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A NOTE ON SHIFT THEORY

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Abstract: Two-sided and one-sided shifts have a main role to extract several examples in some branches, like ergodic theory. In this note our main aim is to generalize them (two-sided and one-sided shifts) and compare the results; in this way we find that if $\phi : \Gamma \to \Gamma$ is one to one, then the the set of all periodic points of the generalized shift $\sigma_\phi : \prod\limits_\Gamma X \to \prod\limits_\Gamma X$ is dense in $\prod\limits_\Gamma X$.

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Preliminaries

We recall the following definitions from [2]:

The function $T : (X, \mathcal{B}, m) \to (X', \mathcal{B}', m')$ of measure spaces is called measurable if for each $D \in \mathcal{B}'$, $T^{-1}(D) \in B$.The measurable function $T : (X, \mathcal{B}, m) \to (X', \mathcal{B}', m')$ of probability spaces is called measure preserving if for each $D \in \mathcal{B}'$, $m(T^{-1}(D)) = m'(D)$. When $T : (X, \mathcal{B}, m) \to (X', \mathcal{B}', m')$ is bijective, measure preserving and $T^{-1} :$ $: (X', \mathcal{B}', m') \to (X, \mathcal{B}, m)$ is measure preserving, then $T : (X, \mathcal{B}, m) \to$ $\to (X', \mathcal{B}', m')$ is called invertible measure preserving. The measure preserving function $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$, with $\mathcal{S}$ as a semi-algebra which generates $\mathcal{B}$, is called

- ergodic if for each $A, B \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}(A) \cap B) = m(A)m(B)$$

(or equivalently for each $D \in \mathcal{B}$, with $D = T^{-1}(D)$ we have $m(D) = 0 \vee$ $\vee\, m(D) = 1$);

- weak-mixing if $\forall A, B \in \mathcal{S}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |m(T^{-i}(A) \cap B) - m(A)m(B)| = 0;$$

- strong-mixing if for each $A, B \in \mathcal{S}$,

$$\lim_{n\to\infty} m(T^{-n}(A) \cap B) = m(A)m(B).$$

In a compact metrisable space X with continuous map $T : X \to X$, for any finite collection $\{\alpha_1, \ldots, \alpha_n\}$ of open covers of X,

$$\bigvee_{1\le i\le n} \alpha_i := \left\{ \bigcap_{1\le i\le n} U_i : \forall i \in \{1, \ldots, n\}\ U_i \in \alpha_i \right\}.$$

If α is an open cover of X, then the entropy of T relative to α is given by

$$h(T, \alpha) := \lim_{n\to\infty} \frac{1}{n} \ln\left(\left| \bigvee_{0\le i\le n-1} T^{-i}(\alpha) \right| \right),$$

and $h(T) := \sup\limits_{\alpha} h(T, \alpha)$ is the topological entropy of T. If $T : X \to X$ is homeomorphism, a finite open cover α of X is a generator for T if for every bisequence $\{A_n\}_{n\in\mathbf{Z}}$ of members of α, $\bigcap\limits_{i\in\mathbf{Z}} T^{-i}(\overline{A}_i)$ contains at most one point of X, in case of existence of a generator for T, T is called expansive. If $T : X \to X$ is expansive and α is a generator for T, then $h(T) = h(T, \alpha)$.

For definition and properties of product of arbitrary $\sigma-$algebras and measure spaces, we refer the interested reader to [1].

Convention. In the following text let Γ be a nonempty index set, $\phi : \Gamma \to \Gamma$ be a map, X be a topological space and $\mathcal{B}_X$ be the $\sigma-$algebra on X generated by open subsets, suppose $Y = \prod\limits_{\Gamma} X$ and $\sigma_\phi : Y \to Y$ be such that $\sigma_\phi\big((x_\gamma)_{\gamma\in\Gamma}\big) = (x_{\phi(\gamma)})_{\gamma\in\Gamma}$ $\Big(\forall (x_\gamma)_{\gamma\in\Gamma} \in \prod\limits_{\Gamma} X\Big)$. For $\eta \in \Gamma$ let $\pi_\eta : \prod\limits_{\Gamma} X \to X$ be the projection map on η's coordinate.

Note: It is evident that for $\mathrm{card}(X) > 1$, σ_ϕ is onto if and only if ϕ is one to one; σ_ϕ is one to one if and only if ϕ is onto; and σ_ϕ is bijective if and only if ϕ is bijective.

Lemma 1. *Let $k \in \mathbf{N} - \{1\}$, $X = \{1, \ldots, k\}$ with discrete topology ($\mathcal{B}_X = = \mathcal{P}(X)$), for each $\gamma \in \Gamma$, $(X, \mathcal{B}_X, m_\gamma)$ be a probability measure space such that $m_\gamma(i) = p_i^\gamma > 0$ ($i \in \{1, \ldots, k\}$) and $(Y, \mathcal{B}', m') = \prod\limits_{\Gamma}(X, \mathcal{B}_X, m_\gamma)$, then:*

(i) *σ_ϕ is measure preserving if and only if ϕ is one to one and $p_i^{\phi(\gamma)} = p_i^\gamma$ ($\forall \gamma \in \Gamma, \forall i \in X$).*

(ii) *σ_ϕ is invertible measure preserving if and only if ϕ is bijective and $p_i^{\phi(\gamma)} = p_i^\gamma$ ($\forall \gamma \in \Gamma, \forall i \in X$).*

Proof. (i). If ϕ is not one to one and $k \geq 2$, then there exist distinct $\gamma_0, \gamma_1 \in \Gamma$ with $\eta := \phi(\gamma_0) = \phi(\gamma_1)$. We have

$$\sigma_\phi^{-1}\Bigg(\Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : x_{\gamma_0} = x_{\gamma_1} = 1\Big\}\Bigg) =$$

$$= \sigma_\phi^{-1}\Bigg(\Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : x_{\gamma_0} = 1\Big\}\Bigg) = \Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : x_\eta = 1\Big\}.$$

Since

$$m\Bigg(\Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : x_{\gamma_0} = x_{\gamma_1} = 1\Big\}\Bigg) =$$

$$= p_1^{\gamma_0} p_1^{\gamma_1} < p_1^{\gamma_0} = m\Bigg(\Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : x_{\gamma_0} = 1\Big\}\Bigg),$$

thus σ_ϕ is not measure preserving.

Now if ϕ is one to one use the fact that for each distinct $\gamma_1, \ldots, \gamma_k \in$ $\in \Gamma$ we have:

$$m\left(\left\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : \forall j \in \{1, \ldots, k\}\ x_{\gamma_j} = i_j\right\}\right) = p_{i_1}^{\gamma_1} \cdots p_{i_k}^{\gamma_k}$$

and

$$m\left(\sigma_\phi^{-1}\left(\left\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : \forall j \in \{1, \ldots, k\}\ x_{\gamma_j} = i_j\right\}\right)\right) =$$
$$= m\left(\left\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X : \forall j \in \{1, \ldots, k\}\ x_{\phi(\gamma_j)} = i_j\right\}\right) =$$
$$= p_{i_1}^{\phi(\gamma_1)} \cdots p_{i_k}^{\phi(\gamma_k)}.$$

(ii). Use (i).

Corollary 2. *Let* $k \in \mathbf{N} - \{1\}$, $X = \{1, \ldots, k\}$ *with discrete topology* ($\mathcal{B}_X = \mathcal{P}(X)$), $(X, \mathcal{B}_X, m)$ *be a probability measure space such that* $m(i) = p_i > 0$ $(i \in \{1, \ldots, k\})$ *and* $(Y, \mathcal{B}', m') = \prod\limits_\Gamma (X, \mathcal{B}_X, m)$, *then:*

(i) σ_ϕ *is measure preserving if and only if* ϕ *is one to one.*

(ii) σ_ϕ *is invertible measure preserving if and only if* ϕ *is bijective.*

Proof. Use Lemma 1.

Theorem 3. *In Lemma 1, let* ϕ *be one to one such that* $p_i^{\phi(\gamma)} = p_i^\gamma$ *and* $\phi^n(\gamma) \neq \gamma$ *for each* $i \in \{1, \ldots, k\}, \gamma \in \Gamma, n \in \mathbf{N}$, *then* σ_ϕ *is ergodic, strong-mixing, and weak-mixing.*

Proof. Ergodicity: Proof is similar to [2, Th. 1.12] in the following way. Suppose $D \in \mathcal{B}'$ and $\sigma_\phi^{-1}(D) = D$. Let $\epsilon > 0$ there exists A in algebra generated by

$$\left\{\prod_{\gamma\in\Gamma} V_\gamma \subseteq \prod_\Gamma X : \exists \gamma_1, \ldots, \gamma_n \in \Gamma \forall \gamma \in \Gamma - \{\gamma_1, \ldots, \gamma_n\} V_\gamma = X\right\}$$

(thus $\Gamma - \{\gamma \in \Gamma |\ \pi_\gamma(A) = X\}$ is finite) such that $m(D\Delta A) < \epsilon$. On the other hand:

$$|m(D) - m(A)| \leq m(D - A) + m(A - D) < \epsilon.$$

Choose $n \in \mathbf{N}$ so large that $\{\gamma \in \Gamma |\ \pi_\gamma(A) \neq X\} \cap \{\gamma \in \Gamma |\ \pi_\gamma(\sigma_\phi^{-n}(A)) \neq$ $\neq X\} = \emptyset$. We have $m(A \cap \sigma_\phi^{-n}(A)) = m(A)m(\sigma_\phi^{-n}(A)) = m(A)^2$. On the other hand $m(D\Delta\sigma_\phi^{-n}(A)) = m(\sigma_\phi^{-n}(D)\Delta\sigma_\phi^{-n}(A)) = m(\sigma_\phi^{-n}(D\Delta A) =$ $= m(D\Delta A) < \epsilon$. By $D\Delta(A \cap \sigma_\phi^{-n}(A)) \subseteq (D\Delta A) \cup (D\Delta\sigma_\phi^{-n}(A))$, we have:

$$m(D - (A \cap \sigma_\phi^{-n}(A)) \leq m(D\Delta(A \cap \sigma_\phi^{-n}(A))) < 2\epsilon,$$

and $|m(D) - m(D)^2| \leq |m(D) - m(A \cap \sigma_\phi^{-n}(A))| + |m(A \cap \sigma_\phi^{-n}(A)) - -m(D)^2| < 2\epsilon + |m(A)^2 - m(D)^2| < 4\epsilon$. Therefore $m(D) = 0$ or $m(D) = 1$ and σ_ϕ is ergodic.

Strong-mixing: Proof is similar to [2, Th. 1.30].

Weak-mixing: With the above argument $\sigma_\phi \times \sigma_\phi$ is ergodic; by [2, Th. 1.24], σ_ϕ is weak-mixing.

Note 4. Let $k \in \mathbf{N} - \{1\}$, $X = \{1, \ldots, k\}$ and Γ be infinite. For $n \in \mathbf{N}$ and $a_1, \ldots, a_n \in X$ let $p_n(a_1, \ldots, a_n) > 0$ be such that:

- $\displaystyle\sum_{a_1 \in X} p_1(a_1) = 1$,
- $\displaystyle p_n(a_1, \ldots, a_n) = \sum_{a_{n+1} \in X} p_{n+1}(a_1, \ldots, a_n, a_{n+1})$.

Let

$$(Y, \mathcal{B}') = \prod_\Gamma (X, \mathcal{P}(X))$$

and $(Y, \mathcal{B}', m)$ be such that for different $\gamma_1, \ldots, \gamma_n \in \Gamma$,

$$m\left(\left\{(x_\gamma)_{\gamma \in \Gamma} \in \prod_\Gamma X \,|\, x_{\gamma_1} = a_1, \ldots, x_{\gamma_n} = a_n\right\}\right) = p_n(a_1, \ldots, a_n)$$

(for $a_1, \ldots, a_n \in X$), then using a similar method described for Lemma 1 we have:

(i) σ_ϕ is measure preserving if and only if ϕ is one to one.

(ii) σ_ϕ is invertible measure preserving if and only if ϕ is bijective.

Theorem 5. *In Note 4 let $P = [p_{ij}]_{1 \leq i,j \leq k}$ be a stochastic matrix, i.e., for each $i, j \in \{1, \ldots, k\}$ we have $p_{ij} \geq 0$, $\sum_{t=1}^{k} p_{it} = 1$, $\sum_{t=1}^{k} p_t p_{tj} = p_j > 0$, and $p_n(a_1, \ldots, a_n) = p_{a_1} p_{a_1 a_2} \cdots p_{a_{n-1} a_n}$. If ϕ is one to one such that for each $n \in \mathbf{N}$ and $\gamma \in \Gamma$ we have $\phi^n(\gamma) \neq \gamma$, then the following statements are equivalent:*

- *σ_ϕ is ergodic;*
- *σ_ϕ is weak-mixing;*
- *σ_ϕ is strong-mixing;*
- *for each $i, j \in \{1, \ldots, k\}$, $p_{ij} = p_j$.*

Proof. If σ_ϕ is ergodic, then

$$p_i p_j = m\left(\left\{(x_\gamma)_{\gamma \in \Gamma} \in \prod_\Gamma X : x_\lambda = i\right\}\right) m\left(\left\{(x_\gamma)_{\gamma \in \Gamma} \in \prod_\Gamma X : x_\lambda = j\right\}\right) =$$

$$= \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} m\Bigg((\sigma_\phi)^{-n}\Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = i\Big\} \cap \cap \Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = j\Big\}\Bigg) =$$

$$= \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} m\Bigg(\Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_{\phi^n(\lambda)} = i\Big\} \cap \cap \Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = j\Big\}\Bigg) =$$

$$= \lim_{N\to+\infty} \frac{1}{N} (\delta_{ij} p_j + (N-1) p_2(i,j)) = p_i p_{ij},$$

thus $p_{ij} = p_j$.

For other parts use a similar method, [2, Th. 1.17], and Cor. 2 (since for each $i, j \in \{1, \ldots, k\}$, $p_{ij} = p_j$, then we have the same measure space).

Theorem 6. *In Note 4 let* $P = [p_{ij}]_{1\le i,j\le k}$ *be a stochastic matrix, i.e., for each* $i, j \in \{1, \ldots, k\}$ *we have* $p_{ij} \ge 0$, $\sum\limits_{t=1}^{k} p_{it} = 1$, $\sum\limits_{t=1}^{k} p_t p_{tj} = p_j > 0$, *and* $p_n(a_1, \ldots, a_n) = p_{a_1} p_{a_1 a_2} \cdots p_{a_{n-1} a_n}$. *If* ϕ *is one to one with out any fix point and* $q > 1$ *then:*

- *If* σ_ϕ *is ergodic, then there exists* $\lambda \in \gamma$, *with* $q = \min\{n \in \mathbf{N} : : \phi^n(\gamma) = \gamma\}$ *if and only if for each* $\lambda \in \gamma$, $q = \min\{n \in \mathbf{N} : \phi^n(\gamma) = \gamma\}$.
- *If* σ_ϕ *is strong-mixing, then for each* $\gamma \in \Gamma$, $\phi^q(\gamma) \neq \gamma$.
- *If* σ_ϕ *is weak-mixing, then there exists* $\lambda \in \gamma$ *with* $q = \min\{n \in \mathbf{N} : : \phi^n(\gamma) = \gamma\}$ *if and only if for each* $\lambda \in \gamma$, $q = \min\{n \in \mathbf{N} : \phi^n(\gamma) = \gamma\}$.

Proof. If σ_ϕ is ergodic, and $\lambda \in \gamma$ is such that $q = \min\{n \in \mathbf{N} : \phi^n(\gamma) = = \gamma\}$, then:

$$p_i p_j = m\Bigg(\Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = i\Big\}\Bigg) m\Bigg(\Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = j\Big\}\Bigg) =$$

$$= \lim_{N\to+\infty} \frac{1}{qN} \sum_{n=0}^{qN-1} m\Bigg((\sigma_\phi)^{-n}\Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = i\Big\} \cap \cap \Big\{ (x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = j\Big\}\Bigg) =$$

$$= \lim_{N\to+\infty} \frac{1}{qN} \sum_{n=0}^{N-1} m\Bigg(\Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_{\phi^n(\lambda)} = i\Big\} \cap$$

$$\cap \Big\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_\lambda = j\Big\}\Bigg) =$$

$$= \lim_{N\to+\infty} \frac{1}{qN}(N\delta_{ij}p_j + (q-1)Np_2(i,j)) =$$

$$= \frac{\delta_{ij}p_j + (q-1)p_2(i,j)}{q} = \frac{\delta_{ij}p_j + (q-1)p_2(j,i)}{q} =$$

$$= \frac{p_j(\delta_{ij} + (q-1)p_{ji})}{q}$$

thus:

$$p_{ij} = \begin{cases} \dfrac{qp_j}{q-1} & i \neq j \\ \dfrac{qp_j - 1}{q-1} & i = j \end{cases},$$

which leads to the desired result (use Th. 5 too).

Lemma 7. *Let X has been occupied with discrete topology and ϕ is one to one, then the set of all periodic points under σ_ϕ are dense in $\prod_\Gamma X$ ($x \in \prod_\Gamma X$ is periodic under σ_ϕ if there exists $n \in \mathbf{N}$ such that $(\sigma_\phi)^n(x) = = (x)$).*

Proof. Suppose $k > 1$, let U be an open neighborhood of $(a_\gamma)_{\gamma\in\Gamma}$ in $\prod_\Gamma X$, there exist distinct $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\prod_{\gamma\in\Gamma} U_\gamma \subseteq U$, where $U_\gamma = \{a_\gamma\}$ for $\gamma = \gamma_1, \ldots, \gamma_n$ and $U_\gamma = X$ otherwise. Without lost of generality we can suppose $l \leq n$ be such that $\{\phi^n(\gamma_i) : n \in \mathbf{Z}\}$s are disjoint sets for $i = 1, \ldots, l$, and $\{\gamma_1, \ldots, \gamma_n\} \subseteq \{\phi^n(\gamma_i) : 1 \leq i \leq l, 0 \leq n \leq p\}$. Define:

$$b_\gamma = \begin{cases} a_\gamma & \begin{array}{c} \gamma \in \{\phi^n(\gamma_i) : 1 \leq i \leq l, 0 \leq n \leq p\} \\ \text{or} \\ (\exists t \in \mathbf{N}\ \phi^t(\gamma) = \gamma) \wedge \gamma \in \displaystyle\bigcup_{i=1,\ldots,l} \{\phi^n(\gamma_i) : n \in \mathbf{Z}\} \end{array} \\ a_{\phi^m(\gamma_i)} & \begin{array}{c} \big(\gamma = \phi^s(\gamma_i), i = 1, ..., l, s \neq 0, ..., p, s \equiv m (\operatorname{mod} p+1), 0 \leq m \leq p\big) \\ \text{and} \\ (\forall t \in \mathbf{N}\ \phi^t(\gamma) \neq \gamma) \end{array} \\ c & \gamma \notin \displaystyle\bigcup_{i=1,\ldots,l} \{\phi^n(\gamma_i) : n \in \mathbf{Z}\} \end{cases},$$

where $c \in X$ is a fix point, then $(b_\gamma)_{\gamma\in\Gamma}$ is a periodic point under σ_ϕ in U.
Theorem 8. *Let ϕ be one to one, then the set of all periodic points under σ_ϕ is dense in $\prod\limits_\Gamma X$.*
Proof. Use Lemma 7.
Theorem 9. *For finite $X = \{1, \ldots, k\}$ and countable Γ we have:*

1. *Suppose $\phi : \Gamma \to \Gamma$ be bijective and for each $n \in \mathbf{N}$, $\gamma \in \Gamma$, $\phi^n(\gamma) \neq \gamma$, moreover there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that*
$$\Gamma = \left\{\phi^i(\gamma_j) : j = 1, \ldots, n, i \in \mathbf{Z}\right\},$$
then $\sigma_\phi : \prod\limits_\Gamma X \to \prod\limits_\Gamma X$ is expansive.

2. *With the same assumptions as in item 1, if for $j = 1, \ldots, n$, $\{\phi^i(\gamma_j) : i \in \mathbf{Z}\}$s are pairwise disjoint, then $\sigma_\phi : \prod\limits_\Gamma X \to \prod\limits_\Gamma X$ has topological entropy $n \ln k$.*

3. *Suppose $\phi : \Gamma \to \Gamma$ be bijective and there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\Gamma = \{\phi^i(\gamma_j) : j = 1, \ldots, n, i \in \mathbf{Z}\}$ and for $j = 1, \ldots, n$, $\{\phi^i(\gamma_j) :$ $: i \in \mathbf{Z}\}$s are pairwise disjoint, then $\sigma_\phi : \prod\limits_\Gamma X \to \prod\limits_\Gamma X$ has topological entropy $m \ln k$, where*
$$m = \left|\left\{j \in \{1, \ldots, n\} : \{\phi^i(\gamma_j) : i \in \mathbf{Z}\} \textit{ is infinite}\right\}\right|.$$

Proof. 1. $\sigma_\phi : \prod\limits_\Gamma X \to \prod\limits_\Gamma X$ is a homeomorphism of compact metrizable spaces. Without less of generality suppose $\{\phi^i(\gamma_j) : i \in \mathbf{Z}\}$ for $j =$ $= 1, \ldots, n$ are pairwise disjoint.
$$\left\{\left\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_\Gamma X : x_{\gamma_1} = i_1, \ldots, x_{\gamma_n} = i_n\right\} : i_1, \ldots, i_n \in \{1, \ldots, k\}\right\}$$
is a generator. Now use [2, Th. 5.22].

2. Use [2, Th. 7.11] and consider the generator introduced in item 1.

Note 10. Let $X = \{1, \ldots, k\}$. If $\Gamma = \mathbf{N}$ and $\phi(n) = n + 1$ ($\forall n \in \mathbf{N}$), then σ_ϕ is called one-sided shift; in addition if $\Gamma = \mathbf{Z}$ and $\phi(n) = n + 1$ ($\forall n \in \mathbf{Z}$), then σ_ϕ is called two-sided shift.

For $\eta, \phi : \Gamma \to \Gamma$, $\sigma_\phi\sigma_\eta = \sigma_\eta\sigma_\phi$ if and only if $|X| \leq 1$ or $\phi\eta =$ $= \eta\phi$. Therefore if $\Gamma = \mathbf{N}$ or $\Gamma = \mathbf{Z}$ and $\phi(n) = n + 1$, $|X| > 1$, then $\sigma_\phi\sigma_\eta = \sigma_\eta\sigma_\phi$ if and only if there exists $n \in \Gamma \cup \{0\}$ such that $\eta = \phi^n$.

Questions. With the same assumptions as in Cor. 2 or Note 4, for one to one ϕ:

What is the centralizer of σ_ϕ?

When σ_ϕ is coalescence?

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