then, for any subspace L of X, there holds $f(L) \leq f(X)$.*

Proof. Since $\tau_{(j)}|L = \tau_{(j)}(L)$, so for any subspace L of X there holds $f(L) \leq f(X)$ if $f = \chi_{(j)}$ or $f = \omega_{(j)}$. In the next we deal with $f = \pi$-$\chi_{(j)}$ or $f = \pi$-$\omega_{(j)}$ only.

In this proof we say that a family $\mathcal{B}$ is an appropriate base if it is π-base for the topology $\tau_{(j)}$ at the point 0 in case $f = \pi$-$\chi_{(j)}$, and it is π-base for the topology $\tau_{(j)}$ in case $f = \pi$-$\omega_{(j)}$.

Suppose that $f(L) > f(X)$ for a subspace L of X. Then there exists an appropriate base $\mathcal{B}$ for X such that $\operatorname{card} \mathcal{B} < f(L)$. Let $\wp$ be a projection onto L and parallel to M. Therefore $\{\wp(B) : B \in \mathcal{B}\}$ is not an appropriate base for L. By Lemma 3.1 the set $\wp(B) \in \tau_{(j)}(L)$ for every $B \in \mathcal{B}$, and $\mathcal{B}$ is not an appropriate base for L. Hence there exists a (j)-open set G or (j)-neighbourhood G of 0, respectively, such that $\wp(B) \setminus G \neq \emptyset$ for every $B \in \mathcal{B}$. Hence $B \setminus (G + M) \neq \emptyset$ for every $B \in \mathcal{B}$. Taking into account that $G + M \in \tau_{(j)}$ we conclude that $\mathcal{B}$ is not an appropriate base. This contradiction proves the validity of the inequality $f(L) \leq f(X)$. $\Diamond$

Lemma 3.2. *The inequality π-$\chi_{(j)}(X) > \dim X$ holds true for every space X.*

Proof. The lemma is obvious in case when X is finite dimensional. Therefore in the next we deal with X such that $\dim X \geq \aleph_0$.

Suppose that π-$\chi_i(X) \leq \dim X$ for an index $i \in \{0,1,2,3\}$ and for some space X. Then there exists π-base for the topology τ_i at 0; let this π-base be $\{V_\alpha : \alpha < \beta\}$, where $\beta \leq \dim X$. We inductively define the set $B = \{b_\alpha : \alpha < \beta\}$ such that $b_1 \in V_1 \setminus \{0\}$ and $b_\alpha \in V_\alpha \setminus \operatorname{Lin}\{b_\gamma : \gamma < \alpha\}$ for $1 < \alpha < \beta$.

The set B is the Hamel base of the subspace $L = \operatorname{Lin} B$ of X. Let $M = \{0\}$ in case $L = X$, and M be the complementary subspace to L in X otherwise. Now, for any $r_\alpha > 0$ such that $r_\alpha b_\alpha \in B_\alpha$, we denote

$$K = \left\{ \sum_{\alpha < \beta} s_\alpha b_\alpha : |s_\alpha| < r_\alpha \right\} + M.$$

It's clear that K is 0-open. Hence K is (j)-open. Since $V_\alpha \setminus K \neq \emptyset$ for

$\alpha < \beta$, so $\{V_\alpha : \alpha < \beta\}$ is not a π-base for the topology τ_i at 0, This contradiction closes the proof. $\Diamond$

From Lemma 3.2 we immediately have

Corollary 3.2. *The inequality* $\chi_{(j)}(X) > \dim X$ *holds for every space* X.

Corollary 3.3. *If* $\dim X \geq \mathfrak{c}$, *then* $w_{(j)} = \chi_{(j)}$.

Proof. Let $\mathcal{B}$ be a base for the topology $\tau_{(j)}$ at the point 0. Then $\sum_{x\in X}(x+\mathcal{B})$ is a base for the topology $\tau_{(j)}$. Therefore $w_{(j)} \leq \chi_{(j)} \cdot \operatorname{card} X$. Since $\chi_{(j)} > \dim X \geq \mathfrak{c}$ and, in accordance with the assumption, we have $\operatorname{card} X = \dim X$, so $w_{(j)} \leq \chi_{(j)}$. It, in view of the obvious inequality $\chi_{(j)} \leq w_{(j)}$, gives the equality $w_{(j)} = \chi_{(j)}$. $\Diamond$

Corollary 3.4. *For every space* X *there holds the equality* $\pi\text{-}\chi_{(j)}(X) = \pi\text{-}w_{(j)}(X)$.

Proof. From Th. 2.1 we have $d_{(j)}(X) = \sup\{\aleph_0, \dim X\}$. In view of the inequality $\pi\text{-}w_{(j)}(X) \leq (\pi\text{-}\chi_{(j)}(X)) \cdot d_{(j)}(X)$ from Lemma 3.2 we get $\pi\text{-}w_{(j)}(X) \leq \pi\text{-}\chi_{(j)}(X)$. This, together with the obvious inequality $\pi\text{-}\chi_{(j)}(X) \leq \pi\text{-}w_{(j)}(X)$, gives the desired equality. $\Diamond$

Theorem 3.2. *If* $\dim X \geq 2$, *then* $\sup\{\mathfrak{c}, \dim X\} < \pi\text{-}\chi_1(X) \leq \chi_1(X) \leq 2^{\sup\{\mathfrak{c}, \dim X\}}$.

Proof. Let L be a subspace of X and $\dim L = 2$. Let's suppose that $\mathcal{B}$ is a π-base for the topology τ_1 at 0 such that $\operatorname{card}\mathcal{B} \leq \mathfrak{c}$. Then, by Lemma 1 [6, p. 241] there exists a set M such that $0 \notin M$, $\operatorname{card} M = \mathfrak{c}$ and $M \cap B \neq \emptyset$ for every $B \in \mathcal{B}$ and each line in X has no more than 2 points laying in M. Therefore $G = X \setminus M$ is 1-open and $B \setminus G \neq \emptyset$ for every $b \in \mathcal{B}$. In consequence, $\mathcal{B}$ is not a π-base for the topology τ_1 at 0. This contradiction implies that $\pi\text{-}\chi_1(X) > \mathfrak{c}$.

By Lemma 3.2 we have $\pi\text{-}\chi_1(X) > \dim X$. This proves the left inequality. It also completes the proof because $\pi\text{-}\chi_1(X) \leq \chi_1(X)$ and $\chi_1(X) \leq 2^{\operatorname{card} X}$. $\Diamond$

Corollary 3.5. *For any space* X *there hold the equalities* $\pi\text{-}\chi_1(X) = \pi\text{-}w_1(X)$ *and* $\chi_1(X) = w_1(X)$.

Proof. The first of above equalities is stated in Cor. 3.4.

The second equality is obvious in case $\dim X = 1$, because $\tau_1(X)$ is the Euclidean topology and, consequently, $\pi\text{-}\chi_1(X) = \chi_1(X) = w_1(X) = \pi\text{-}w_1(X) = \aleph_0$. If $\dim X \geq 2$, then by Th. 3.4 we have $\operatorname{card} X < \pi\text{-}\chi_1(X)$. By the obvious inequality $w_1(X) \leq \operatorname{card} X \cdot \chi_1(X)$ we get $\chi_1(X) \leq w_1(X) \leq \chi_1(X)$, so $w_1(X) = \chi_1(X)$. $\Diamond$

As in [1, p. 115], the family $\mathcal{D} \subset \mathbb{N}^{\mathbb{N}}$ is called a *dominating family* if for each $f \in \mathbb{N}^{\mathbb{N}}$ there exists a function $g \in \mathcal{D}$ such that $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$. If this inequality holds true for all $n \in \mathbb{N}$, then let us call the family $\mathcal{D}$ a *strongly dominating family.* As in [1, p. 115], the minimal cardinality of a dominating family is denoted by $\mathfrak{d}$. In [1, p. 119] it is shown that the cardinality of strongly dominating family is also equal to $\mathfrak{d}$.

Lemma 3.3. *Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of X. Let U be a 0-open neighbourhood of the point 0. Then there exists the sequence (ε_n) of positive numbers such that $V = \left\{x = \sum\limits_{n\in\mathbb{N}} \alpha_n b_n : |\alpha_n| < \varepsilon_n \text{ for } n \in \mathbb{N}\right\} \subset U$.*

Proof. It is obvious that there exists $\varepsilon_1 > 0$ such that $\langle -\varepsilon_1 b_1, \varepsilon_1 b_1\rangle \subset U \cap$ $\cap \mathbb{R} b_1$. Now we suppose that there exist positive ε_k, where $k = 1, 2, \ldots, n$, such that $V_n = \left\{x = \sum\limits_{k=1}^{n} \alpha_k b_k : |\alpha_k| \leq \varepsilon_k \text{ for } k = 1, 2, \ldots, n\right\} \subset U$. We will show that there exists $\varepsilon_{n+1} > 0$ such that

$$V_{n+1} = V_n + \langle -\varepsilon_{n+1} b_{n+1}, \varepsilon_{n+1} b_{n+1}\rangle \subset U.$$

In the Euclidean topology in $L_{n+1} = \mathrm{Lin}\,\{b_k : k = 1, 2, \ldots, n+1\}$ the set V_n is compact, $L_{n+1} \setminus U$ is closed and $V_n \cap (L_{n+1} \setminus U) = \emptyset$. Hence $r = \rho\,(V_n, L_{n+1} \setminus V) > 0$. Taking $\varepsilon_{n+1} < r$ we have $V_{n+1} \subset U$. By induction, there exists the sequence (ε_n) of positive numbers such that $\overline{V} = \left\{x = \sum\limits_{n\in\mathbb{N}} \alpha_n b_n : |\alpha_n| \leq \varepsilon_n \text{ for } n \in \mathbb{N}\right\} \subset U$ and it makes the proof complete. $\Diamond$

Corollary 3.6. *Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of the space X and let G be a non-empty 0-open set. Then there exist $m \in \mathbb{N}$, $u^{(n)} \in \mathbb{Q}$ for $n = 1, 2, \ldots, m$ and $f \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\sum_{n=1}^{m} u^{(n)} b_n + \left\{\sum_{n\in\mathbb{N}} \alpha_n b_n : |\alpha_n| < \frac{1}{f(n)}\right\} \subset G.$$

Proof. Since the set $\left\{\sum\limits_{n\in\mathbb{N}} \alpha_n b_n : \alpha_n \in \mathbb{Q}\right\}$ is 0-dense, so there exist $m \in \mathbb{N}$ and $u^{(n)} \in \mathbb{Q}$ for $n = 1, 2, \ldots, m$ such that $u = \sum\limits_{n=1}^{m} u^{(n)} b_n \in G$. From Lemma 3.3 there exists a sequence (ε_n) of positive numbers such that $\left\{\sum\limits_{n\in\mathbb{N}} \alpha_n b_n : |\alpha_n| < \varepsilon_n\right\} \subset G - u$. Taking a function $f \in \mathbb{N}^{\mathbb{N}}$ such that $f(n) > \frac{1}{\varepsilon_n}$ and it easily implies the validity of the corollary. $\Diamond$

In particular, we have

Corollary 3.7. *If $0 \in G$ and G is 0-open set, then there exists $f \in \mathbb{N}^{\mathbb{N}}$ such that* $\left\{ \sum\limits_{n \in \mathbb{N}} \alpha_n b_n : |\alpha_n| < \frac{1}{f(n)} \right\} \subset G$.

Lemma 3.4. *Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of X. For every $f \in \mathbb{N}^{\mathbb{N}}$ we define the set* $K_f = \left\{ \sum\limits_{n \in \mathbb{N}} \alpha_n b_n : |\alpha_n| < \frac{1}{f(n)} \right\}$. *Let $\mathcal{V}$ be a π-base for the topology τ_0 at 0. For every $V \in \mathcal{V}$ there are defined $n_V \in \mathbb{N}$, $u_V \in \mathbb{Q}^{n_V}$ and $f_V \in \mathbb{N}^{\mathbb{N}}$ such that* $G_V = \sum\limits_{n=1}^{n_V} u_V^{(n)} b_n + K_{f_V} \subset V$, *where $u_V^{(n)}$ denotes the n-th coordinate of u_V. Then the family $\{f_V : V \in \mathcal{V}\}$ contains a dominating family in $\mathbb{N}^{\mathbb{N}}$.*

Proof. Suppose that $\{f_V : V \in \mathcal{V}\}$ does not contain a dominating family in $\mathbb{N}^{\mathbb{N}}$. Then there exists a function $g \in \mathbb{N}^{\mathbb{N}}$ such that for every $V \in \mathcal{V}$ there exists $k_V \in \mathbb{N}$ such that $g(k_V) > f_V(k_V)$ and $k_V > n_V$. It implies that $G_V \setminus K_g \neq \emptyset$ for every $V \in \mathcal{V}$. Since K_g is 0-open and contains 0, so $\mathcal{V}$ is not a π-base for τ_0 at 0. This contradiction shows that $\{f_V : V \in \mathcal{V}\}$ contains a dominating family in $\mathbb{N}^{\mathbb{N}}$. $\Diamond$

Theorem 3.3. (1) *If X is finite dimensional, then* $\chi_0(X) = \pi\text{-}\chi_0(X) = \pi\text{-}w_0(X) = w_0(X) = \aleph_0$.

(2) *If* $\dim X = \aleph_0$, *then* $\chi_0(X) = \pi\text{-}\chi_0(X) = \pi\text{-}w_0(X) = w_0(X) = \mathfrak{d}$.

(3) *If* $\dim X > \aleph_0$, *then* $\dim X < \pi\text{-}w_0(X) = \pi\text{-}\chi_0(X) \leq \chi_0(X) \leq \leq w_0(X) \leq 2^{\dim X}$ *and* $\mathfrak{d} \leq \pi\text{-}w_0(X) = \pi\text{-}\chi_0(X) \leq \chi_0(X) \leq w_0(X)$.

Proof. (1) holds true because $\tau_0(X)$ is Euclidean.

(2). Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of X and let $\mathcal{B}$ be π-base for topology τ_0. Let $U_{m,u,f} = \sum\limits_{n=1}^{m} u^{(n)} b_n + \left\{ \sum\limits_{n \in \mathbb{N}} \alpha_n b_n : |\alpha_n| < \frac{1}{f(n)} \right\}$, where $m \in \mathbb{N}$, $u \in \mathbb{Q}^m$, $u^{(n)}$ is the n-th coordinate of u and $f \in \mathbb{N}^{\mathbb{N}}$. By Cor. 3.6 for each $V \in \mathcal{B}$ there exist $m_V \in \mathbb{N}$, $u_V \in \mathbb{Q}^{m_V}$ and $f_V \in \mathbb{N}^{\mathbb{N}}$ such that $U_{m_V, u_V, f_V} \subset V$. On behalf of Lemma 3.4 we have $\operatorname{card}\{f_V : V \in \mathcal{B}\} \geq \mathfrak{d}$. Since $m_V \in \mathbb{N}$ and $u_V \in \mathbb{Q}^{m_V}$, so we easily conclude that $\operatorname{card}\mathcal{B} \geq \mathfrak{d}$, hence $\pi\text{-}w_0(X) \geq \mathfrak{d}$. By Cor. 3.4 we have $\pi\text{-}w_0(X) = \pi\text{-}\chi_0(X) \geq \mathfrak{d}$.

Now let $\mathcal{F}$ be a strongly dominating family in $\mathbb{N}^{\mathbb{N}}$ such that $\operatorname{card}\mathcal{F} = \mathfrak{d}$. We put $V_f = \left\{ \sum\limits_{n \in \mathbb{N}} \alpha_n b_n : |\alpha_n| < \frac{1}{f(n)} \right\}$ for $f \in \mathcal{F}$. Cor. 3.7 implies that $\{V_f : f \in \mathcal{F}\}$ is a base for the topology τ_0 at 0. Hence the family $\{x + V_f : f \in \mathcal{F}, x \in X\}$ is the base for the topology τ_0. Denote $L_n = = \operatorname{Lin}\{b_k : k = 1, 2, \ldots, n\}$, $M_n = \operatorname{Lin}\{b_k : k > n\}$, and for given $x \in X \setminus \{0\}$ let n_x denote a natural number such that $x \in L_{n_x}$. Let's take $x = \sum_{n\in\mathbb{N}} \alpha_n b_n \in X \setminus \{0\}$, where $a_n \in \mathbb{R}$. Then there exist $r_k, s_k \in \mathbb{Q}$, where $k = 1, 2, \ldots, n_x$, such that $r_k < \alpha_k < s_k$ and $U_{x,f} = \sum_{k=1}^{n_x}(r_k, s_k)b_k \subset$ $\subset (x + V_f) \cap L_{n_x}$. Therefore

$$(*) \qquad x \in U_{x,f} + (V_f \cap M_{n_x}) \subset x + V_f.$$

It's clear that $U_{x,f} + (V_f \cap M_{n_x}) \in \tau_0$.

Let $\mathcal{B}_0 = \{V_f : f \in \mathcal{F}\}$ and

$$\mathcal{B}_n = \left\{ \sum_{k=1}^{n}(r_k, s_k)b_k + (V_f \cap M_n) : r_k, s_k \in \mathbb{Q},\ r_k < s_k,\ f \in \mathcal{F} \right\}$$

for $n \in \mathbb{N}$. It's obvious that $\mathcal{B}_n \subset \tau_0$ and $\operatorname{card}\mathcal{B}_n = \aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$ for $n \in \mathbb{N}\cup$ $\cup \{0\}$. From the inclusion $(*)$ it follows that $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$ is the base for the topology τ_0. Obviously, $\operatorname{card}\mathcal{B} = \mathfrak{d}$ and, consequently, $w_0(X) \leq \mathfrak{d}$.

Since $\pi\text{-}\chi_0(X) \leq \chi_0(X) \leq w_0(X)$, so $\pi\text{-}\chi_0(X) = \chi_0(X) = w_0(X) =$ $= \mathfrak{d}$. The equality $\pi\text{-}w_0(X) = \pi\text{-}\chi_0(X)$ is stated in Cor. 3.4.

(3). The inequality $\dim X < \pi\text{-}\chi_0(X)$ is stated in Lemma 3.2. The inequality $\chi_0(X) \leq 2^{\dim X}$ follows from Cor. 3.1. The inequality $\pi\text{-}\chi_0(X) \geq \mathfrak{d}$ follows from the part (2) and Fact 3.1. On the behalf of Cor. 3.4 and the obvious inequality $\pi\text{-}\chi_0(X) \leq \chi_0(X) \leq w_0(X)$ we see the desired inequalities are satisfied. It closes the proof. $\Diamond$

Theorem 3.4. *For every space X there hold the equalities $\pi\text{-}\chi_3(X) =$ $= \pi\text{-}w_3(X) = \pi\text{-}w_0(X) = \pi\text{-}\chi_0(X)$.*

Proof. $\operatorname{Int}_0(G) \neq \emptyset$ for every nonempty set $G \in \tau_3$. Therefore, if $\mathcal{B}$ is a π-base for the topology τ_0, then it is also a π-base for τ_3. It proves that $\pi\text{-}w_3(X) \leq \pi\text{-}w_0(X)$. On the other side, if $\mathcal{B}$ is a π-base for the topology τ_3, then for every $G \in \tau_3$ there exists a set $B \in \mathcal{B}$ such that $B \subset G$. Consequently, $\operatorname{Int}_0 B \subset G$ and $\{\operatorname{Int}_0 B : B \in \mathcal{B}\}$ is a π-base for the topology τ_0 and $\pi\text{-}w_0(X) \leq \pi\text{-}w_3(X)$. Therefore $\pi\text{-}w_0(X) = \pi\text{-}w_3(X)$.

Applying Cor. 3.4 we get the equalities $\pi\text{-}\chi_0(X) = \pi\text{-}w_0(X) =$ $= \pi\text{-}w_3(X) = \pi\text{-}\chi_3(X)$ and it makes the proof complete. $\Diamond$

In aim to determine the values of the character of the Klee topology and the directional topology, and the value of π-character of τ_2 we introduce the notion of the isolated direction.

Definition 3.1. Let G be an j-open set and let $x \in G$. A semiline, denoted by P, is called an *isolated direction* of the set G for the point x if there are satisfied following conditions:

1° the semiline P has its origin at x,
2° there exist a j-component U of $G \setminus \{x\}$ and a point $y \in P \setminus \{x\}$ such that the segment $(x, y\rangle \subset U$,
3° there does not exist an element $z \in U \setminus \hat{P}$ such that $\langle x, z\rangle \subset U$, where $\hat{P}$ is the line containing the semiline P.

Obviously, the set $G \setminus \{x\}$ is j-open if G is j-open. Since every two different j-components of $G \setminus \{x\}$ are disjoint, so for the point x the cardinality of all isolated directions is not greater than the cellularity $c_j(X)$. So, by Th. 2.1, we immediately get

Corollary 3.8. *In every j-open set there exist at most* $\sup\{\aleph_0, \dim X\}$ *isolated directions.*

Fact 3.2. *For every semiline $P \subset X$ there exists a 3-open set G such that P is its isolated direction and* $\mathrm{Int}_0 G = G \setminus \{x\}$, *where x is the origin point of P.*

Proof. Let L be a subspace of X such that $\mathrm{card}\,(P \cap L) \leq 1$ and $\mathrm{codim}\, L = 1$. Let $\{b_t : t \in T\}$ be a Hamel base of L. First we will show that there exists a 3-open set satisfying the requirements concerning the semiline $P - x$.

Let $y \in (P - x) \setminus L$ and $F = \Big\{\alpha y + \sum\limits_{t \in T} \alpha_t b_t : \sum\limits_{t \in T} \alpha_t^2 = \alpha^4, \alpha > 0\Big\}$. For every finite dimensional space M containing y the set $(F \cup \{0\}) \cap M$ is 0-closed. Hence $F \cup \{0\}$ is 0-closed and $G_1 \setminus \{0\} \subset \mathrm{Int}_0 G_1$, where $G_1 = X \setminus F$. Since $0 \in \mathrm{Cl}_0 F$, so $\mathrm{Int}_0 G_1 = G_1 \setminus \{0\}$.

Let $E = \bigcup\limits_{z \in F} (0, z\rangle$ and $L_- = \{\alpha y + u : \alpha \leq 0, u \in L\}$. It is easy to see the set $H = (P - y) \cup \frac{1}{2} E \cup L_-$ is 0-closed, $H \setminus \{0\} \subset \mathrm{Int}_0 G_1$ and $0 \in \mathrm{Cor}\, H$. It says that the sets $\mathrm{Int}_0 G_1$ and $H \setminus \{0\}$ form the Klee pair for the point 0. In consequence, G_1 is 3-open.

Moreover, if $\dim M = 2$, then $(F \cup \{0\}) \cap M$ is composed of parts of two parabolas. Every one of them is tangent to the semiline $P - x$ at the point 0. These parts lay on different sides of the line $\mathbb{R}y$. Hence $P - x$ is the isolated direction for the set G_1 at the point 0. Since the translation f_x, where $f_x(v) = x + v$, is a homeomorphism from (X, τ_0) onto the same space, so $G_1 + x$ is the set G mentioned in the thesis. $\Diamond$

Lemma 3.5. *If $2 \leq \dim X \leq \aleph_0$, $\mathcal{P}$ denotes the family of all semilines in X beginning at the point 0, G_P, where $P \in \mathcal{P}$, denotes a 3-open set such that P is its isolated direction at 0, then* $\mathrm{card}\, \mathcal{B} \geq \mathfrak{c}$, *where $\mathcal{B}$ is* *a family of* 1*-neighbourhoods of the point* 0 *such that for every* $P \in \mathcal{P}$ *there exists* $B \in \mathcal{B}$ *and* $B \subset G_P$.

Proof. It's clear that if $B \in \mathcal{B}$ and $B \subset G_P$ then P is the isolated direction for $\mathrm{Int}_1 B$. Since $\mathrm{card}\,\mathcal{P} = \mathfrak{c}$ and for every $B \in \mathcal{B}$, the set $\mathrm{Int}_1 B$ may have at most countable many isolated directions at 0, so $\mathrm{card}\,\mathcal{B} \geq \mathfrak{c}$. $\Diamond$

In virtue of the inclusions $\tau_3 \subset \tau_2 \subset \tau_1$, from Lemma 3.5 it immediately follows

Corollary 3.9. *If* $\dim X \geq 2$ *and* $\mathcal{B}$ *is a base for the topology* τ_i *at* 0*, where* $i = 2, 3$*, then* $\mathrm{card}\,\mathcal{B} \geq \mathfrak{c}$.

Theorem 3.5. *For every space* X *there holds* $w_3(X) = \chi_3(X)$*. Moreover,* (1) *if* $\dim X \geq 2$ *and* $2^{\dim X} \leq \mathfrak{c}$*, then* $\chi_3(X) = \mathfrak{c}$*,*

(2) *if* $2^{\dim X} > \mathfrak{c}$*, then* $\dim X < \chi_3(X)$ *and* $\mathfrak{c} \leq \chi_3(X) \leq 2^{\dim X}$.

Proof. First we consider Case (1). By Cor. 3.9 we have $\chi_3(X) \geq \mathfrak{c}$. By Cor. 3.1 we have $\chi_3(X) \leq 2^{\dim X} \leq \mathfrak{c}$. Therefore (1) holds true.

The first inequality in (2) follows from Cor. 3.2. The left part of the second inequality is implied by (1) and Fact 3.1, the right one follows from Cor. 3.1.

Since $\chi_3(X) \geq \sup\{\mathfrak{c}, \dim X\} = \mathrm{card}\,X$ and $w_3(X) \leq \mathrm{card}\,X \cdot \cdot\, \chi_3(X)$, so $w_3(X) \leq \chi_3(X)$. By the obvious inequality $w_3(X) \geq \chi_3(X)$ we get the equality $w_3(X) = \chi_3(X)$. $\Diamond$

Lemma 3.6. *Let* L *be a subspace of* X*,* $\mathrm{codim}\, L = 1$*,* $b \in X \setminus L$*. Let* f *be a directionally continuous function on* L *such that* $f(L) \subset \langle 0, 1\rangle$*. Let* $b_1 \in L \setminus \{0\}$*,* $L' = \mathrm{Lin}\,\{b, b_1\}$ *and* $F = \{\alpha b + \alpha_1 b_1 : |\alpha_1| = \alpha^2, \alpha > 0, \alpha_1 \in \in \mathbb{R}\}$*. Then for* $a \in f^{-1}(\langle 0, 1))$ *there exists a directionally continuous function* f_a *in* L *such that* $f_a | L = f$*,* $f_a(X) \subset \langle 0, 1\rangle$ *and* $(X \setminus L) \cap \cap f_a^{-1}(1) = a + F$.

Proof. Since every translation, i.e., the function $p_y : X \to X$ defined for every $y \in X$ by the formula $p_y(X) = x + y$, is a homeomorphism of the space (X, τ_2) onto itself, so without the loss of generality we can work with $a = 0$. Let $B = \{b_t : t \in T\}$ be a Hamel base of the space L and let $1 \in T$. Let's denote $\|y\| = \sqrt{\sum\limits_{t \in T} \alpha_t^2}$ for $y = \sum\limits_{t \in T} \alpha_t b_t$, where $\alpha_t \in \mathbb{R}$, and $F_1 = \{\alpha b + y : \alpha^2 = \|y\|, y \in L, \alpha > 0\}$. It's clear that $F_1 \cap L' = F$ and $F_1 \cup \{0\}$ are 0-closed.

Now we define the function f_1 on the set $H = \{\alpha b + y : \alpha^2 \geq \geq \|y\|, y \in L, \alpha > 0\}$. Accordingly to [7, p. 57], the space (X, τ_0) is hereditary normal. Since $\mathbb{R}_+ b \setminus \{0\}$ and F are disjoint 0-closed sets in H, so there exists in H a function 0-continuous, i.e., continuous in τ_0, such that $f_1(H) \subset \langle f(0), 1\rangle$, $f_1(x) = 1$ for $x \in F_1$ and $f_1(x) = f(0)$ for $x \in \mathbb{R}_+ +b \setminus \{0\}$. We extend f_1 to the set $-\mathbb{R}_+ b + L$ by the formula $f_1(\alpha b + y) = = f(y)$, where $\alpha < 0$. At last we extend it to the set $(\mathbb{R}_+ b + L) \setminus (H \cup L)$ by the formula $f_1(\alpha b + y) = \beta + (1-\beta) f(y)$, where $\beta = \alpha \|y\|^{-1/2}$. One can check that $f_1|P \setminus \{0\}$ is continuous if P is an arbitrary line in X and P is equipped with the Euclidean topology. We will show that the function $f|P$ is also continuous on every line in X. The continuity is obvious if $P \subset L$ or $P = \mathbb{R}b$. If $0 \in P \neq \mathbb{R}b$ and $P \cap L = \{0\}$, then there exist $\alpha_0 > 0$ and $z \in L \setminus \{0\}$ such that $\langle 0, \alpha_0 b + z\rangle \subset P \cap X \setminus H$. In consequence, there exist a point $y_0 \in L \setminus \{0\}$ and a positive number θ such that $\alpha b + y \in P_+$ iff $\alpha = \theta \|y\|$ and $y = \|y\| y_0$, where $y_0 \in P_+$, P_+ is the semiline beginning at 0 and contained in the line P.

It is easy to check that $\lim\limits_{r \to 0^+} f_1(r(\theta b + y_0)) = f(0)$. Therefore the restriction $f_1|P$ is continuous. Hence f_1 is directionally continuous.

In the space $-\mathbb{R}_+ b + L$ we define the function f_1 by the formula $f_2(\alpha b + y) = \exp(\alpha)$, where $\alpha \leq 0$ and $y \in L$. For any $x \in L'$ we put $f_2(x) = 1$. For $y \in \operatorname{Lin}(B \setminus \{b_1\})$ we set $f_2(\alpha b + \alpha_1 b_1 + y) = \exp(-\alpha \|y\|_1)$, where $\alpha > 0$, $\alpha_1 \in \mathbb{R}$ and $\|y\|_1 = \sqrt{\sum\limits_{t \in T \setminus \{1\}} \alpha_t^2}$ for $y = \sum\limits_{t \in T \setminus \{1\}} \alpha_t b_t$, where $\alpha_t \in \mathbb{R}$. The function f_2 is 0-continuous in X and $f_2^{-1}(1) = L \cup L'$.

The function $f_0 = f_1 f_2$ is directionally continuous. Moreover, $f_0|L = f$, $f_0^{-1}(1) = f^{-1}(1) \cup F$. It makes the proof complete. $\Diamond$

Lemma 3.7. *Let $\{b_t : t \in T\}$ be a Hamel base of X, $\mathcal{T}$ – the family of all finite subsets of T, the empty set excluded and let $L_S = \operatorname{Lin}\{b_t : t \in S\}$, where $S \in \mathcal{T}$. Then there exists a set A such that*

(1) *$A \cap L_S$ is dense in the Euclidean topology in L_S for every $S \in \mathcal{T}$,*

(2) $\operatorname{card}(A \cap L_S) = \aleph_0$,

(3) *if $x, y \in A$, $x \neq y$ and $x = \sum\limits_{t \in T} \alpha_t b_t$, $y = \sum\limits_{t \in T} \beta_t b_t$, where $\alpha_t, \beta_t \in \mathbb{R}$, then for every $t \in T$ the equality $\alpha_t = \beta_t$ implies $\alpha_t = \beta_t = 0$.*

Proof. Let $\mathcal{T}_n = \{S \in \mathcal{T} : \operatorname{card} S = n\}$ and $L'_S = \bigcup\{L_{S'} : S' \subsetneq S\}$, where $n \in \mathbb{N}$ and $S \in \mathcal{T}$. We inductively define the sets A_n, $n \in \mathbb{N}$, such that $A_n \subset A_{n+1}$ and for every $n \in \mathbb{N}$ there hold the conditions (1)–(3) with A and $\mathcal{T}$ replaced by A_n and $\mathcal{T}_n$, resp.

It's obvious that for every $S \in \mathcal{T}_1$ there exists a set $A_S \subset L_S$ which is countable and dense in the Euclidean topology in L_S. Hence it is clear that the set $A_1 = \bigcup_{S\in\mathcal{T}_1} A_S$ fulfills the conditions (1)–(3) with A and $\mathcal{T}$ replaced by A_1 and $\mathcal{T}_1$, resp.

Now let's suppose that (1)–(3) are satisfied with A and $\mathcal{T}$ replaced by A_n and $\mathcal{T}_n$, resp. Let's take a set $S \in \mathcal{T}_{n+1}$. For a set $Z \subset L_S$ we define the set $D_S(Z) = Z + \bigcup_{s\in S} L_{S\setminus\{s\}}$. It's clear that if $\operatorname{card} Z \leq \aleph_0$ then $L_S \setminus D_S(Z)$ is dense in L_S in the Euclidean topology.

Let $\{B_n : n \in \mathbb{N}\}$ be a base of the Euclidean topology in L_S. Now we can inductively define the sequence $(a_{S,n}) \subset L_S \setminus L'_S$ such that $a_{S,1} \in B_1 \cap (L_S \setminus D_S(A_n \cap L_S))$ and $a_{S,n+1} \in B_{n+1} \cap (L_S \setminus D_S((A_n \cap L_S) \cup \cup \{a_1, a_2, \ldots, a_n\}))$. We apply this procedure for every $S \in \mathcal{T}_{n+1}$. It's obvious that the set $A_{n+1} = A_n \cup \{a_{S,k} : S \in \mathcal{T}_{n+1}, k \in \mathbb{N}\}$ satisfies the conditions (1)–(3) with A and $\mathcal{T}$ replaced by A_{n+1} and $\mathcal{T}_{n+1}$, resp.

This way we constructed the family $\{A_n : n \in \mathbb{N}\}$ of sets satisfying appropriately conditions (1)–(3). In consequence, the set $A = \bigcup_{n=1}^{\infty} A_n$ fulfills (1)–(3) and it closes the proof. $\Diamond$

Lemma 3.8. *Let $\{b_\alpha : 1 \leq \alpha < \gamma\}$ be a Hamel base of the space X, where γ is the initial ordinal number for* $\dim X \geq \aleph_0$. *Let's denote $L_\beta = \operatorname{Lin}\{b_\alpha : 1 \leq \alpha < \beta\}$ for $\beta < \gamma$ and $M_\alpha = \operatorname{Lin}\{b_1, b_{\alpha+1}\}$ for $1 \leq \alpha < \gamma$. Let $c_\alpha \in M_\alpha \setminus \mathbb{R}b_1$ for every α such that $1 \leq \alpha \leq \gamma$. If A is the set investigated in Lemma 3.7, then there exists a bijection $\mu : \{\alpha : : 1 \leq \alpha < \gamma\} \to A \setminus \mathbb{R}b_1$ for which there exists a directionally continuous function f satisfying following conditions:*

(1) $f(X) \subset \langle 0, 1\rangle$,

(2) $f^{-1}(1) = \bigcup_{1\leq\alpha<\gamma} F_\alpha$, *where* $F_\alpha = a_\alpha + \{rc_\alpha + sb_1 : r^2 = |s|, r > 0, s \in \mathbb{R}\}$ *and* $a_\alpha = \mu(\alpha)$.

Proof. Immediately from the definition of set A it follows that

$$\operatorname{card}(A \setminus \mathbb{R}b_1) = \dim X.$$

Let ν be a bijection from $\{\alpha : 1 \leq \alpha < \gamma\}$ onto $A \setminus \mathbb{R}b_1$. Instead of $\nu(\alpha)$ we write a'_α. We introduce $I = \{a'_\alpha : 1 \leq \alpha < \gamma\}$. We will construct a transfinite sequence $\{a_\alpha : 1 \leq \alpha < \gamma\}$ such that

(a) for every $a \in A \setminus \mathbb{R}b_1$ there exists $\alpha < \gamma$ such that $a_\alpha = a$,

(b) $a_1, a_2 \in L_3 \cap (A \setminus \mathbb{R}b_1)$,

(c) $a_\alpha \in L_{\alpha+1} \cap (A \setminus \mathbb{R}b_1)$, if $3 \leq \alpha < \gamma$.

The condition (b) is fulfilled when a_1, a_2 are two first elements in the set $I \cap L_3$. Let's suppose that we already have a_α with $\alpha \leq \beta$, $2 \leq \beta < \gamma$, satisfying (b) and (c). Since $(A \cap L_{\beta+1}) \setminus \bigcup_{\delta<\beta} L_{\delta+1} \neq \emptyset$, so $I \setminus \{a_\alpha : 1 \leq \leq \alpha < \beta\} \neq \emptyset$ and we denote its first element belonging to $L_{\beta+1}$ by a_β. It's clear that for every $a' \in I$ there exists α such that $1 \leq \alpha < \gamma$ and $a' = a_\alpha$. Therefore there exists a transfinite sequence $\{a_\alpha : 1 \leq \alpha < \gamma\}$ satisfying conditions (a)–(c).

Now, for every β, where $3 \leq \beta \leq \gamma$, we define on L_β a directionally continuous function f_β such that

(α) $f_\beta(L_\beta) \subset \langle 0, 1\rangle$,

(β) $f_\beta | L_\alpha = f_\alpha$ for $3 \leq \alpha < \beta < \gamma$,

(γ) $f^{-1}_{\beta+1}(1) \setminus L_\beta = \emptyset$ if β is a limit number and $\aleph_0 \leq \beta < \gamma$,

(δ) $f^{-1}_{\beta+2}(1) = F_\beta$ for $2 \leq \beta < \gamma$, where F_β is the set mentioned in the thesis,

(ε) $f^{-1}_\beta(1) \cap (A \setminus \mathbb{R}b_1) = \emptyset$ if $3 \leq \beta < \gamma$.

We set $f_3 = 0$. Let's suppose that for any β where $4 \leq \beta < \gamma$, and for every ordinal number δ, where $3 \leq \delta < \beta$, there is defined a directionally continuous function f_δ fulfilling the conditions (α)–(ε). Now, we define a directionally continuous functions f_β satisfying (α)–(ε).

If β is a limit number, we put $f_\beta(x) = f_\delta(x)$ for $x \in L_\delta$. It's clear that f_β is directionally continuous and fulfills (α) and (β), so it fulfills all conditions (α)–(ε).

If $\beta = \varphi + 1$ and φ is a limit number, we construct the function g_φ on L_β such that $g_\varphi(rb_\varphi + y) = f_\varphi(y)$, where $y \in L_\varphi$. Moreover, we take the function h_φ on L_β such that $h_\varphi(rb_\varphi + y) = \exp(-r^2)$, where $y \in L_\varphi$. It's clear that both functions, g_φ and h_φ, are directionally continuous on L_β. Hence their product $f_\beta = g_\varphi h_\varphi$ is the directionally continuous function. It's easy to verity that f_β fulfills conditions (α)–(ε).

Now let $\beta = \varphi + 2$, where φ is an ordinal number (not necessarily a limit one). From the condition (c) it follows that $a_\varphi \in L_{\varphi+1} \subset L_\beta$. It's clear that now we can apply Lemma 3.6 with X, L, f, b, b_1 and a substituted by L_β, $L_{\varphi+1}$, $f_{\varphi+1}$, c_φ, b_1 and a_φ, respectively. By this Lemma there exists in L_β a directionally continued function f_β satisfying conditions (α), (β) and (δ). At last we will show that the condition (ε) holds. Let $a_\varphi = \sum_{1\leq\alpha\leq\varphi} \theta_\alpha b_\alpha$, where $\theta_\alpha \in \mathbb{R}$. Since $a_\varphi \in A \setminus \mathbb{R}b_1$, so there exists α_0 such that $1 < \alpha_0 \leq \varphi$ and $\theta_{\alpha_0} \neq 0$. Let's suppose that there exists an element $a \in (a_\varphi + M_\varphi) \cap (A \setminus \mathbb{R}b_1)$ and $a \neq a_\varphi$ Then

$a = \sum_{1 \le \alpha \le \varphi+1} \theta'_\alpha b_\alpha$ and $\theta'_\alpha = \theta_\alpha$ if $1 < \alpha \le \varphi$. Therefore $\theta'_{\alpha_0} = \theta_{\alpha_0}$ and it is contradictory to the condition (3) in Lemma 3.7. Consequently, $(a_\varphi + M_\varphi) \cap (A \setminus \mathbb{R}b_1) = \{a_\varphi\}$. Taking into account that $F_\varphi \subset a_\varphi + M_\varphi$ and $a_\varphi \notin F_\varphi$ we have that $F_\varphi \cap (A \setminus \mathbb{R}b_1) = \emptyset$. It implies that $f_\beta^{-1}(1) \cap$ $\cap (A \setminus L_{\varphi+1}) = \emptyset$. Since the function $f_{\varphi+1}$ satisfies the condition (ε), so f_β does it, too.

This way, by the transfinite induction, we proved that on X there exists a directionally continued function f fulfilling the condition (1) and such that $f|L_\alpha = f_\alpha$ for every $3 \le \alpha < \gamma$.

Since $f_3 = 0$ and functions f_α satisfy conditions (α)-(ε) for $3 \le \alpha <$ $< \gamma$, so f satisfies the condition (2). $\Diamond$

Corollary 3.10. *If X is infinite dimensional, then there exists a 2-open set G such that for any finite dimensional space L and every $x \in X$ the set $(x + L) \setminus G$ is nowhere dense in the Euclidean topology in $x + L$ and* $\mathrm{Int}_0 G = \emptyset$.

Proof. We keep denotations used in the proof of Lemma 3.8 and we put $G = f^{-1}(\langle 0, 1))$. From the condition (2) in Lemma 3.8 it follows that each line P in X has at most 3 common points with the set $f^{-1}(1)$. Hence $P \cap f^{-1}(1)$ is nowhere dense in the Euclidean topology in P.

If $2 \le \dim L < \aleph_0$, then for every $x \in X$ there exist at most finitely many ordinal numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $(x + L) \cap F_{\alpha_k} \ne \emptyset$ and $1 \le \alpha_k < \gamma$ for $k = 1, 2, \ldots, n$. Therefore $(x + L) \cap f^{-1}(1)$ is nowhere dense in the Euclidean topology in $x + L$.

From the definition of the set A it follows that $A \setminus \mathbb{R}b_1$ is 0-dense in X. Since for every $a \in A \setminus \mathbb{R}b_1$ there exists a two-dimensional subspace L of X such that a is the accumulation point of the set $f^{-1}(1) \cap (a + L)$, so $A \setminus \mathbb{R}b_1 \subset \mathrm{Cl}_0(X \setminus G)$. Since $A \setminus \mathbb{R}b_1$ is 0-dense in X, so $\mathrm{Cl}_0(X \setminus G) = X$. Hence $\mathrm{Int}_0 G = \emptyset$. $\Diamond$

Theorem 3.6. *For any space X there hold the equalities $\chi_2(X) = \omega_2(X)$ and π-$\chi_2(X) = \pi$-$\omega_2(X)$, as well as*

(1) $\chi_2(X) = \pi\text{-}\chi_2(X) = \aleph_0$ *if* $\dim X = 1$,

(2) $\pi\text{-}\chi_2(X) = \aleph_0$ *if* $\dim X < \aleph_0$,

(3) $\chi_2(X) = \mathfrak{c}$ *if* $\dim X \ge 2$ *and* $2^{\dim X} \le \mathfrak{c}$,

(4) $\pi\text{-}\chi_2(X) = \mathfrak{c}$ *if* $\dim X \ge \aleph_0$ *and* $2^{\dim X} = \mathfrak{c}$,

(5) $\chi_2(X) \ge \pi\text{-}\chi_2(X) \ge \mathfrak{c}$ *if* $2^{\dim X} > \mathfrak{c}$,

(6) $\dim X < \pi\text{-}\chi_2(X) \le \chi_2(X) \le 2^{\dim X}$ *if* $\dim X > \mathfrak{c}$.

Proof. First we deal with points (1)–(6).

(1) is obvious.

(2). Since (X, τ_2) is regular for arbitrary X, so for every 2-open set G and every $x \in G$ there exists a 2-open set G' such that $x \in G'$ and $\mathrm{Cl}_2 G' \subset G$. From Cor. 2 in [6, p. 245] it follows that there exists an open set U in the Euclidean topology in X such that $x \in \mathrm{Cl}_0 U$ and $U \subset \mathrm{Cl}_2 G'$. Therefore each base for the Euclidean topology in X is a π-base for the topology τ_2 at 0. Hence π-$\chi_0(X) = \aleph_0$.

(3). From Cor. 3.9 it follows that $\chi_2(X) \geq \mathfrak{c}$ if $\dim X \geq 2$. If $2^{\dim X} \leq \mathfrak{c}$, then from Cor. 3.1 it follows that $\chi_2(X) \leq \mathfrak{c}$. In consequence, if $\dim X \geq 2$ and $2^{\dim X} \leq \mathfrak{c}$, then $\chi_2(X) = \mathfrak{c}$.

(4). Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of the space X and let A be the set as in Lemma 3.7. Taking into account the definition of A it is easy to state that there exists a sequence $(a_n) \subset X$ such that $\{a_n : n \in \mathbb{N}\} = A \setminus \mathbb{R} b_1$, $a_1 \in L_2$ and $a_n \in L_n$ for $n \geq 2$, where $L_n = \mathrm{Lin}\{b_1, b_2, \ldots, b_n\}$. Let $c_{n,\theta} = \theta b_1 + b_{n+2}$, where $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$. Since the assumptions of Lemma 3.8 are fulfilled, so for every $\theta \in \mathbb{R}$ there exists a directionally continuous function f_θ defined in this Lemma for the sequences (a_n) and (c_n), where $c_n = c_{n,\theta}$ and $n \in \mathbb{N}$. It's easy to see that the semiline $a_n + \mathbb{R}_+ c_{n,\theta}$ is the isolated direction of the set $f_\theta^{-1}(\langle 0, 1)) \cap M_n$ in the space $(M_n, \tau_2 | M_n)$ where $M_n = a_n + \mathrm{Lin}\{b_1, b_{n+2}\}$.

Let $\mathcal{B}$ be a π-base for the topology $\tau_2(X)$ at 0. We will show that for every $B \in \mathcal{B}$ the set $\Theta_B = \{\theta \in \mathbb{R} : B \subset G_\theta\}$, where $G_\theta = f_\theta^{-1}(\langle 0, 1))$, is at most countable. Arguing as in part (2) we conclude that for every L_n there exists a set $U \subset B$ which is open in the Euclidean topology in L_n. Since $(A \setminus \mathbb{R} b_1) \cap U = \emptyset$ for $n = 1$, so in the next we deal with $n \geq 2$. Therefore, for $B \subset G_\theta$, and every $a \in (A \setminus \mathbb{R} b_1) \cap U$ there exists $n_a \in \mathbb{N}$ such that the semiline $a + \mathbb{R}_+ c_{n_a,\theta}$ is the isolated direction of the set $B \cap M_{n_a}$ for the point a in the topology $\tau_2 | M_{n_2}$. Let Θ_B^* denote the set of all real θ such that $a + \mathbb{R}_+ c_{n_a,\theta}$ is the isolated direction of the set $B \cap M_{n_a}$ for every point $a \in U \cap (A \setminus \mathbb{R} b_1)$ in the topology $\tau_2 | M_{n_a}$. Since $B \cap M_{n_a}$ has at most countable many isolated directions for every point a, so $\mathrm{card}\, \Theta_B^* \leq \aleph_0$. Taking into account that $\Theta_B \subset \Theta_B^*$ we have $\mathrm{card}\, \Theta_B \leq \aleph_0$. Since $\mathrm{card}\{G_\theta : \theta \in \mathbb{R}\} = \mathfrak{c}$, so $\mathrm{card}\, \mathcal{B} \geq \mathfrak{c}$. Therefore π-$\chi_2(X) \geq \mathfrak{c}$.

If $\dim X \geq \aleph_0$, then by Fact 3.1 it follows that π-$\chi_2(x) \geq \mathfrak{c}$. If $2^{\dim X} \leq \mathfrak{c}$, so from (3) it follows that π-$\chi_2(x) \leq \mathfrak{c}$. It proves (4).

(5) is the consequence of (4) and Fact 3.1.

(6). The left inequality is stated in Cor. 3.2, the right one is implied by Cor. 3.1 because $2^{\dim X} > \mathfrak{c}$.

As points (1)–(6) are proved, we notice that the equality $\pi\text{-}\chi_2(X) = \pi\text{-}w_2(X)$ is stated by Cor. 3.4. The equality $\chi_2(X) = w_2(X)$ is obvious if $\dim X = 1$. If $\dim X > 1$, from (2)–(6) we have $\chi_2(X) \geq \operatorname{card} X$. Since $w_2(X) \leq \chi_2(X) \cdot \operatorname{card} X = \chi_2(X)$, so $w_2(X) = \chi_2(X)$. $\Diamond$

Theorem 3.7. $o_0(X) = ro_0(X) = \sup\{\mathfrak{c}, 2^{\dim X}\}$.

Proof. Since $\mathcal{T}_0(X)$ is Euclidean topology if $\dim X < \aleph_0$, so $o_0(X) = \mathfrak{c}$.

In the next we deal with X such that $\dim X \geq \aleph_0$. Let $\mathcal{B}$ be a base for the topology τ_0 such that $\operatorname{card} \mathcal{B} = w_0(X)$ and $\{b_t : t \in T\}$ be a Hamel base of X. Let's denote the family of all finite subsets of T by $\mathcal{T}$, and $L_S = \operatorname{Lin}\{b_t : t \in S\}$ for every $S \in \mathcal{T}$. We take $G \in \tau_0$ and $\mathcal{G} \subset \mathcal{B}$ such that $\bigcup \mathcal{G} = G$. For arbitrary $S \in \mathcal{T}$ we can choose a countable subfamily $\mathcal{G}_S \subset \mathcal{G}$ such that $L_S \cap \bigcup \mathcal{G}_S = L_S \cap G$. Let $\mathcal{G}' = \bigcup_{S \in \mathcal{T}} \mathcal{G}_S$. Therefore $\bigcup \mathcal{G}' = G$. Since $\operatorname{card} \mathcal{T} = \dim X$, so $\operatorname{card} \mathcal{G}' = \dim X$. Hence $o_0(X) \leq w_0(X)^{\dim X}$. Accordingly with Th. 3.3 we have $o_0(X) \leq (\dim X)^{\dim X} = 2^{\dim X}$.

Since $o_0(X) \geq ro_0(X)$ so, by Th. 2.4, $o_0(X) \geq 2^{\dim X}$. From both above inequalities we have $o_0(X) = 2^{\dim X}$.

Reassuming, $o_0(X) = \sup\{\mathfrak{c}, 2^{\dim X}\}$ for X of arbitrary dimension. $\Diamond$

Theorem 3.8. *$o_j(X) = \mathfrak{c}$ if $\dim X = 1$, and $o_j(X) = 2^{\sup\{\mathfrak{c}, \dim X\}}$ otherwise.*

Proof. In case $\dim X = 1$ the thesis is obvious. Investigating the case $\dim X \geq 2$ we put $S = \left\{ \sum_{t \in T} \alpha_t b_t : \sum_{t \in T} \alpha_t^2 = 1 \text{ and } \alpha_t \in \mathbb{R} \right\}$, where $\{b_t : t \in T\}$ is a Hamel base of X. Obviously, S is 0-closed. Taking $x \in S$ we see that the set $G_x = \{x\} \cup (X \setminus S)$ is 3-open. Since $\operatorname{card} S = \operatorname{card} X$, so $o_3(X) \geq 2^{\operatorname{card} X}$. At the same time $o_3(X) \leq 2^{\operatorname{card} X}$, hence $o_3(X) = 2^{\sup\{\mathfrak{c}, \dim X\}}$. Taking into account that $\tau_3 \subset \tau_2 \subset \tau_1$ and $o_j(X) \leq 2^{\operatorname{card} X}$ we get the thesis. $\Diamond$

4. Tightness

It's obvious that there holds

Fact 4.1. *For every X and every $x \in X$ we have $t_{(j)}(x, X) = t_{(j)}(0, X)$, where $t_i(x, X)$ denotes the i-tightness of a point x in the topological space (X, τ_i).*

Since the tightness of the sequential space is equal $\aleph_0$ (see [2, p. 87]) and both τ_0 and τ_1 are sequential (it is proved in [8]), so there holds

Theorem 4.1. $t_0(X) = t_1(X) = \aleph_0$.

We will show that the analogous result takes place for the Klee topology. Before stating this result we give

Lemma 4.1. *Let $U \in \tau_0(X)$, $x \notin U$ and $V = U \cup \{x\}$. If $V \cap L \in \tau_3(L)$ for every finite dimensional subspace L of infinite dimensional space X then $V \in \tau_3(X)$.*

Proof. Without the loss of generality we can work with $x = 0$. Let L be an arbitrary finite dimensional subspace of X. Since $V \cap L \in \in \tau_3|L$, so there exists in Euclidean topology a closed set $F_L \subset L$ such that $(U \cap L, F_L \setminus \{0\})$ is the Klee pair for the point 0. Let $F_L^* = \{y \in \in F_L : \langle 0, y\rangle \subset F_L\}$. For arbitrary sequence $(z_n) \subset F_L^*$, which is convergent in the Euclidean topology to z_0, the segment $\langle 0, z_0\rangle$ is contained in $\mathrm{Cl} \bigcup_{n=1}^{\infty} \langle 0, z_n\rangle \subset F_L$. It shows that $z_0 \in F_L^*$, so $F_L^* \cup \{0\}$ is 0-closed.

Applying the transfinite induction we will show that there exists a 0-closed set F such that $0 \in \mathrm{Cor}\, F$ and $F \setminus \{0\} \subset U$.

In this aim let γ be an initial number for $\dim X$ and let $\{b_\alpha : 1 \leq \leq \alpha < \gamma\}$ be a Hamel base of X. Let $X_\beta = \mathrm{Lin}\,\{b_\alpha : 1 \leq \alpha < \beta\}$, where $1 < \beta \leq \gamma$. We go to show that for every β such that $1 < \beta \leq \gamma$ there exists a 0-closed set F_β satisfying three following conditions:

(1) $F_\beta \setminus \{0\} \subset X_\beta \cap U$,

(2) $0 \in \mathrm{Cor}_{X_\beta} F_\beta$,

(3) $F_{\beta_2} \cap X_{\beta_1} = F_{\beta_1}$ if $\beta_1 < \beta_2 < \gamma$.

For $\beta = 2$ we have $X_\beta = \mathbb{R} b_1$ and, obviously, there exists a set F_1 satisfying conditions (1) and (2). Now let's assume that for any $\beta > 2$ and for all $\beta' < \beta$ there exist 0-closed sets $F_{\beta'}$ which satisfy the conditions (1)–(3). In the next we consider two cases: β is a limit number or it is not.

If β is a limit number, we put $F_\beta = \bigcup_{\beta' < \beta} F_{\beta'}$. It's clear that F_β satisfies conditions (1) and (3). Moreover, F_β is 0-closed and in aim to prove it we take an arbitrary finite dimensional subspace $L \subset X_\beta$. Then there exists $\beta_0 < \beta$ such that $L \subset X_{\beta_0}$. Hence $L \cap F_{\beta_0}$ is 0-closed. In virtue of the equalities $L \cap F_\beta = L \cap X_{\beta_0} \cap F_\beta = L \cap F_{\beta_0}$ we see that $L \cap F_\beta$ is 0-closed. Since L was chosen arbitrarily, so F_β is 0-closed.

For every $y \in X_\beta$ there exists $\beta_0 < \beta$ such that $y \in X_{\beta_0}$. By (2),

$0 \in \mathrm{Cor}_{X_{\beta_0}}(F_\beta \cap X_{\beta_0})$ and, furthermore, there exists $z \in X_\beta$ such that $(0,z) \subset (0,y) \cap F_\beta$. It implies that $0 \in \mathrm{Cor}_{X_\beta} F_\beta$. Hence F_β satisfies (1)–(3) in the case when β is a limit number.

Now we deal with the case when β is not a limit number. Let $\beta = \beta_0 + 1$.

First we will show that there exists a 0-closed set E satisfying conditions (1)–(2) with $F_\beta = E$. The existence of such a set is obvious if $\beta < \aleph_0$. In the opposite situation, $\beta \geq \aleph_0$, we have card $\beta_0 = \mathrm{card}\beta$. We take an automorphism h of the space X such that

$$h(\{b_\alpha : \alpha < \beta_0\}) = \{b_\alpha : \alpha \leq \beta_0\}.$$

Then the set $U' = h^{-1}(U)$ is 0-open and $(U' \cup \{0\}) \cap L \in \tau_3|L$ for every finite dimensional L. By the inductive assumption, in X_{β_0} there exist 0-closed sets F'_α, where $2 \leq \alpha \leq \beta_0$, such that conditions (1)–(3) are satisfied with U' instead of U and F'_α replacing F'_β. Therefore $E = h\left(F'_{\beta_0}\right)$ is 0-closed in X_β and, in consequence, it satisfies conditions (1)–(2), where F_β is replaced by E. We define

$$H = \left\{ x = \sum_{\beta' < \beta} r_\beta b_\beta : \bigforall_{\beta' < \beta_0} |r_{\beta_0}| \geq r^2_{\beta'} \right\}.$$

Since for every finite subset $P \subset \{\beta' : \beta' \leq \beta\}$ the intersection $\mathrm{Lin}\,\{b_{\beta'} : \beta' \in P\} \cap H$ is closed in the Euclidean topology, so H is 0-closed. It's obvious that $H \cap X_{\beta_0} = \{0\}$ and for every $y \in X_\beta \setminus X_{\beta_0}$ there exists z such that $(0,z) \subset (0,y) \cap H$. The set $F_\beta = (E \cap H) \cup F_{\beta_0}$ is 0-closed and satisfies conditions (1)–(3). In this way we inductively proved that there exists a 0-closed set $F = F_\gamma$ such that $0 \in \mathrm{Cor}\, F$ and $F \setminus \{0\} \subset U$. It means that $(U, F \setminus \{0\})$ is a Klee pair for the point 0. This way we proved that V is 3-open. $\Diamond$

Let's notice that Lemma 4.1 does not hold for an arbitrary V such that $V \cap L \in \tau_3(L)$ for every finite dimensional $L \subset X$. This is shown in following

Example 4.1. Let $\{b_n : n \in \mathbb{N}\}$ be a Hamel base of X. For $m \in \mathbb{N} \setminus \{1\}$ we introduce $L_m = \mathrm{Lin}\,\{b_1, b_m\}$, $X_m = \mathrm{Lin}\,\{b_n : n \leq m\}$, $F^*_m = \{r_1 b_1 + r_m b_m : r^2_m \leq |r_1| \leq 4r^2_m\}$, $F_m = F^*_m + \frac{1}{m} b_1$, $G = X \setminus \bigcup_{m=2}^{\infty} F_m$, $H = G \cup \left\{\frac{1}{n+1} b_1 : n \in \mathbb{N}\right\}$ and $G_m = (X_m \cap H) \setminus \left\{\frac{1}{n+1} b_1 : n = 1, 2, \ldots, m-1\right\}$. It is clear that G_m is open in X_m in the Euclidean topology. We will show that $H \cap X_m \in \tau_3(X_m)$. To do it we put $J^*_k = \left(\mathbb{R} b_k \cup \left\{r_1 b_1 + r_k b_k : |r_1| \geq 5 r^2_k\right\}\right) + \mathrm{Lin}\left\{b_n : n = 2, 3, \ldots, m \text{ and } n \neq k\right\}$. It is easy to see that J_k^* is closed in the Euclidean topology in X_m and $0 \in \mathrm{Cor}_{X_m} J_k^*$ for every $k = 2, 3, \ldots, m$. For $k = 2, 3, \ldots, m$ we put $J_k = \left(J_k^* + \frac{1}{k} b_1\right) \cap$ $\cap\, B_k$, where B_k denotes the closed ball in X_m with the center at $\frac{1}{k} b_1$ and the radius $\frac{1}{(k+1)^2}$. Now, we notice that J_k is closed in the Euclidean topology and $\frac{1}{k} b_1 \in \mathrm{Cor}_{X_m} J_k$. Moreover, $J_k \setminus \left\{\frac{1}{k} b_1\right\} \subset G_m \subset H \cap X_m$. Since $G_m \in \tau_0(X_m)$ and $\frac{1}{k} b_1 \in \mathrm{Cor}_{X_m} J_k$, so $\left(G_m, J_k \setminus \left\{\frac{1}{k} b_1\right\}\right)$ is a Klee pair for the point $\frac{1}{k} b_1$ in the space X_m. It implies that $H \cap X_m \in \tau_3 | X_m$. In consequence, the set $H \cap L$, where L is a finite dimensional space, is 3-open in L.

At last we show that $H \notin \tau_3$. In this aim we notice that $\mathrm{Int}_0 H \subset$ $\subset G$. Therefore $\left\{\frac{1}{n+1} b_1 : n \in \mathbb{N}\right\} \cap \mathrm{Int}_0 H = \emptyset$. Hence $0 \notin \mathrm{Cor}\, G$ and, consequently, $0 \notin \mathrm{Int}_3 H$ and it states that $H \notin \tau_3$.

Now we can give the announced result which is analogous to Th. 4.1.

Theorem 4.2. *$t_3(X) = \aleph_0$.*

Proof. From Fact 4.1 it is enough to consider $t_3(0, X)$.

Let $A \subset X \setminus \{0\}$, $0 \in \mathrm{Cl}_3 A$ and $B = \mathrm{Cl}_0 A$. Since $\mathrm{Cl}_3 A \subset B$, so $0 \in B$.

Let's consider the case when there exists $x \in X \setminus \{0\}$ such that there exists a sequence $(x_n) \subset B \cap \mathbb{R}x$ convergent to 0. Then, by Th. 4.1 and Fact 4.1, there exist countable sets $A_n \subset A$ for $n \in \mathbb{N}$ such that $x_n \in \mathrm{Cl}_0 A_n$. It implies that $0 \in \mathrm{Cl}_0\Big(\bigcup_{n=1}^{\infty} A_n\Big)$ and, consequently, there does not exist an 0-open set G and an element $z \in G \setminus \{0\}$ such that $G \cap \bigcup_{n=1}^{\infty} A_n = \emptyset$ and $(0, z) \subset G \cap \mathbb{R}x$. Therefore $t_3(0, A) = \aleph_0$.

In the next we investigate the case when for every $x \in X \setminus \{0\}$ there exists $r_x > 0$ such that $(r_x x, 0) \subset (X \setminus B) \cap \mathbb{R}x$. This investigation is made below in two parts: I) if $\dim X < \aleph_0$ and II) if $\dim X \geq \aleph_0$.

Part I ($\dim X < \aleph_0$). In this part the closure, the interior, the convergence etc. are in the Euclidean topology, if any other topology is not indicated. In the same manner we write, e.g., $\mathrm{Cl}\, A$ and $\mathrm{Int}\, A$ instead of $\mathrm{Cl}_0 A$ and $\mathrm{Int}_0 A$, resp. Moreover, in this part of this proof we denote $\hat{x} = \frac{x}{\|x\|}$ for $x \in X \setminus \{0\}$.

Let's denote $K = K(0, 1)$, $\bar{K} = \mathrm{Cl}\, K$ and $S = \bar{K} \setminus K$. For a set $H \subset$ $\subset X$ such that $0 \in \mathrm{Cor}\, (H \cup \{0\})$ we define the function $\omega_H : S \to (0, 1\rangle$ as follows: for every $x \in S$ the segment $(0, \omega_H(x) x)$ is a component of $\mathrm{Int}_{\mathbb{R}x}(H \cap (0, 1) x)$.

Now, for a certain ordinal number β we construct the families

$\{C_n^\alpha : n \in \mathbb{N}, 1 \leq \alpha \leq \beta\}$, $\{D^\alpha : 1 \leq \alpha \leq \beta\}$ and $\{E^\alpha : 1 \leq \alpha \leq \leq \beta\}$, of sets such that

(a) $C_1^1 = D^1 = S$,

(b) D^α and C_n^α are non-empty sets, for $\alpha < \beta$ and every $n \in \mathbb{N}$,

(c) $C_n^\alpha = \mathrm{Cl}\left\{x \in C_1^\alpha : \omega_{X\setminus B}(x) \leq \frac{1}{n}\right\}$,

(d) $D^{\alpha+1} = \bigcap\limits_{n=1}^{\infty} C_n^\alpha$,

(e) $D^\alpha = \bigcap\limits_{\alpha'<\alpha} C_1^{\alpha'}$ if α is a limit number,

(f) $D^\alpha = C_1^\alpha \cup E^\alpha$, where the set C^α is perfect and the set E^α is countable,

(g) for every $\alpha < \beta$ the set $C_1^{\alpha+1}$ is boundary in the space (C_1^α, τ), where τ denotes the Euclidean topology in C_1^α, except for the case when $\alpha + 1 = \beta$ and the condition (h2) holds,

(h) there holds true one of following conditions

1) $\mathrm{card}\, D^\beta \leq \aleph_0$,

2) $\mathrm{Int}_{C_1^{\beta'}} C_1^\beta \neq \emptyset$ for $\beta' + 1 = \beta$,

3) $C_n^\beta = \emptyset$ for any $n \in \mathbb{N}$.

In the next the set C_1^α is denoted by C^α.

Let's assume that for an ordinal number $\alpha \leq \beta$ and for every $\alpha' < \alpha$ there are already constructed sets $C^{\alpha'}$, $D^{\alpha'}$, $E^{\alpha'}$, and $C_n^{\alpha'}$ with $n \in \mathbb{N}$ satisfying conditions (a)–(g).

Now, according to (d) and (e), we construct the set D^α. If $\mathrm{card}\, D^\alpha \leq \leq \aleph_0$ then $\alpha = \beta$ and we put $C^\alpha = C_n^\alpha = \emptyset$ for $n \in \mathbb{N}$. In the opposite case, since D^α is closed, so by Cantor–Bendixson Theorem [2, p. 84] we have the decomposition $D^\alpha = C^\alpha \cup E^\alpha$, where C^α is perfect in the space $(D^\alpha, \tau_0|D^\alpha)$ and $\mathrm{card}\, E^\alpha \leq \aleph_0$. If $\alpha = \alpha_1 + 1$ and $\mathrm{Int}_{C^{\alpha_1}} C^\alpha \neq \emptyset$ then $\alpha = \beta$. If C^α is boundary in C^{α_1} or α is a limit number then the sets C_n^α are constructed by (c). If there exists $m \in \mathbb{N}$ such that $C_n^\alpha = \emptyset$ for all $n > m$ then we have $\alpha = \beta$. But if $C_n^\alpha \neq \emptyset$ for every $n \in \mathbb{N}$ then the sets C^α, D^α and C_n^α are already constructed and $\alpha < \beta$. This way we inductively constructed the families of sets satisfying conditions (a)–(h) for a certain ordinal number β.

Since every C^α is closed and there holds the implication $\alpha_1 < \alpha_2 < < \beta \Rightarrow C^{\alpha_2} \subset C^{\alpha_1}$, so $\mathrm{card}\,\{C^\alpha : \alpha < \beta\} \leq \aleph_0$ and this way $\mathrm{card}\, \beta \leq \aleph_0$.

Now our proceeding is depending on which case of (h1)–(h3) takes place.

In case (h2), i.e., when $\operatorname{Int}_{C^{\beta'}} C^\beta \neq \emptyset$ for $\beta' + 1 = \beta$, we take a set U open in $\tau_0|C^{\beta'}$ and such that $\operatorname{Cl} U \subset C^\beta$, and we take a subset $W \subset U$ such that $\operatorname{card} W = \aleph_0$ and $\operatorname{Cl} W = \operatorname{Cl} U$.

Since $C^\beta \subset \bigcap\limits_{n=1}^{\infty} C_n^{\beta'}$, so $W \subset U \subset C_n^{\beta'}$ for every $n \in \mathbb{N}$. In consequence, for every $w \in W$ there exists a sequence $\big(z_n^{(w)}\big) \subset (B \cap \operatorname{Con} U) \setminus \{0\}$ such that $z_n^{(w)} \to 0$ and $\hat{z}_n^{(w)} \to w$. Let's denote $Z = \big\{z_n^{(w)} : n \in \mathbb{N}, w \in W\big\}$. Since $Z \subset B$, so for every $z \in Z$ there exists a sequence $\big(x_n^{(z)}\big) \subset A$ such that $x_n^{(z)} \to z$. Now we define the set $A^* = \big\{x_n^{(z)} : n \in \mathbb{N}, z \in Z\big\}$. Let's assume that there exists a closed set F such that $0 \in \operatorname{Cor} F$ and $F \setminus \{0\} \subset X \setminus \operatorname{Cl} A^*$. For every $m \in \mathbb{N}$ the set $F_m = \big\{y \in \operatorname{Cl} U : \frac{1}{m} \leq \omega_F(y)\big\}$ is closed because for every convergent sequence $(y_n) \subset F_m$, where $m \in \mathbb{N}$, the segments $\big\langle \frac{1}{m} y_n, 0\big\rangle \subset F \cap \frac{1}{m}\bar{K}$ and it implies that $\big\langle \frac{1}{m} y_0, 0\big\rangle \subset \operatorname{Cl} \bigcup\limits_{n=1}^{\infty} \big\langle \frac{1}{m} y_n, 0\big\rangle$, so the segment $\big\langle \frac{1}{m} y_0, 0\big\rangle \subset F \cap \frac{1}{m}\bar{K}$, where y_0 denotes the limit of (y_n). Since $\operatorname{Cl} U$ is a Baire set of the 2nd category in $C^{\beta'}$ i.e., it is not a countable sum of nowhere dense sets and $\bigcup\limits_{n=1}^{\infty} F_n = \operatorname{Cl} U$, so there exists $n_0 \in \mathbb{N}$ such that $\operatorname{Int}_{C^{\beta'}} F_{n_0} \neq \emptyset$. Let $V \in \tau_0|C^{\beta'}$ and $V \subset F_{n_0}$. Since $V \cap W \neq \emptyset$, so $\frac{1}{n}K \cap \operatorname{Con} V \cap Z \neq \emptyset$ for every $n \in \mathbb{N}$. Since $\frac{1}{n}K \cap \operatorname{Con} V \subset F$ for every $n > n_0$, so $F \cap Z \neq \emptyset$. Therefore $(F \setminus \{0\}) \cap \operatorname{Cl} A^* \neq \emptyset$, so there does not exists a closed set F such that $0 \in \operatorname{Cor} F$ and $(F \setminus \{0\}) \subset X \setminus \operatorname{Cl} A^*$. It means that the sets $F \setminus \{0\}$ and $X \setminus \operatorname{Cl} A^*$ do not form a Klee pair for 0. It follows that there does not exist a 3-neighbourhood of 0 disjoint with the set A^*. It implies that $0 \in \operatorname{Cl}_3 A^*$. Reassuming, we proved for a set A there exists its countable subset A^* such that $0 \in \operatorname{Cl}_3 A^*$.

Now we consider cases (h1) and (h3). First we define a function $\varphi : S \to \big(0, \frac{2}{3}\big\rangle$. The definition of φ will be separately given on non-empty sets E^α and $C_n^\alpha \setminus C_{n+1}^\alpha$ for $n \in \mathbb{N}$ and $\alpha \leq \beta$.

Let $\delta(\alpha)$ denote the rank of the set D^α. It is easy to see that $E^\alpha = \bigcup\limits_{0 \leq \delta < \delta(\alpha)} E_\delta^\alpha$, where $E_\delta^\alpha = (D^\alpha)^{(\delta)} \setminus (D^\alpha)^{(\delta+1)}$ and $(D^\alpha)^{(\gamma)}$ denotes the derived set of the order γ of the set D^α. Taking into account that $C^\alpha = (D^\alpha)^{\delta(\alpha)}$ we define the family

$$\mathcal{D} = \big\{(D^\alpha)^{(\delta)} : \alpha \leq \beta \text{ and } 0 \leq \delta \leq \delta(\alpha)\big\} \cup \big\{C_n^\alpha : \alpha \leq \beta \text{ and } n \in \mathbb{N} \setminus \{1\}\big\}.$$

It's easy to see that $\mathcal{D}$ is well-ordered by the inclusion $\supset$. Obviously, $\rho_{G_1}(x) \leq \rho_{G_2}(x)$ if $G_2 \subset G_1$ and $\rho_G(x) > 0$ if $x \notin G$, where G_1, G_2,

$G \in \mathcal{D}$. For $\alpha \leq \beta$ we define

$$\varphi(x) = \begin{cases} \frac{1}{3}\rho_T(x)\omega_{X\setminus B}(x) & \text{if } T = (D^\alpha)^{(\delta+1)} \quad \text{and } x \in E^\alpha_\delta, \\ \frac{1}{5n}\rho_T(x) & \text{if } T = C^\alpha_{n+1} \quad \text{and } x \in C^\alpha_n \setminus C^\alpha_{n+1}. \end{cases}$$

As the function φ on the sphere S is defined, we will show that the set $F = \bigcup_{x\in S} \left\langle 0, \varphi_{(x)}x \right\rangle$ is closed. Let's denote $F^* = \{\varphi(x)x : x \in S\}$ and let's take the sequence $(x_n) \subset F^*$ convergent to a point x_0.

If $x_0 = 0$ then it's obvious that $x_0 \in F$. We will prove that $x_0 \in F$ also when $x_0 \neq 0$. In this aim we denote by G_1 the first set in the family $\mathcal{D}$ such that $\hat{x}_0 \notin G_1$.

We will show that $G = \bigcap\{G' \in \mathcal{D} : G_1 \subset G'\} \in \mathcal{D}$. First let's notice that $G_1 \neq D^\alpha$ for $\alpha < \beta$. Let's suppose that it does not take place. Then $\hat{x}_0 \in C^{\alpha'}_n$ for every $\alpha' < \alpha$ and $n \in \mathbb{N}$. From conditions (d) and (e) we have that $\hat{x}_0 \in D^\alpha$, and it contradicts that $G_1 = D^\alpha$. If $G_1 = (D^\alpha)^{(\delta)}$ for some $\alpha < \beta$ and $1 \leq \delta \leq \delta(\alpha)$ then δ is not a limit number (and one can check it as above), so $\delta = \gamma + 1$ and $G = (D^\alpha)^{(\gamma)}$. If $G_1 = C^\alpha_n$ for some $\alpha < \beta$ and $n \in \mathbb{N}$ then or $G = C^\alpha_{n-1}$ either $G = (D^\alpha)^{(\delta(\alpha))}$ depending on n is greater than or equal to 1, respectively.

The above constructed set G is the last set in $\mathcal{D}$ such that $\hat{x}_0 \in G$.

It is enough to examine two cases:

(i) $\hat{x}_n \notin G$ for all $n \in \mathbb{N}$,

(ii) $\hat{x}_n \in G$ for all $n \in \mathbb{N}$.

In Case (i), by definition of the function φ and from the equality $\|x_n\| = \varphi(\hat{x}_n)$, we have $||x_n|| < \varrho_G(\hat{x}_n)$. Taking into account that $\hat{x}_0 \in G$ we have $||x_n|| \to 0$, so $x_0 = 0$. It is in the contrary with the assumption $x_0 \neq 0$, so Case (i) takes no place.

Now we consider Case (ii). Since the set G_1 is closed, so if the sequence $(\hat{x}_n) \subset G$ is convergent to $\hat{x}_0$ then $\hat{x}_n \in G \setminus G_1$ for all but finitely many natural n. There are two possibilities: either $\hat{x}_0 \in E^\alpha_\delta$ or $\hat{x}_0 \in C^\alpha_m \setminus C^\alpha_{m+1}$ for any $m \in \mathbb{N}$, where $\alpha \leq \beta$ and $0 \leq \delta < \delta(\alpha)$. Since E^α_δ is composed of isolated points, so if $\hat{x}_0 \in E^\alpha_\delta$, then $\hat{x}_n = \hat{x}_0$ for all but finitely many $n \in \mathbb{N}$. Hence $x_n = x_0$ for all but finitely many $n \in \mathbb{N}$ and, furthermore, $x_0 \in F$. If $\hat{x}_0 \in C^\alpha_m \setminus C^\alpha_{m+1}$ then $\hat{x}_n \in C^\alpha_m \setminus C^\alpha_{m+1}$ for almost all $m \in \mathbb{N}$. Since the function $\rho_{C^\alpha_{m+1}}$ is continuous, so $\rho_{C^\alpha_{m+1}}(\hat{x}_n) \to \rho_{C^\alpha_{m+1}}(\hat{x}_0)$. From the definition of the function φ it follows that $\varphi(\hat{x}_n) \to \varphi(\hat{x}_0)$. Hence, since $(x_n) \subset F^*$, i.e., $x_n = \varphi(\hat{x}_n)\hat{x}_n$ for every $n \in \mathbb{N}$, and $\hat{x}_n \to \hat{x}_0$, so the sequence (x_n) converges to $\hat{x}_0 = \varphi(\hat{x}_0)\hat{x}_0 \in F^*$. This way we proved that $x_0 \in F$ if $\hat{x}_0 \in E^\alpha_\delta$

$(0 \leq \delta < \delta(\alpha)$ and $\alpha \leq \beta)$ and also if $\hat{x}_0 \in C_n^\alpha \setminus C_{n+1}^\alpha$ $(n \in \mathbb{N}, \alpha \leq$ $\leq \beta)$. In view of the equality

$$(*) \qquad \bigcup_{1\leq\alpha\leq\beta} \; \bigcup_{0\leq\delta<\delta(\alpha)} E_\delta^\alpha \cup \bigcup_{1\leq\alpha\leq\beta} \; \bigcup_{n=1}^{\infty} \left(C_n^\alpha \setminus C_{n+1}^\alpha\right) = S$$

we have $x_0 \in F$. This way we proved that F is closed. Now we prove that $F \setminus \{0\} \subset X \setminus B$. If $x \in C_n^\alpha \setminus C_{n+1}^\alpha$, where $1 \leq \alpha \leq \beta$ and $n \in \mathbb{N}$, then $\omega_{X\setminus B}(x) \geq \frac{1}{n+1}$. Therefore $(0, \varphi(x)x) \subset \left(0, \frac{2}{5n}x\right) \subset \left(0, \omega_{X\setminus B}(x)x\right)$ and, consequently, $(0, x\rangle \cap F \subset X \setminus B$. If $x \in E_\delta^\alpha$, where $1 \leq \alpha \leq \beta$ and $0 \leq \delta < \delta(\alpha)$, then $(0, \varphi(x)x) \subset \left(0, \omega_{X\setminus B}(x)x\right)$ and, as above $(0, x\rangle \cap$ $\cap F \subset X \setminus B$. This way, in view of $(*)$, we have $F \setminus \{0\} \subset X \setminus B$. From $0 \in \mathrm{Cor}F$ it follows that $(X \setminus B, F \setminus \{0\})$ is a Klee pair for the point 0. Hence there exists a 3-neighbourhood of 0 disjoint with $B \setminus \{0\}$ and therefore so it is disjoint with A. Consequently, $0 \notin \mathrm{Cl}_3 A$ and it contradicts the assumption that $0 \in \mathrm{Cl}_3 A$.

This statement completes the proof of Part I because we showed that if $0 \in \mathrm{Cl}_3 A$ then there exists a countable subset of A such that its 3-closure contains 0.

Part II $(\dim X \geq \aleph_0)$. Since $0 \in \mathrm{Cl}_3\,(B \setminus \{0\})$, so, by Lemma 4.1, it follows that there exists a finite dimensional space L such that $0 \in$ $\in \mathrm{Cl}_3\,((B \setminus \{0\}) \cap L)$.

Let $B_1 \subset L$ be a countable and 0-dense set in $B \cap L$. Since $B =$ $= \mathrm{Cl}_0 A$, so, by Th. 4.1, for every $x \in B_1$ there exists a countable set $A_x \subset A$ such that $x \in \mathrm{Cl}_0 A_x$. We put $A_0 = \bigcup_{x\in B_1} A_x$. Then $B \cap L =$ $= \mathrm{Cl}_0 B_1 \subset \mathrm{Cl}_0 A_0$. Let's assume that there exists $V_1 \in \tau_0$ such that

$$(**) \qquad V = \{0\} \cup V_1 \in \tau_3$$

and $V_1 \cap A_0 = \emptyset$. Then $V_1 \cap \mathrm{Cl}_0 A_0 = \emptyset$, so $V_1 \cap ((B \cap L) \setminus \{0\}) =$ $= \emptyset$ and, consequently, $V \cap ((B \cap L) \setminus \{0\}) = \emptyset$. It implies that $0 \notin$ $\mathrm{Cl}_3\,((B \cap L) \setminus \{0\})$. This way we obtained the contradiction. Therefore $V \cap A_0 \neq \emptyset$ for an arbitrary set V of the form $(**)$, where $V_1 \in \tau_0$. Since every 3-neighbourhood G of the point 0 contains the set V of the form $(**)$, so $G \cap A_0 \neq \emptyset$. In consequence, $0 \in \mathrm{Cl}_3 A_0$ and therefore $t_3(0, A) = \aleph_0$. $\diamondsuit$

What concerns the tightness of the directional topology, we obviously have $\aleph_0 \leq t_2(X) \leq \sup\{\mathfrak{c}, \dim X\}$. It could be interesting to get the answer to the following

Question 4.1. *Is it possible to determine more precisely the value* $t_2(X)$*?*

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