UNIFORM TYPE HYPERSPACES

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Abstract: If \((X, \mathcal{Q})\) is a quasi-uniform space, then in the hyperspace \(\mathcal{P}_0(X)\) of all non-empty subsets of \(X\) we investigate the several quasi-uniformities related with the Bourbaki–Hausdorff quasi-uniformity ([5], [10], [11], [12]). We show that if \((X, \mathcal{Q})\) is a quasi-uniform monoid (conoid), then \(\mathcal{P}_0(X)\) with respect to the corresponding algebraic operations and quasi-uniformities is again a quasi-uniform monoid (conoid). Moreover, it is demonstrated that if \((X, \mathcal{Q})\) is a quasi-uniform conoid, then in case of the hyperspace \(\mathcal{P}_c(X)\) of all non-empty convex subsets of \(X\) the scalar multiplication on positive real numbers has some nice continuity properties.

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1. Preliminary concepts

1.1. Uniform type spaces

\( \mathbb{R} \) will denote the set of real numbers and \( \mathbb{R}_+ := [0, \infty[ \). The set \( \mathbb{R} \) and its subsets (including \( \mathbb{R}_+ \) and the unit segment \( [0,1] \)) will be supposed to be endowed with the usual topology \( \varepsilon \). For a topological space \((X, \tau)\) we denote by \( N_\tau(x) \) the collection of all neighborhoods of a point \( x \in X \). For the considered topologies and topological spaces no separation axioms are required in advance.

Fix a non-empty set \( X \), a subset of \( X \times X \) is called a \emph{(binary)} relation on \( X \). The relations will be denoted by \( P, Q, R \), etc. We write:

\[
\Delta_X := \{ (x,x) \in X \times X \mid x \in X \},
\]

\[
\top(P) := \{ (y,x) \in X \times X \mid (x,y) \in P \},
\]

\[
P \circ Q := \{ (x,y) \in X \times X \mid \exists z \in X \text{ such that } (x,z) \in Q, (z,y) \in P \}.
\]

The relation \( \top(P) \) is called the \emph{converse relation} of \( P \). Instead of \( \top(P) \) the notation \( P^{-1} \) also is used. A relation \( P \) is called reflexive if \( \Delta_X \subset P \) and symmetric if \( \top(P) = P \).

For a collection \( Q \) of relations on \( X \), we write \( Q^\top := \{ \top(Q) \mid Q \in \mathcal{Q} \} \) and we say that \( Q \) is \emph{symmetric} if \( Q = Q^\top \). The relation \( P \circ Q \) is called the \emph{composition} of relations \( P \) and \( Q \).

For \( x \in X \) and \( E \subset X \) we set \( P[x] := \{ y \in X \mid (x,y) \in P \} \) and \( P[E] := \bigcup_{x \in E} P[x] \).

We recall the usual terminology from the theory of quasi-uniform spaces (see, e.g.,[6], [14], [13]):

- A filter \( Q \) consisting of reflexive relations on \( X \) is a
  - Local Quasi-uniformity if \( \forall x \in X, \forall Q \in \mathcal{Q}, \exists P \in \mathcal{Q} \text{ such that } P \circ P[x] \subset Q[x] \).
  - Local Uniformity if \( Q \) is a symmetric local quasi-uniformity.
  - Quasi-Uniformity if \( \forall Q \in \mathcal{Q}, \exists P \in \mathcal{Q} \text{ such that } P \circ P \subset Q \).
  - Uniformity if \( Q \) is a symmetric quasi-uniformity.

If \( Q \) is a quasi-uniformity, the filter \( Q^\top \) is a quasi-uniformity too. However, if \( Q \) is a local quasi-uniformity, then \( Q^\top \) may not be a local quasi-uniformity. A local quasi-uniformity \( Q \) is called \emph{bilocal quasi-uniformity} if \( Q^\top \) is a local quasi-uniformity as well (cf. [2]).

The pair \((X, Q)\) is called a local quasi-uniform space (a local uniform space, a quasi-uniform space, a uniform space) when \( Q \) is a local
quasi-uniformity (a local uniformity, a quasi-uniformity, a uniformity) and the members of \( \mathcal{Q} \) are called entourages.\(^1\)

Every uniform type structure \( \mathcal{Q} \) induces in \( X \) the topology \( \tau_{\mathcal{Q}} \) for which
\[
\{ \mathcal{Q}[x] \mid \mathcal{Q} \in \mathcal{Q} \} = \mathcal{N}_{\tau_{\mathcal{Q}}}(x), \ \forall x \in X.
\]
For a quasi-uniformity \( \mathcal{Q} \) the topologies \( \tau_{\mathcal{Q}} \) and \( \tau_{\mathcal{Q}^T} \) may be distinct.

A uniform type structure \( \mathcal{Q} \) is called compatible with a topology \( \tau \) if \( \tau_{\mathcal{Q}} = \tau \).

We say that a (local) quasi-uniformity \( \mathcal{Q} \) is
1. weakly locally symmetric at \( x \in X \) if for every \( \mathcal{Q} \in \mathcal{Q} \) there is a symmetric entourage \( S \in \mathcal{Q} \) such that \( S[x] \subset \mathcal{Q}[x] \);
2. weakly locally symmetric or point-symmetric if \( \mathcal{Q} \) is weakly locally symmetric at \( x \) for every \( x \in X \);
3. locally symmetric at \( x \in X \) if for every \( \mathcal{Q} \in \mathcal{Q} \) there is a symmetric entourage \( S \in \mathcal{Q} \) such that \( S \circ S[x] \subset \mathcal{Q}[x] \);
4. locally symmetric if \( \mathcal{Q} \in \mathcal{Q} \) is locally symmetric at \( x \) for every \( x \in X \).

Let \( X \) be a set and \( (\mathcal{Q}_i)_{i \in I} \) be a non-empty family of uniform type structures in \( X \). For this family, in the partially ordered set of all filters over \( X \times X \), always exist the least upper bound \( \vee_{i \in I} \mathcal{Q}_i \) and the greatest lower bound \( \wedge_{i \in I} \mathcal{Q}_i \). They are uniform type structures of the same type of \( \mathcal{Q}_i \) (see [1] or [3]). Moreover \( \{ \cap_{i \in J} \mathcal{Q}_i \mid \mathcal{Q}_i \in \mathcal{Q}_i, J \) finite \( \subset I \} \) is a base of \( \vee_{i \in I} \mathcal{Q}_i \).

For a given bilocal quasi-uniformity \( \mathcal{Q} \) we denote \( \mathcal{Q}^V = \mathcal{Q} \vee \mathcal{Q}^T \) and \( \mathcal{Q}^\wedge = \mathcal{Q} \wedge \mathcal{Q}^T \). It is known that \( \mathcal{Q}^V \) is the coarsest local uniformity containing \( \mathcal{Q} \) and \( \mathcal{Q}^\wedge \) is the finest local uniformity contained into \( \mathcal{Q} \).

A local quasi-uniform space \( (X, \mathcal{Q}) \) is called precompact if \( \forall \mathcal{Q} \in \mathcal{Q} \exists F \) finite \( \subset X \) such that \( X = \mathcal{Q}[F] \).

If \( (X, \mathcal{P}) \) and \( (Y, \mathcal{Q}) \) are local quasi-uniform spaces and \( \mathcal{F} \subset Y^X \) is a non-empty family of mappings, then \( \mathcal{F} \) is called \((\mathcal{P}, \mathcal{Q})\)-uniformly equicontinuous if
\[
\forall \mathcal{Q} \in \mathcal{Q}, \exists \mathcal{P} \in \mathcal{P} \text{ such that } (f(x_1), f(x_2)) \in \mathcal{Q}, \ \forall (x_1, x_2) \in \mathcal{P}, \ \forall f \in \mathcal{F}.
\]

**Proposition 1.1.** Let \( X \) and \( Y \) be nonempty sets, \( \mathcal{F} \subset Y^X \) a nonempty family of mappings, \( (\mathcal{P}_i)_{i \in I} \) a nonempty family of local quasi-uniformities on \( X \) and \( (\mathcal{Q}_i)_{i \in I} \) a nonempty family of local quasi-uniformities on \( Y \). Assume that \( \forall i \in I, \mathcal{F} \) is \((\mathcal{P}_i, \mathcal{Q}_i)\)-uniformly equicontinuous. Then:

\(^1\)Some authors use the term “vicinity” instead of entourage.
1.2. Uniform type semigroups and monoids

A semigroup is a pair \((X, +)\), where \(X\) is a non-empty set and \(+ : X \times X \to X\) is an associative binary operation. A **monoid** is a triplet \((X, +, \theta)\), where \((X, +)\) is a semigroup which has the neutral element \(\theta\). If \((X, +)\) is a semigroup (monoid) in \(X \times X\) we define a semigroup operation componentwise.

As usual, for non-empty subsets \(A, B\) of a semigroup \(A + B\) will stand for their algebraic or Minkowski sum \(\{a + b | a \in A, b \in B\}\).

A monoid (semigroup) \(X\) which is also a topological space is called a **topological monoid** if \(+\) is continuous with respect to the product topology in \(X \times X\) and the topology of \(X\).

A monoid (semigroup) \(X\) equipped with a local quasi-uniformity (bilocal quasi-uniformity, quasi-uniformity, local uniformity, uniformity) \(Q\) is called a **local quasi-uniform** (bilocal quasi-uniform, quasi-uniform, local uniform, uniformity) **monoid** (semigroup) if \(+\) is uniformly continuous with respect to the product quasi-uniformity \(Q \otimes Q\) and \(Q\).

**Lemma 1.2.** Let \((X, +, \theta)\) be a monoid, \(Q\) be a local quasi-uniformity.

a) The following statements are equivalent:
   (i) \((X, Q)\) is a local quasi-uniform monoid.
   (ii) \(\forall Q \in Q \exists P \in Q\) such that \(P + P \subset Q\).

b) If \((X, Q)\) is a bilocal quasi-uniform monoid, then \((X, Q^\top)\) also is.

c) If \((X, Q)\) is a (bilocal) quasi-uniform monoid, then \((X, Q^\lor)\) is a (local) uniform monoid.

1.3. Uniform type conoids

A **conoid** is an Abelian monoid \((X, +, \theta)\) for which an external operation
\[
m : X \times \mathbb{R}_+ \to X, \ m(x, \alpha) = x \cdot \alpha
\]
is defined with the properties:
\[
A.1 \quad (x_1 + x_2) \cdot \alpha = x_1 \cdot \alpha + x_2 \cdot \alpha \quad \forall x_1, x_2 \in X, \ \forall \alpha \in \mathbb{R}_+;
A.2 \quad (x \cdot \alpha_1) \cdot \alpha_2 = x \cdot (\alpha_1 \cdot \alpha_2) \quad \forall x \in X, \ \forall \alpha_1, \alpha_2 \in \mathbb{R}_+;
A.3 \quad x \cdot (\alpha_1 + \alpha_2) = x \cdot \alpha_1 + x \cdot \alpha_2 \quad \forall x \in X, \ \forall \alpha_1, \alpha_2 \in \mathbb{R}_+;
A.4 \quad x \cdot 1 = x \quad \forall x \in X.
\]
In the literature a conoid is also called an abstract convex cone [16], a cone [9], a semi-vector space [15], or a semilinear space [7], [8], [17], etc. In [1] the conoids were introduced to develop a integration scheme in quasi-uniform spaces, these structures also have been studies in [4].

If \((X, +, \theta, m)\) is a conoid then in \(X \times X\) we define a conoid structure componentwise.

Let \((X, +, \theta, m)\) be a conoid, \(K\) be a non-empty subset of \(X\), \(\alpha \in \mathbb{R}_+\) and \(A\) be an element of \(\mathbb{R}_+\). We write
\[
K \cdot \alpha := \{ x \cdot \alpha — x \in K \} \quad \text{and} \quad K \cdot A := \{ x \cdot \alpha — x \in K, \alpha \in A \}.
\]
Let \((X, +, \theta, m)\) be a conoid, \(K\) be a subset of \(X\) and \(b\) be an element of \(X\). \(K\) is called:

1. Convex if either \(K\) is empty, or \(K \cdot \alpha + K \cdot (1 - \alpha) \subset K\), for every \(\alpha \in [0, 1]\).
2. Balanced if either \(K\) is empty, or \(K \cdot [0, 1] \subset K\).

**Remark 1.3.** Let \((X, +, \theta, m)\) be a conoid.

1. \(X\) itself is convex, balanced and radial.
2. If \(K\) is a non-empty convex subset of \(X\), then \(K \cdot (\alpha + \beta) = K \cdot \alpha + K \cdot \beta\), \(\alpha, \beta \in \mathbb{R}_+\).
3. The intersection of any non-empty family of convex (balanced) subsets of a conoid is convex (balanced).

As usual, we denote \(\text{co}(K)\) the convex hull of a subset \(K \subset X\).

**Definition 1.4.** A conoid \((X, +, \theta, m)\) equipped with a local quasi-uniformity (bilocal quasi-uniformity, quasi-uniformity, local uniformity, uniformity) \(Q\) is called a local quasi-uniform (bilocal quasi-uniform, quasi-uniform, local uniform, uniform) conoid if \((X, +, \theta, Q)\) is a local quasi-uniform monoid. It is denoted by \((X, +, \theta, m, Q)\).

Therefore a local quasi-uniform conoid is simply a local quasi-uniform monoid which algebraically is a conoid.

We shall say that a local quasi-uniform conoid \((X, +, \theta, m, Q)\) is

- **locally convex** if \(Q\) admits a base consisting of convex entourages;
- **locally balanced** if \(Q\) admits a base consisting of balanced entourages.

**Remark 1.5.** Let \((X, +, \theta, m, Q)\) a bilocal quasi-uniform conoid.

1. \((X, +, \theta, m, Q^\uparrow)\) is a bilocal quasi-uniform conoid (see 1.2).
2. \((X, +, \theta, m, Q^\wedge), (X, +, \theta, m, Q^\vee)\) are local uniform conoids (see 1.2).

For every \(x \in X\), and for every \(\alpha \in \mathbb{R}_+\) we will consider the mappings
Denoting by $\mathcal{E}_+$ the usual uniformity on $\mathbb{R}_+$, we say that the external operation of a local quasi-uniform conoid $(X, +, \theta, m, \mathcal{Q})$ is

- $UC_{\ell}$ if $m_x$ is $(\mathcal{E}_+, \mathcal{Q})$-uniformly continuous $\forall x \in X$;
- $UC_t$ if $m_\alpha$ is $\mathcal{Q}$-uniformly continuous $\forall \alpha \in \mathbb{R}_+$;
- $C_{\ell,0}$ if $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q})$-continuous at 0 $\forall x \in X$;
- $C_\ell$ if $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q})$-continuous on $\mathbb{R}_+$ $\forall x \in X$;
- $C_{t,\theta}$ if $m_\alpha$ is $\tau_\mathcal{Q}$-continuous at $\theta$ $\forall \alpha \in \mathbb{R}_+$;
- $C_t$ if $m_\alpha$ is $\tau_\mathcal{Q}$-continuous on $X$ $\forall \alpha \in \mathbb{R}_+$;
- $JC_{(\theta,0)}$ if $m$ is $(\tau \otimes \mathbf{e}, \tau_\mathcal{Q})$-continuous at $(\theta, 0)$;
- $JC$ if $m$ is $(\tau_\mathcal{Q} \otimes \mathbf{e}, \tau_\mathcal{Q})$-continuous everywhere.

Let $(X, +, \theta, m)$ be a conoid. A local quasi-uniformity $\mathcal{Q}$ on $X$ is called homogeneous if $Q \cdot \alpha \in \mathcal{Q}$ $\forall Q \in \mathcal{Q}$, $\forall \alpha > 0$.

**Proposition 1.6.** Let $(X, +, \theta, m, \mathcal{Q})$ be a bilocal quasi-uniform conoid such that $m$ is $C_{\ell,0}$. The following statements are valid:

a) $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q})$-right-continuous $\forall x \in X$.

b) If $Q^\top$ is weakly locally symmetric at $\theta$, then $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q}^\top)$-continuous at 0 $\forall x \in X$.

c) If $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q}^\top)$-continuous at 0 $\forall x \in X$, then $m$ is $UC_{\ell}$.

d) If $Q^\top$ is weakly locally symmetric at $\theta$, then $m$ is $UC_{\ell}$.

e) If $Q$ is a uniformity, then $m$ is $C_{\ell,0}$ if and only if $m$ is $UC_{\ell}$.

**Proof.**

a) Fix $x \in X$ and $\alpha \in \mathbb{R}_+$, $\alpha > 0$ and $Q \in \mathcal{Q}$. Since $+$ is $(\tau_\mathcal{Q} \otimes \tau_\mathcal{Q}, \tau_\mathcal{Q})$-continuous at $(x \cdot \alpha, \theta)$ and $x \cdot \alpha = x \cdot \alpha + \theta$, there exists $R \in \mathcal{Q}$ such that $R[x \cdot \alpha] + R[\theta] \subset Q[x \cdot \alpha]$.

Since $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q})$-continuous at 0 there exists $\varepsilon > 0$ such that $x \cdot t \in R[\theta]$ $\forall t \in [0, \varepsilon[$. Then:

$$x \cdot (\alpha + t) = x \cdot \alpha + x \cdot t \in R[x \cdot \alpha] + R[\theta] \subset Q[x \cdot \alpha] \quad \forall t \in [0, \varepsilon]$$

and the $(\mathbf{e}, \tau_\mathcal{Q})$-right-continuity of $m_x$ at $\alpha$ is proved.

b) Obvious.

c) Fix $x \in X$. Since $(X, \mathcal{Q})$ is a bilocal quasi-uniform semigroup, there exists $R \in \mathcal{Q}$ such that $R + R \subset \mathcal{Q}$. Since $m_x$ is $(\mathbf{e}, \tau_\mathcal{Q} \lor \tau_\mathcal{Q}^\top)$-continuous at 0 there exists $\varepsilon > 0$ such that $m_x([0, \varepsilon]) \subset R[\theta] \cap T(R)[\theta]$, i.e.,

\[(\theta, x \cdot t) \in R \quad \text{and} \quad (x \cdot t, \theta) \in R, \forall t \in [0, \varepsilon].\]
Take $\alpha, \beta \in \mathbb{R}_+$ with $|\alpha - \beta| < \varepsilon$ and let us show that $(x \cdot \alpha, x \cdot \beta) \in Q$.

If $\alpha < \beta$, then $\beta = \alpha + t$ with $t := \beta - \alpha \in [0, \varepsilon]$. This and (*) imply:

$$(x \cdot \alpha, x \cdot \beta) = (x \cdot \alpha + \theta, (x \cdot \alpha + x \cdot t)) 
= (x \cdot \alpha, x \cdot \alpha + (\theta, x \cdot t)) \in R + R \subset Q.$$ 

If $\alpha > \beta$, then $\alpha = \beta + t$ with $t := \alpha - \beta \in [0, \varepsilon]$. This and (*) imply:

$$(x \cdot \alpha, x \cdot \beta) = (x \cdot t + x \cdot \beta, (x \cdot \beta, x \cdot \beta)) 
= (x \cdot t, (x \cdot \beta, x \cdot \beta)) \in R + R \subset Q.$$ 

Consequently, $\alpha, \beta \in \mathbb{R}_+, |\alpha - \beta| < \varepsilon \Rightarrow (x \cdot \alpha, x \cdot \beta) \in Q$

and so, $m_x$ is $(\mathcal{E}^+, Q)$- uniformly continuous.

d) Follows from b) and c).

e) Follows from d). \diamondsuit

2. Uniform type hyperspaces

Let $X$ be a nonempty set and $\mathcal{P}_0(X)$ be the collection of all nonempty subsets of $X$. For each relation $Q$ on $X$, set

$$Q^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) | B \subset Q[A]\},$$

$$Q^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) | A \subset \top(Q)[B]\},$$

$$Q^* := Q^+ \cap Q^-.$$

Remark 2.1. Let $P, Q$ be relations on $X$, then:

1. $\top(Q^-) = (\top(Q))^+$ and $\top(Q^+) = (\top(Q))^-$.
2. $(P \cup Q)^+ = P^+ \cup Q^+$.
3. $(P \cup Q)^- = P^- \cup Q^-.
4. (P \cap Q)^+ \subset P^+ \cap Q^+
5. (P \cap Q)^- \subset P^- \cap Q^-.
6. (P \cap Q)^* \subset P^* \cap Q^*.

For a local quasi-uniformity $Q$ on $X$ let

- $Q^+$ be the filter generated by $\{Q^+ | Q \in Q\},$
- $Q^-$ be the filter generated by $\{Q^- | Q \in Q\},$
- $Q^* := Q^+ \lor Q^-.$

Remark 2.2. If $(X, Q)$ is a local quasi-uniform space, then

1. $(Q^-)^\top = (Q^\top)^+$, and $(Q^+)^\top = (Q^\top)^-$;
2. $(Q^*)^\top = (Q^\top)^*.$

Proposition 2.3. Let $(X, Q)$ be a quasi-uniform space. The following statements are true:

(a) (cf. [5, 10]) $Q^+, Q^-$ and $Q^*$ are quasi-uniformities.
(b) If $\mathcal{Q}$ is a uniformity, then $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are conjugate quasi-uniformities, and $\mathcal{Q}^*$ is a uniformity on $\mathcal{P}_0(X)$.

**Proof.** (a) Fix $\mathfrak{P} \in \mathcal{Q}^+$. There exists $P \in \mathcal{Q}$ such that $P^+ \subset \mathfrak{P}$. Since $\mathcal{Q}$ is a quasi-uniformity there is $Q \in \mathcal{Q}$ such that $Q \circ Q \subset P$. Let us show that $Q^+ \circ Q^+ \subset P^+$:

Take $(A, B) \in Q^+ \circ Q^+$. There is a $C$ such that $(A, C) \in Q^+$ and $(C, B) \in Q^+$. For each $b \in B$ there is $c \in C$ such that $(c, b) \in Q$ and there is $a \in A$ such that $(a, c) \in Q$. It follows that $(a, b) \in Q \circ Q \subset P$ and so, $b \in P[a] \subset P[A]$. Hence $B \subset P[A]$ and $(A, B) \in P^+$.

The other cases are analogous.

(b) Follows from Rem. 2.1(1). $\Diamond$

The quasi-uniformities $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are called, respectively, the *upper* and *lower Hausdorff quasi-uniformities* on $\mathcal{P}_0(X)$ associated with $\mathcal{Q}$.

The quasi-uniformity $\mathcal{Q}^*$ is called *Hausdorff (or Bourbaki) quasi-uniformity* on $\mathcal{P}_0(X)$ associated with $\mathcal{Q}$.

The next proposition shows that an analogue of Prop. 2.3(a) is not true for bilocal quasi-uniformities.

**Proposition 2.4.** Let $(X, \mathcal{Q})$ be a bilocal quasi-uniform space. Then:

a) $\mathcal{Q}^+$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.

b) $\mathcal{Q}^-$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.

c) $\mathcal{Q}^*$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.

**Proof.** Let $X = \{0, 1/2, 1, 1/3, 2, \ldots, 1/n, \ldots\}$ and

$$Q_n = \Delta \cup \left\{ \left( 0, \frac{1}{i} \right) : i \geq n \right\} \cup \left\{ \left( \frac{1}{i+1}, \frac{1}{i} \right) : i \geq n \right\}.$$  

First we will see that $Q_0 = \{Q_n : n \in \mathbb{N}\}$ is base of a bilocal quasi-uniformity $\mathcal{Q}$ (cf. [1]).

- $Q_{n+1} \subset Q_n$, for each $n \in \mathbb{N}$, therefore $\mathcal{Q}$ is a filter base on $X \times X$.
- $\Delta \subset Q_n$, for every $n \in \mathbb{N}$.
- Observe that:

$$Q_n[0] = \left\{ 0, \frac{1}{n}, \frac{1}{n+1}, \ldots \right\} \text{ and } Q_n \circ Q_n[0] = \left\{ 0, \frac{1}{n}, \frac{1}{n+1}, \ldots \right\}$$

hence $Q_n \circ Q_n[0] = Q_n[0]$. Now, let $n \geq 1$ and $k \geq 1$, we have that:

$$Q_k \circ Q_k \left[ \frac{1}{k} \right] = \left\{ \frac{1}{k} \right\} \text{ hence } Q_k \circ Q_k \left[ \frac{1}{k} \right] \subset Q_n \left[ \frac{1}{k} \right].$$

- Notice that:

$$\top(Q_n) = \Delta \cup \left\{ \left( \frac{1}{i}, 0 \right) : i \geq n \right\} \cup \left\{ \left( \frac{1}{i+1}, \frac{1}{i} \right) : i \geq n \right\}, \quad n = 1, 2, 3 \ldots$$
Observe that: $-\top(Q_n)[0] = \{0\}$ and $\top(Q_n) \circ \top(Q_n)[0] = \{0\}$ hence $\top(Q_n) \circ \top(Q_n)[0] = \top(Q_n)[0]$, $n = 1, 2, \ldots$ and for $n, k \in \mathbb{N}$ we have:

\[
\top(Q_{k+1}) \circ \top(Q_{k+1}) \left[ \frac{1}{k} \right] = \left[ \frac{1}{k} \right]
\]

hence $\top(Q_{k+1}) \circ \top(Q_{k+1}) \left[ \frac{1}{k} \right] \subset \top(Q_n) \left[ \frac{1}{k} \right]$.

a) Let $A = \{\frac{1}{3}, \frac{1}{6}, \ldots, \frac{1}{3n}, \ldots\}$, let us see that

\[
Q^+_m \circ Q^+_m[A] \not\subset Q^+_1[A], \forall m \in \mathbb{N}
\]

with

\[
Q^+_1[A] = P_0 \left( \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{3n-1}, \frac{1}{3n}, \ldots \right\} \right) \cup \emptyset.
\]

We have:

\[
\left\{ (A, \left\{ \frac{1}{3m-1} \right\}) \in Q^+_m \right. \\
\left. \left\{ \left\{ \frac{1}{3m-1} \right\}, \left\{ \frac{1}{3m-2} \right\} \right\} \in Q^+_m \right. \\
\]

Therefore $\left\{ \frac{1}{3m-2} \right\} \in Q^+_m \circ Q^+_m[A] \forall m \in \mathbb{N}$, but $\left\{ \frac{1}{3m-2} \right\} \not\in Q^+_1[A]$.

b) Let $A = \{\frac{1}{3}, \frac{1}{6}, \ldots, \frac{1}{3n}, \ldots\}$, let we us see that

\[
Q^-_m \circ Q^-_m[A] \not\subset Q^-_1[A], \forall m \in \mathbb{N}
\]

with

\[
Q^-_1[A] = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{3n-1}, \frac{1}{3n}, \ldots \right\}.
\]

Consider

\[
B_m = \left\{ \frac{1}{3k} : 1 \leq k < m \right\} \cup \left\{ \frac{1}{3k-2} : k \geq m \right\}
\]

and

\[
C_m = \left\{ \frac{1}{3k} : 1 \leq k < m \right\} \cup \left\{ \frac{1}{3m-1}, \frac{1}{3m+2}, \frac{1}{3m+5}, \ldots \right\}.
\]

We have

\[
\left\{ (A, C_m) \in Q^-_m \right. \\
\left. \left\{ (C_m, B_m) \in Q^-_m \right. \\
\right.
\]

hence $B_m \in Q^-_m \circ Q^-_m[A], \forall m \in \mathbb{N}$, but $B_m \not\in Q^-_1[A]$.

c) Let $A = \{\frac{1}{3}, \frac{1}{6}, \ldots, \frac{1}{3n}, \ldots\}$, then by a) and b) we have

\[
Q^+_1[A] = (Q^+_1 \cap Q^-_1)[A] \subset Q^+_1[A] \cap Q^-_1[A] =
\]

\[
= \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{3n-1}, \frac{1}{3n}, \ldots \right\}
\]

Let $C_m$ and $B_m$ the sets defined in b). We have also
Proposition 2.5. Let \( T \). Abreu, E. Corbacho and V. Tarieladze \( B \) hence \( Q \land P \), then

\[
\left\{ \begin{array}{l}
(A, C_m) \in Q_m^+ \cap Q_m^- \\
(C_m, B_m) \in Q_m^+ \cap Q_m^- ,
\end{array} \right.
\]

hence \( B_m \in (Q_m^+ \cap Q_m^-) \circ (Q_m^+ \cap Q_m^-)[A], \ \forall m \in \mathbb{N}, \) but \( B_m \notin Q_1^+[A]. \) %

Taking into account Rem. 2.1 it is easy to prove the following:

**Proposition 2.6.** Let \( Q \) and \( P \) be quasi-uniformity on \( X \). Then:

1. \( P^* \lor Q^* \subset (P \lor Q)^* . \)
2. \( (P \land Q)^* \subset P^* \land Q^* . \)
3. If the set \( \{ P \lor Q \mid P \in \mathcal{P}, \ Q \in \mathcal{Q} \} \) is a quasi-uniform base of \( Q \land \mathcal{P} \), then
   a) \( \{ (P \lor Q)^+ \mid P^+ \in \mathcal{P}^+, \ Q^+ \in \mathcal{Q}^+ \} = \{ P^+ \cup Q^+ \mid P^+ \in \mathcal{P}^+ , \ Q^+ \in \mathcal{Q}^+ \} \) and both are quasi-uniform bases. Consequently, \( Q^+ \land P^+ = (Q \land \mathcal{P})^+ . \)
   b) \( \{ (P \lor Q)^- \mid P^- \in \mathcal{P}^-, \ Q^- \in \mathcal{Q}^- \} = \{ P^- \cup Q^- \mid P^- \in \mathcal{P}^-, \ Q^- \in \mathcal{Q}^- \} \) and both are quasi-uniform bases. Consequently, \( Q^- \land P^- = (Q \land \mathcal{P})^- . \)
   c) \( \{ (P \lor Q)^* \mid P^* \in \mathcal{P}^*, \ Q^* \in \mathcal{Q}^* \} = \{ P^* \cup Q^* \mid P^* \in \mathcal{P}^*, \ Q^* \in \mathcal{Q}^* \} \) are quasi-uniform bases and \( Q^* \land P^* = (Q \land \mathcal{P})^* . \)
4. In particular, we have
   a) \( (Q^*)^\lor \subset (Q^\lor)^* . \)
   b) \( (Q_\land)^* \subset (Q^\land)^* . \)
   c) When \( \{ \top (Q) \lor Q \mid Q \in \mathcal{Q} \} \) is base of \( Q_\land \) then \( (Q^*)_\land = (Q_\land)^* . \)

The following proposition shows that the local symmetry is preserved for singletons.

**Proposition 2.6.** Let \( (X, \mathcal{Q}) \) be a weakly locally symmetric quasi-uniform space. Then:

a) \( (\mathcal{P}_0(X), \mathcal{Q}^-) \) is weakly locally symmetric at \( \{ x \} \), \( \forall x \in X ; \)

b) \( (\mathcal{P}_0(X), \mathcal{Q}^+) \) is weakly locally symmetric at \( \{ x \} \), \( \forall x \in X ; \)

c) \( (\mathcal{P}_0(X), \mathcal{Q}^*) \) is weakly locally symmetric at \( \{ x \} \), \( \forall x \in X . \)

**Proof.** a) Fix \( \Omega \in \mathcal{Q}^- . \) There exists \( Q \in \mathcal{Q} \) such that \( Q^- \subset \Omega . \) For a \( x \in X \) there is a symmetric entourage \( S \in \mathcal{Q} \) such that \( S[x] \subset Q[x] . \)

Let \( B \in S^-[\{ x \}] , \) then there is a \( b \in B \) such that

\[
(x, b) \in S \text{ hence } (x, b) \in Q.
\]

Therefore \( \{ x \}, B \in \mathcal{Q}^- \) and so \( B \in \mathcal{Q}^-[\{ x \}] . \)

b) Is analogous to a).

c) Follows from a) and b) because the supremum of a family of weakly locally symmetric quasi-uniformities is weakly locally symmetric. %
**Proposition 2.7.** Let \((X, \mathcal{Q})\) be a locally symmetric quasi-uniform space. We have:

a) \((\mathcal{P}_0(X), \mathcal{Q}^-)\) is locally symmetric at \(\{x\}\), \(\forall x \in X\).

b) \((\mathcal{P}_0(X), \mathcal{Q}^+)\) is locally symmetric at \(\{x\}\), \(\forall x \in X\).

c) \((\mathcal{P}_0(X), \mathcal{Q}^*)\) is locally symmetric at \(\{x\}\), \(\forall x \in X\).

**Proof.** a) Fix \(Q \in \mathcal{Q}^-.\) There exists \(Q \in \mathcal{Q}^-\) such that \(Q^- \subset Q^-.\) For a \(x \in X\) there is a symmetric entourage \(S \in \mathcal{Q}^-\) such that \(S \circ S[\{x\}] \subset Q[\{x\}].\)

Let \(B \in S^- \circ S^-[\{x\}],\) then there is a \(C \subset X\) such that \((\{x\}, C) \in S^-\) and \((C, B) \in S^-\).

Then for each \(c \in C\) there is a \(b \in B\) such that \((x, c) \in S\) and \((c, b) \in S\).

Hence, there is \(b \in B\) such that \((x, b) \in S \circ S^-\) then \((x, b) \in Q^-\).

b) Is analogous to a).

c) Follows from a) and b) because the supremum of family of locally symmetric quasi-uniformities is weakly locally symmetric. \(\Box\)

### 2.1. Hyperspaces with algebraic structures

If \((X, +, \theta)\) is a monoid, then \(\mathcal{P}_0(X)\) is a monoid as well with respect to the internal operation

\[
+ : \mathcal{P}_0(X) \times \mathcal{P}_0(X) \to \mathcal{P}_0(X)
\]

\[
(A, B) \mapsto A + B
\]

and the neutral element \(\{\theta\}\).

**Theorem 2.8.** Let \((X, +, \theta, \mathcal{Q})\) be a quasi-uniform monoid, then \((\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^-), \quad (\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^+)\) and \((\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^*)\) are quasi-uniform monoids.

**Proof.** Fix \(Q \in \mathcal{Q}^+.\) There exists \(Q \in \mathcal{Q}\) such that \(Q^+ \subset \mathcal{Q}^+.\) Since + is uniformly continuous, there is a entourage \(P\) such that \(P + P \subset Q.\)

Observe that:

- if \((A_1, B_1) \in Q^+\) then \(B_1 \subset P[A_1];\)
- if \((A_2, B_2) \in Q^+\) then \(B_2 \subset P[A_2].\)

Then

\[
B_1 + B_2 \subset P[A_1] + P[A_2] \subset P[A_1 + A_2] \subset Q[B_1 + B_2].
\]

Hence

\[
P^+ + P^+ \subset Q^+.
\]
In the same way it is easy to see that $+$ is also uniformly continuous with respect to $Q^-$. Since $+$ is uniformly continuous with respect $Q^+$ and $Q^-$, by Prop. 1.1 it is also uniformly continuous with respect to $Q^*$. ♦

Let $(X, +, \theta, m)$ be a conoid. The external operation $m$ can be extended to $\mathcal{P}_0(X)$ in a natural manner:

$$m : \mathcal{P}_0(X) \times \mathbb{R}_+ \to \mathcal{P}_0(X)$$

$$(A, \alpha) \mapsto A \cdot \alpha$$

The structure $(\mathcal{P}_0(X), +, \{\theta\}, m)$ may not be a conoid, because, in general, property A.3 may fail.

Denote $\mathcal{P}_c(X)$ be the collection of all convex members of $\mathcal{P}_0(X)$. By Rem. 1.3(2) the structure $(\mathcal{P}_c(X), +, \{\theta\}, m)$ is a conoid. This is an important example of conoid. Observe that, since $X + X = X$, this conoid is not cancellative provided $X \neq \{\theta\}$.

Let $\mathcal{Q}$ be a quasi-uniformity in a conoid $(X, +, \theta, m)$. We denote $Q^+_c, Q^-_c$ and $Q^*_c$ the induced quasi-uniformities on $\mathcal{P}_c(X)$ by the quasi-uniformities $Q^+$, $Q^-$ and $Q^*$.

The following result is a particular case of Th. 2.8.

**Corollary 2.9.** Let $(X, +, \theta, m, \mathcal{Q})$ be a quasi-uniform conoid, then $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}^-), (\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}^+_c)$ and $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}^*_c)$ are quasi-uniform conoids.

**Proposition 2.10.** Let $(X, +, \theta, m)$ be a conoid, and $\mathcal{Q}$ be a quasi-uniformity on $X$.

a) If $\mathcal{Q}$ is locally convex, then $Q^-_c, Q^+_c$ and $Q^*_c$ are locally convex.

b) If $\mathcal{Q}$ is locally balanced, then $Q^-_c, Q^+_c$ and $Q^*_c$ are locally balanced.

**Proof.** a) Fix $\mathfrak{P} \in Q^+_c$. There exists a convex $P \in \mathcal{Q}$ such that $P^+ \subset \mathfrak{P}$. Fix $(A_1, B_1), (A_2, B_2) \in P^+$, we have that $B_1 \subseteq P[A_1]$ and $B_2 \subseteq P[A_2]$.

For each $b_1 \in B_1, b_2 \in B_2$ there is a $a_1 \in A_1, a_2 \in A_2$ such that

$$(a_1, b_1) \in P \text{ and } (a_2, b_2) \in P,$$

since $P$ is a convex entourage then

$$(a_1 \cdot \alpha + a_2 \cdot \beta, b_1 \cdot \alpha + b_2 \cdot \beta) \in P \text{ with } \alpha + \beta = 1.$$ 

Therefore $b_1 \cdot \alpha + b_2 \cdot \beta \in P[a_1 \cdot \alpha + a_2 \cdot \beta] \Rightarrow B_1 \cdot \alpha + B_2 \cdot \beta \in P[A_1 \cdot \alpha + A_2 \cdot \beta]$. Then

$$(A_1, B_1) \cdot \alpha + (A_2, B_2) \cdot \beta \in P^+ \text{ with } \alpha + \beta = 1.$$ 

In a similar way we can prove that the lower quasi-uniformity $Q^-_c$, is locally convex too.
Since $Q^*_c = Q^+_c \vee Q^-_c$, then $Q^*_c$ has also a base consisting of convex sets.

b) Now we will prove that if $P$ is a balanced entourage then $P^+$ is also balanced. Let $(A,B) \in P^+$, then

\[ B \subset P[A] \Rightarrow \forall b \in B \exists a \in A \text{ such that} \]

\[ (a,b) \in P \Rightarrow (a \cdot t, b \cdot t) \in P, \forall t \in [0,1], \]

hence $B \cdot t \subset P[A \cdot t]$ with $t \in [0,1]$.

In a similar way we can prove that the lower quasi-uniformity is locally balanced too.

Since $Q^*_c = Q^+_c \vee Q^-_c$, then $Q^*_c$ has also a base consisting of balanced sets. \(\diamond\)

In the following propositions we study the stability of the partial continuity of the action on the hyperspace $P_r(X)$.

We begin with the maps $m_\alpha : P_r(X) \to P_r(X)$.

**Proposition 2.11.** Let $(X,+,\theta,m)$ be a conoid and $Q$ be a quasi-uniformity for which $m$ is $C_{r,\theta}$. Then $m$ is $C_{r,\{\theta\}}$ in the conoids $(P_r(X),+,\{\theta\},m,Q^-_c)$, $(P_r(X),+,\{\theta\},m,Q^+_c)$ and $(P_r(X),+,\{\theta\},m,Q^*_c)$.

**Proof.** Fix $Q \in Q$ and $\alpha \in \mathbb{R}_+$. Since $m_\alpha$ is $\tau_{Q^-}$ continuous at $\theta$, there is a $P \in Q$ such that $P[\theta] \cdot \alpha \subset Q[\theta]$. Let $B \subset P^{-\{\theta\}}$, then there is $b \in B$ such that

\[ (\theta,b) \in P \Rightarrow (\theta,b \cdot \alpha) \in Q \Rightarrow \{\theta\} \subset \top(Q)[b \cdot \alpha]. \]

Thus $B \cdot \alpha \in Q^{-\{\theta\}}$.

In the same way we can prove that $m_\alpha$ is $\tau_{Q^+_c}$-continuous at $\{\theta\}$, and using the previous results and Prop. 1.1 we can conclude that $m$ is also $C_{r,\{\theta\}}$ in $(P_r(X),+,\{\theta\},m,Q^*_c)$. \(\diamond\)

**Proposition 2.12.** Let $(X,+,\theta,m)$ be a conoid and $Q$ be a quasi-uniformity for which $m$ is $UC_r$. Then $m$ is $UC_r$ in the conoids $(P_r(X),+,\{\theta\},m,Q^-_c)$, $(P_r(X),+,\{\theta\},m,Q^+_c)$ and $(P_r(X),+,\{\theta\},m,Q^*_c)$.

**Proof.** Fix $Q \in Q$ and $\alpha \in \mathbb{R}_+$. Since $m_\alpha$ is $Q$-uniformly continuous, there is an entourage $P$ such that $P \cdot \alpha \subset Q$.

If $B \subset P[A]$ then for each $b \in B$ there is a $a \in A$ such that

\[ (a,b) \in P \Rightarrow (a \cdot \alpha,b \cdot \alpha) \in Q \Rightarrow b \cdot \alpha \subset Q[a \cdot \alpha], \]

then

\[ b \cdot \alpha \subset \bigcup_{a \in A} Q[a \cdot \alpha] = Q[A \cdot \alpha]. \]

Hence $B \cdot \alpha \subset Q[A \cdot \alpha]$. Thus $P^+ \cdot \alpha \subset Q^+$. 
The case \((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)\) is analogous, and using the previous results and Prop. 1.1, we can prove that \(m\) is \(UC_r\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)\). ◊

Now we study the maps \(m_A : \mathbb{R}_+ \to \mathcal{P}_c(X), A \in \mathcal{P}_c(X)\).

**Proposition 2.13.** Let \((X, +, \theta, m)\) be a conoid and \(Q\) a quasi-uniformity on \(X\). If \(m\) is \(C_{\ell,0}\) then 

a) \(m\) is \(C_{\ell,0}\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\).

b) If \((X, Q)\) is a locally balanced, precompact quasi-uniform space, then:

i) \(m\) is \(C_{\ell,0}\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\).

ii) This item is a consequence of the last statements and Prop. 1.1.

**Proof.** a) Let \(A\) be a non-empty convex subset of \(X\), and fix \(Q \in \mathcal{Q}\). Let 
\(x \in A\). As \(m_A\) is \(\tau_Q\)-continuous at 0, there is \(\varepsilon > 0\) such that 
\((\theta, x \cdot t) \in Q\), 
\(\forall t \in [0, \varepsilon]\). Then 
\[\{\theta\} \subset T(Q)[A \cdot t], \forall t \in [0, \varepsilon],\]

hence 
\[A \cdot t \in Q^{-\{\theta\}}, \forall t \in [0, \varepsilon].\]

b) i) Let \(A\) be a convex subset of \(X\). Fix \(P \in \mathcal{Q}\). There is a balanced entourage \(Q\) such that \(Q \circ Q \subset P\). Since \((X, Q)\) is precompact, there is a finite subset 
\(F = \{x_1, x_2, \ldots, x_n\} \subset X\) such that 
\(A \subset \bigcup_{i=1}^n Q[x_i]\).

Since for \(i \leq n\) the map \(m_{x_i}\) is continuous, there is \(\varepsilon_{x_i} \subset [0, 1]\) such that 
\[\begin{align*}
(\theta, x_i \cdot t) & \in Q, \forall t \in [0, \varepsilon_{x_i}], \\
A \cdot t & \subset P[\{\theta\}]\end{align*}\]

Put \(\varepsilon = \min\{\varepsilon_{x_i} | 1 \leq i \leq n\}\).

For all \(x \in A\), there is \(i \leq n\) such that \((x_i, x) \in Q\). Since \(Q\) is balanced,
\[\begin{align*}
(x_i \cdot t, x \cdot t) & \in Q, \forall t \in [0, \varepsilon] \subset [0, 1], \\
\forall x \in A, (\theta, x \cdot t) & \in Q \circ Q \subset P, \forall t \in [0, \varepsilon],
\end{align*}\]

and so, \(A \cdot t \subset P[\{\theta\}]\) and \(A \cdot t \in P^+[\{\theta\}]\).

ii) This item is a consequence of the last statements and Prop. 1.1. ◊

**Proposition 2.14.** Let \((X, +, \theta, m, \mathcal{Q})\) be a uniform conoid.

a) \(m\) is \(C_{\ell,0}\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)\) if and only if \(m\) is \(UC_\ell\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)\).

b) \(m\) is \(C_{\ell,0}\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\) if and only if \(m\) is \(UC_\ell\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\).

c) \(m\) is \(C_{\ell,0}\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\) if and only if \(m\) is \(UC_\ell\) in 
\((\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)\).
Proof. The statements follow from Prop. 1.6(e). ♦

**Corollary 2.15.** Let \((X, +, \theta, m, Q)\) be a uniform conoid. If \(m\) is \(C_{\ell,0}\), then

a) \(m\) is \(UC_{\ell}\) in \((P_c(X), +, \{\theta\}, m, Q_c^-)\).

b) If \((X, Q)\) is a locally balanced, precompact quasi-uniform space, then:

i) \(m\) is \(UC_{\ell}\) in \((P_c(X), +, \{\theta\}, m, Q_c^+);\)

ii) \(m\) is \(UC_{\ell}\) in \((P_c(X), +, \{\theta\}, m, Q_c^*).\)

**Proof.** The statements follows from Props. 2.13 and 2.14. ♦

At last we study the joint continuity of the action \(m : P_c(X) \times \mathbb{R}_+ \to P_c(X)\).

**Proposition 2.16.** Let \((X, +, \theta, m)\) be a conoid and \(Q\) a quasi-uniformity on \(X\) for which \(m\) is \(JC(\theta, 0)\). Then \(m\) is \(JC(\{\theta\}, 0)\) in the conoids \((P_c(X), +, \{\theta\}, m, Q^{-*})\), \((P_c(X), +, \{\theta\}, m, Q_+^*)\) and \((P_c(X), +, \{\theta\}, m, Q^*).\)

**Proof.** Fix \(Q \in Q\). Since \(m\) is continuous at \((\theta, 0)\), there are \(P \in Q\) and \(\varepsilon > 0\) such that

\[P[\theta] \cdot t \subset Q[\theta], \quad \forall t \in [0, \varepsilon[.\]

Let \(B \subset P^{-*}[\{\theta\}].\) There is \(b \in B\) such that

\[(\theta, b) \in P \Rightarrow (\theta, b \cdot t) \in Q \Rightarrow \{\theta\} \subset T(Q)[b \cdot t], \forall t \in [0, \varepsilon[.\]

Thus

\[B \cdot t \subset Q^{-*}[\{\theta\}], \forall t \in [0, \varepsilon[.\]

The others cases are analogous. ♦

**Open questions 2.17.** Let \((X, +, m, Q)\) be a quasi-uniform conoid.

1) If \(m\) is \(C_r\) in \((X, +, m, Q)\) can we say that \(m\) is \(C_r\)

\[(P_c(X), +, m, Q_c^-), (P_c(X), +, m, Q_c^+)\) or \((P_c(X), +, m, Q_c^*)?\)

2) If \(m\) is \(JC\) in \((X, +, m, Q)\) can we say that \(m\) is \(JC\) in

\[(P_c(X), +, m, Q_c^-), (P_c(X), +, m, Q_c^+)\) or \((P_c(X), +, m, Q_c^*)?\)

**References**


