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UNIFORM TYPE HYPERSPACES

Teresa **Abreu**

Escola Superior de Gestão, Instituto Politécnico do Cávado e do Ave. Barcelos. Portugal

Eusebio **Corbacho**

Departamento de Matematica Aplicada 1, E. T. S. E. de Telecomunicacion, Universidad de Vigo, Vigo, Spain

Vaja **Tarieladze**

Niko Muskhelishvili Institute of Computational Mathematics, Tbilisi–0193, Georgia

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Abstract: If $(X, \mathcal{Q})$ is a quasi-uniform space, then in the hyperspace $\mathcal{P}_0(X)$ of all non-empty subsets of X we investigate the several quasi-uniformities related with the Bourbaki–Hausdorff quasi-uniformity ([5], [10], [11], [12]). We show that if $(X, \mathcal{Q})$ is a quasi-uniform monoid (conoid), then $\mathcal{P}_0(X)$ with respect to the corresponding algebraic operations and quasi-uniformities is again a quasi-uniform monoid (conoid). Moreover, it is demonstrated that if $(X, \mathcal{Q})$ is a quasi-uniform conoid, then in case of the hyperspace $\mathcal{P}_c(X)$ of all non-empty convex subsets of X the scalar multiplication on positive real numbers has some nice continuity properties.

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E-mail addresses: tabreu@ipca.pt, corbacho@uvigo.es, tar@gw.acnet.ge,
visit01@uvigo.es

1. Preliminary concepts

1.1. Uniform type spaces

$\mathbb{R}$ will denote the set of real numbers and $\mathbb{R}_+ := [0, \infty[$. The set $\mathbb{R}$ and its subsets (including $\mathbb{R}_+$ and the unit segment $[0,1]$) will be supposed to be endowed with the usual topology $\mathfrak{e}$.

For a topological space (X, τ) we denote by $\mathcal{N}_\tau(x)$ the collection of all neighborhoods of a point $x \in X$. For the considered topologies and topological spaces no separation axioms are required in advance.

Fix a non-empty set X, a subset of $X \times X$ is called *a (binary) relation* on X. The relations will be denoted by P, Q, R, etc. We write:

$$\Delta_X := \{(x,x) \in X \times X \mid x \in X\},$$

$$\top(P) := \{(y,x) \in X \times X \mid (x,y) \in P\},$$

$$P \circ Q := \{(x,y) \in X \times X \mid \exists z \in X \text{ such that } (x,z) \in Q, (z,y) \in P\}.$$

The relation $\top(P)$ is called *the converse relation* of P. Instead of $\top(P)$ the notation P^{-1} also is used. A relation P is called reflexive if $\Delta_X \subset P$ and symmetric if $\top(P) = P$.

For a collection $\mathcal{Q}$ of relations on X, we write $\mathcal{Q}^\top := \{\top(Q) \mid Q \in \in \mathcal{Q}\}$ and we say that $\mathcal{Q}$ is *symmetric* if $\mathcal{Q} = \mathcal{Q}^\top$. The relation $P \circ Q$ is called the *composition* of relations P and Q.

For $x \in X$ and $E \subset X$ we set $P[x] := \{y \in X \mid (x,y) \in P\}$ and $P[E] := \bigcup_{x \in E} P[x]$.

We recall the usual terminology from the theory of quasi-uniform spaces (see, e.g.,[6], [14], [13]):

A filter $\mathcal{Q}$ consisting of reflexive relations on X is a

- *Local Quasi-uniformity* if $\forall x \in X$, $\forall Q \in \mathcal{Q}$, $\exists P \in \mathcal{Q}$ such that $P \circ P[x] \subset Q[x]$.
- *Local Uniformity* if $\mathcal{Q}$ is a symmetric local quasi-uniformity.
- *Quasi-Uniformity* if $\forall Q \in \mathcal{Q} \quad \exists P \in \mathcal{Q}$ such that $P \circ P \subset Q$.
- *Uniformity* if $\mathcal{Q}$ is a symmetric quasi-uniformity.

If $\mathcal{Q}$ is a quasi-uniformity, the filter $\mathcal{Q}^\top$ is a quasi-uniformity too. However, if $\mathcal{Q}$ is a local quasi-uniformity, then $\mathcal{Q}^\top$ may not be a local quasi-uniformity. A local quasi-uniformity $\mathcal{Q}$ is called *bilocal quasi-uniformity* if $\mathcal{Q}^\top$ is a local quasi-uniformity as well (cf. [2]).

The pair $(X, \mathcal{Q})$ is called a local quasi-uniform space (a local uniform space, a quasi-uniform space, a uniform space) when $\mathcal{Q}$ is a local quasi-uniformity (a local uniformity, a quasi-uniformity, a uniformity) and the members of $\mathcal{Q}$ are called entourages.[1]

Every uniform type structure $\mathcal{Q}$ induces in X the topology $\tau_{\mathcal{Q}}$ for which

$$\{Q[x] \mid Q \in \mathcal{Q}\} = \mathcal{N}_{\tau_{\mathcal{Q}}}(x), \ \forall x \in X.$$

For a quasi-uniformity $\mathcal{Q}$ the topologies $\tau_{\mathcal{Q}}$ and $\tau_{\mathcal{Q}^\top}$ may be distinct.

A uniform type structure $\mathcal{Q}$ is called *compatible with a topology* τ *if* $\tau_{\mathcal{Q}} = \tau$.

We say that a (local) quasi-uniformity $\mathcal{Q}$ is

(1) *weakly locally symmetric* at $x \in X$ if for every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S[x] \subset Q[x]$;

(2) *weakly locally symmetric* or *point-symmetric* if $\mathcal{Q}$ is weakly locally symmetric at x for every $x \in X$;

(3) *locally symmetric* at $x \in X$ if for every $Q \in \mathcal{Q}$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S \circ S[x] \subset Q[x]$;

(4) *locally symmetric* if $Q \in \mathcal{Q}$ is locally symmetric at x for every $x \in X$.

Let X be a set and $(\mathcal{Q}_i)_{i \in I}$ be a non-empty family of uniform type structures in X. For this family, in the partially ordered set of all filters over $X \times X$, always exist the least upper bound $\vee_{i\in I}\mathcal{Q}_i$ and the greatest lower bound $\wedge_{i\in I}\mathcal{Q}_i$. They are uniform type structures of the same type of $\mathcal{Q}_i$ (see [1] or [3]). Moreover $\{\cap_{i\in J} Q_i \mid Q_i \in \mathcal{Q}_i, \ J \text{ finite} \subset I\}$ is a base of $\vee_{i\in I}\mathcal{Q}_i$.

For a given bilocal quasi-uniformity $\mathcal{Q}$ we denote $\mathcal{Q}^\vee = \mathcal{Q} \vee \mathcal{Q}^\top$ and $\mathcal{Q}_\wedge = \mathcal{Q} \wedge \mathcal{Q}^\top$. It is known that $\mathcal{Q}^\vee$ is the *coarsest local uniformity* containing $\mathcal{Q}$ and $\mathcal{Q}_\wedge$ is the *finest local uniformity* contained into $\mathcal{Q}$.

A local quasi-uniform space $(X, \mathcal{Q})$ is called *precompact* if $\forall Q \in \mathcal{Q}$ $\exists F$ finite $\subset X$ such that $X = Q[F]$.

If $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are local quasi-uniform spaces and $\mathcal{F} \subset Y^X$ is a non-empty family of mappings, then $\mathcal{F}$ is called $(\mathcal{P}, \mathcal{Q})$-*uniformly equicontinuous* if

$$\forall Q \in \mathcal{Q}, \ \exists P \in \mathcal{P} \text{ such that } (f(x_1), f(x_2)) \in Q, \forall (x_1, x_2) \in P, \ \ \forall f \in \mathcal{F}.$$

Proposition 1.1. *Let X and Y be nonempty sets, $\mathcal{F} \subset Y^X$ a nonempty family of mappings, $(\mathcal{P}_i)_{i\in I}$ a nonempty family of local quasi-uniformities on X and $(\mathcal{Q}_i)_{i\in I}$ a nonempty family of local quasi-uniformities on Y. Assume that $\forall i \in I$, $\mathcal{F}$ is $(\mathcal{P}_i, \mathcal{Q}_i)$-uniformly equicontinuous. Then:*

[1] Some authors use the term "vicinity" instead of entourage.

a) $\mathcal{F}$ *is* $(\vee_{i\in I}\mathcal{P}_i, \vee_{i\in I}\mathcal{Q}_i)$*-uniformly equicontinuous.*
b) $\mathcal{F}$ *is* $(\wedge_{i\in I}\mathcal{P}_i, \wedge_{i\in I}\mathcal{Q}_i)$*-uniformly equicontinuous.*

1.2. Uniform type semigroups and monoids

A semigroup is a pair $(X,+)$, where X is a non-empty set and $+ : X\times X \to X$ is an associative binary operation. *A monoid* is a triplet $(X,+,\theta)$, where $(X,+)$ is a semigroup which has the neutral element θ. If $(X,+)$ is a semigroup (monoid) in $X\times X$ we define a semigroup operation componentwise.

As usual, for non-empty subsets A, B of a semigroup $A+B$ will stand for their algebraic or Minkowski sum $\{a+b \,|\, a\in A,\ b\in B\}$.

A monoid (semigroup) X which is also a topological space is called a topological monoid if $+$ is continuous with respect to the product topology in $X\times X$ and the topology of X.

A monoid (semigroup) X equipped with a local quasi-uniformity (bilocal quasi-uniformity, quasi-uniformity, local uniformity, uniformity) $\mathcal{Q}$ is called *a local quasi-uniform (bilocal quasi-uniform, quasi-uniform, local uniform, uniformity) monoid (semigroup)* if $+$ is uniformly continuous with respect to the product quasi-uniformity $\mathcal{Q}\otimes\mathcal{Q}$ and $\mathcal{Q}$.

Lemma 1.2. *Let* $(X,+,\theta)$ *be a monoid,* $\mathcal{Q}$ *be a local quasi-uniformity.*

a) *The following statements are equivalent:*
  (i) $(X,\mathcal{Q})$ *is a local quasi-uniform monoid.*
  (ii) $\forall Q\in\mathcal{Q}\ \exists P\in\mathcal{Q}$ *such that* $P+P\subset Q$.

b) *If* $(X,\mathcal{Q})$ *is a bilocal quasi-uniform monoid, then* $(X,\mathcal{Q}^{\top})$ *also is.*

c) *If* $(X,\mathcal{Q})$ *is a (bilocal) quasi-uniform monoid, then* $(X,\mathcal{Q}^{\vee})$ *is a (local) uniform monoid.*

1.3. Uniform type conoids

A *conoid* is an Abelian monoid $(X,+,\theta)$ for which an external operation

$$m : X\times\mathbb{R}_+ \to X,\ m(x,\alpha) = x\cdot\alpha$$

is defined with the properties:

$A.1$ $\quad (x_1+x_2)\cdot\alpha = x_1\cdot\alpha + x_2\cdot\alpha \quad \forall x_1,x_2\in X, \quad \forall\alpha\in\mathbb{R}_+;$
$A.2$ $\quad (x\cdot\alpha_1)\cdot\alpha_2 = x\cdot(\alpha_1\cdot\alpha_2) \quad \forall x\in X, \quad \forall\alpha_1,\alpha_2\in\mathbb{R}_+;$
$A.3$ $\quad x\cdot(\alpha_1+\alpha_2) = x\cdot\alpha_1 + x\cdot\alpha_2 \quad \forall x\in X, \quad \forall\alpha_1,\alpha_2\in\mathbb{R}_+;$
$A.4$ $\quad x\cdot 1 = x \quad \forall x\in X.$

In the literature a conoid is also called an *abstract convex cone* [16], a *cone* [9], a *semi-vector space* [15], or a *semilinear space* [7], [8], [17], etc. In [1] the conoids were introduced to develop a integration scheme in quasi-uniform spaces, these structures also have been studies in [4].

If $(X, +, \theta, m)$ is a conoid then in $X \times X$ we define a conoid structure componentwise.

Let $(X, +, \theta, m)$ be a conoid, K be a non-empty subset of X, $\alpha \in \mathbb{R}_+$ and A non-empty subset of $\mathbb{R}_+$. We write

$$K \cdot \alpha := \{x \cdot \alpha — \ x \in K\} \quad \text{and} \quad K \cdot A := \{x \cdot \alpha — x \in K,\, \alpha \in A\}.$$

Let $(X, +, \theta, m)$ be a conoid, K be a subset of X and b be an element of X. K is called:

(1) *Convex* if either K is empty, or $K \cdot \alpha + K \cdot (1 - \alpha) \subset K$, for every $\alpha \in [0, 1]$.

(2) *Balanced* if either K is empty, or $K \cdot [0, 1] \subset K$.

Remark 1.3. Let $(X, +, \theta, m)$ be a conoid.

(1) X itself is convex, balanced and radial.

(2) If K is a non-empty convex subset of X, then $K \cdot (\alpha + \beta) = = K \cdot \alpha + K \cdot \beta$, $\alpha, \beta \in \mathbb{R}_+$.

(3) The intersection of any non-empty family of convex (balanced) subsets of a conoid is convex (balanced).

As usual, we denote $co(K)$ the convex hull of a subset $K \subset X$.

Definition 1.4. A conoid $(X, +, \theta, m)$ equipped with a local quasi-uniformity (bilocal quasi-uniformity, quasi-uniformity, local uniformity, uniformity) $\mathcal{Q}$ is called a *local quasi-uniform* (*bilocal quasi-uniform*, *quasi-uniform*, *local uniform*, *uniform*) *conoid* if $(X, +, \theta, \mathcal{Q})$ is a local quasi-uniform monoid. It is denoted by $(X, +, \theta, m, \mathcal{Q})$.

Therefore a local quasi-uniform conoid is simply a local quasi-uniform monoid which algebraically is a conoid.

We shall say that a local quasi-uniform conoid $(X, +, \theta, m, \mathcal{Q})$ is

- *locally convex* if $\mathcal{Q}$ admits a base consisting of convex entourages;
- *locally balanced* if $\mathcal{Q}$ admits a base consisting of balanced entourages.

Remark 1.5. Let $(X, +, \theta, m, \mathcal{Q})$ a bilocal quasi-uniform conoid.

(1) $(X, +, \theta, m, \mathcal{Q}^{\top})$ is a bilocal quasi-uniform conoid (see 1.2).

(2) $(X, +, \theta, m, \mathcal{Q}_{\wedge})$, $(X, +, \theta, m, \mathcal{Q}^{\vee})$ are local uniform conoids (see 1.2).

For every $x \in X$, and for every $\alpha \in \mathbb{R}_+$ we will consider the mappings

$$m_x: \begin{array}{ccc} \mathbb{R}_+ & \to & X \\ \alpha & \mapsto & x\cdot\alpha \end{array} \qquad \text{and} \qquad m_\alpha: \begin{array}{ccc} X & \to & X \\ x & \mapsto & x\cdot\alpha. \end{array}$$

Denoting by $\mathcal{E}_+$ the usual uniformity on $\mathbb{R}_+$, we say that the external operation of a local quasi-uniform conoid $(X,+,\theta,m,\mathcal{Q})$ is

- UC_ℓ if m_x is $(\mathcal{E}_+,\mathcal{Q})$-uniformly continuous $\forall x\in X$;
- UC_r if m_α is $\mathcal{Q}$-uniformly continuous $\forall \alpha\in\mathbb{R}_+$;
- $C_{\ell,0}$ if m_x is $(\mathfrak{e},\tau_{\mathcal{Q}})$-continuous at 0 $\forall x\in X$;
- C_ℓ if m_x is $(\mathfrak{e},\tau_{\mathcal{Q}})$-continuous on $\mathbb{R}_+$ $\forall x\in X$;
- $C_{r,\theta}$ if m_α is $\tau_{\mathcal{Q}}$-continuous at θ $\forall\alpha\in\mathbb{R}_+$;
- C_r if m_α is $\tau_{\mathcal{Q}}$-continuous on X $\forall\alpha\in\mathbb{R}_+$;
- $JC_{(\theta,0)}$ if m is $(\tau\otimes\mathfrak{e},\tau_{\mathcal{Q}})$-continuous at $(\theta,0)$;
- JC if m is $(\tau_{\mathcal{Q}}\otimes\mathfrak{e},\tau_{\mathcal{Q}})$-continuous everywhere.

Let $(X,+,\theta,m)$ be a conoid. A local quasi-uniformity $\mathcal{Q}$ on X is called *homogeneous* if

$$Q\cdot\alpha\in\mathcal{Q} \qquad \forall Q\in\mathcal{Q}, \quad \forall\alpha>0.$$

Proposition 1.6. *Let $(X,+,\theta,m,\mathcal{Q})$ be a bilocal quasi-uniform conoid such that m is $C_{\ell,0}$. The following statements are valid:*

a) *m_x is $(\mathfrak{e},\tau_{\mathcal{Q}})$-right-continuous $\forall x\in X$.*

b) *If $\mathcal{Q}^\top$ is weakly locally symmetric at θ, then m_x is $(\mathfrak{e},\tau_{\mathcal{Q}^\top})$-continuous at 0 $\forall x\in X$.*

c) *If m_x is $(\mathfrak{e},\tau_{\mathcal{Q}^\top})$- continuous at 0 $\forall x\in X$, then m is UC_ℓ.*

d) *If $\mathcal{Q}^\top$ is weakly locally symmetric at θ, then m is UC_ℓ.*

e) *If $\mathcal{Q}$ is a uniformity, then m is $C_{\ell,0}$ if and only if m is UC_ℓ.*

Proof. a) Fix $x\in X$ and $\alpha\in\mathbb{R}_+$, $\alpha>0$ and $Q\in\mathcal{Q}$. Since $+$ is $(\tau_{\mathcal{Q}}\otimes\tau_{\mathcal{Q}},\tau_{\mathcal{Q}})$-continuous at $(x\cdot\alpha,\theta)$ and $x\cdot\alpha=x\cdot\alpha+\theta$, there exists $R\in\mathcal{Q}$ such that $R[x\cdot\alpha]+R[\theta]\subset Q[x\cdot\alpha]$.

Since m_x is $(\mathfrak{e},\tau_{\mathcal{Q}})$-continuous at 0 there exists $\varepsilon>0$ such that $x\cdot t\in R[\theta] \quad \forall t\in[0,\varepsilon[$. Then:

$$x\cdot(\alpha+t)=x\cdot\alpha+x\cdot t\in R[x\cdot\alpha]+R[\theta]\subset Q[x\cdot\alpha] \quad \forall t\in[0,\varepsilon[$$

and the $(\mathfrak{e},\tau_{\mathcal{Q}})$-right-continuity of m_x at α is proved.

b) Obvious.

c) Fix $x\in X$. Since $(X,\mathcal{Q})$ is a bilocal quasi-uniform semigroup, there exists $R\in\mathcal{Q}$ such that $R+R\subset Q$. Since m_x is $(\mathfrak{e},\tau_{\mathcal{Q}}\vee\tau_{\mathcal{Q}^\top})$-continuous at 0 there exists $\varepsilon>0$ such that

$$m_x([0,\varepsilon[)\subset R[\theta]\cap T(R)[\theta],$$

i.e.,

$$(*) \qquad (\theta,x\cdot t)\in R \quad \text{and} \quad (x\cdot t,\theta)\in R,\ \forall t\in[0,\varepsilon[.$$

Take $\alpha, \beta \in \mathbb{R}_+$ with $|\alpha - \beta| < \varepsilon$ and let us show that $(x \cdot \alpha, x \cdot \beta) \in Q$.

If $\alpha < \beta$, then $\beta = \alpha + t$ with $t := \beta - \alpha \in [0, \varepsilon[$. This and $(*)$ imply:

$(x\cdot\alpha, x\cdot\beta) = (x\cdot\alpha+\theta, x\cdot\alpha+x\cdot t) = (x\cdot\alpha, x\cdot\alpha)+(\theta, x\cdot t) \in R+R \subset Q.$

If $\alpha > \beta$, then $\alpha = \beta + t$ with $t := \alpha - \beta \in [0, \varepsilon[$. This and $(*)$ imply:

$(x\cdot\alpha, x\cdot\beta) = (x\cdot t+x\cdot\beta, \theta+x\cdot\beta) = (x\cdot t, \theta)+(x\cdot\beta, x\cdot\beta) \in R+R \subset Q.$

Consequently,

$$\alpha, \beta \in \mathbb{R}^+,\ |\alpha - \beta| < \varepsilon \;\Gamma\!\!\Rightarrow (x \cdot \alpha, x \cdot \beta) \in Q$$

and so, m_x is $(\mathcal{E}^+, \mathcal{Q})$- uniformly continuous.

d) Follows from b) and c).

e) Follows from d). $\Diamond$

2. Uniform type hyperspaces

Let X be a nonempty set and $\mathcal{P}_0(X)$ be the collection of all nonempty subsets of X. For each relation Q on X, set

$$Q^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) \,|\, B \subset Q[A]\},$$
$$Q^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) \,|\, A \subset \top(Q)[B]\},$$
$$Q^* := Q^+ \cap Q^-.$$

Remark 2.1. Let P, Q be relations on X, then:

(1) $\top(Q^-) = (\top(Q))^+$ and $\top(Q^+) = (\top(Q))^-$.

(2) $(P \cup Q)^+ = P^+ \cup Q^+$.

(3) $(P \cup Q)^- = P^- \cup Q^-$.

(4) $(P \cap Q)^+ \subset P^+ \cap Q^+$.

(5) $(P \cap Q)^- \subset P^- \cap Q^-$.

(6) $(P \cap Q)^* \subset P^* \cap Q^*$.

For a local quasi-uniformity $\mathcal{Q}$ on X let

- $\mathcal{Q}^+$ be the filter generated by $\{Q^+ \,|\, Q \in \mathcal{Q}\}$,
- $\mathcal{Q}^-$ be the filter generated by $\{Q^- \,|\, Q \in \mathcal{Q}\}$,
- $\mathcal{Q}^* := \mathcal{Q}^+ \vee \mathcal{Q}^-$.

Remark 2.2. If $(X, \mathcal{Q})$ is a local quasi-uniform space, then

(1) $(\mathcal{Q}^-)^\top = (\mathcal{Q}^\top)^+$, and $(\mathcal{Q}^+)^\top = (\mathcal{Q}^\top)^-$;

(2) $(\mathcal{Q}^*)^\top = (\mathcal{Q}^\top)^*$.

Proposition 2.3. *Let $(X, \mathcal{Q})$ be a quasi-uniform space. The following statements are true:*

(a) (cf. [5, 10]) *$\mathcal{Q}^+$, $\mathcal{Q}^-$ and $\mathcal{Q}^*$ are quasi-uniformities.*

(b) *If $\mathcal{Q}$ is a uniformity, then $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are conjugate quasi-uniformities, and $\mathcal{Q}^*$ is a uniformity on $\mathcal{P}_0(X)$.*

Proof. (a) Fix $\mathfrak{P} \in \mathcal{Q}^+$. There exists $P \in \mathcal{Q}$ such that $P^+ \subset \mathfrak{P}$. Since $\mathcal{Q}$ is a quasi-uniformity there is $Q \in \mathcal{Q}$ such that $Q \circ Q \subset P$. Let us show that $Q^+ \circ Q^+ \subset P^+$:

Take $(A, B) \in Q^+ \circ Q^+$. There is a C such that $(A, C) \in Q^+$ and $(C, B) \in Q^+$. For each $b \in B$ there is $c \in C$ such that $(c, b) \in Q$ and there is $a \in A$ such that $(a, c) \in Q$. It follows that $(a, b) \in Q \circ Q \subset P$ and so, $b \in P[a] \subset P[A]$. Hence $B \subset P[A]$ and $(A, B) \in P^+$.

The other cases are analogous.

(b) Follows from Rem. 2.1(1). $\Diamond$

The quasi-uniformities $\mathcal{Q}^+$ and $\mathcal{Q}^-$ are called, respectively, the *upper* and *lower Hausdorff quasi-uniformities* on $\mathcal{P}_0(X)$ associated with $\mathcal{Q}$.

The quasi-uniformity $\mathcal{Q}^*$ is called *Hausdorff (or Bourbaki) quasi-uniformity* on $\mathcal{P}_0(X)$ associated with $\mathcal{Q}$.

The next proposition shows that an analogue of Prop. 2.3(a) is not true for bilocal quasi-uniformities.

Proposition 2.4. *Let $(X, \mathcal{Q})$ be a bilocal quasi-uniform space. Then:*

a) *$\mathcal{Q}^+$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.*

b) *$\mathcal{Q}^-$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.*

c) *$\mathcal{Q}^*$ may not be a local quasi-uniformity on $\mathcal{P}_0(X)$.*

Proof. Let $X = \{0, 1, \frac{1}{2}, \dots \frac{1}{n}, \dots\}$ and

$$Q_n = \Delta \cup \left\{ \left(0, \frac{1}{i}\right) : i \geq n \right\} \cup \left\{ \left(\frac{1}{i+1}, \frac{1}{i}\right) : i \geq n \right\}.$$

First we will see that $\mathcal{Q}_0 = \{Q_n : \ n \in \mathbb{N}\}$ is base of a bilocal quasi-uniformity $\mathcal{Q}$ (cf. [1]).

- $Q_{n+1} \subset Q_n$, for each $n \in \mathbb{N}$, therefore $\mathcal{Q}$ is a filter base on $X \times X$.
- $\Delta \subset Q_n$, for every $n \in \mathbb{N}$.
- Observe that:

$$Q_n[0] = \left\{0, \frac{1}{n}, \frac{1}{n+1} \dots \right\} \text{ and } Q_n \circ Q_n[0] = \left\{0, \frac{1}{n}, \frac{1}{n+1} \dots \right\}$$

hence $Q_n \circ Q_n[0] = Q_n[0], n = 1, 2, \dots$. Now, let $n \geq 1$ and $k \geq 1$, we have that:

$$Q_k \circ Q_k\left[\frac{1}{k}\right] = \left\{\frac{1}{k}\right\} \text{ hence } Q_k \circ Q_k\left[\frac{1}{k}\right] \subset Q_n\left[\frac{1}{k}\right].$$

- Notice that:

$$\top(Q_n) = \Delta \cup \left\{ \left(\frac{1}{i}, 0\right) : i \geq n \right\} \cup \left\{ \left(\frac{1}{i}, \frac{1}{i+1}\right) : i \geq n \right\}, \quad n = 1, 2, 3 \dots$$

Observe that: $-\top(Q_n)[0] = \{0\}$ and $\top(Q_n) \circ \top(Q_n)[0] = \{0\}$ hence $\top(Q_n) \circ \top(Q_n)[0] = \top(Q_n)[0]$, $n = 1, 2, \dots$ and for $n, k \in \mathbb{N}$ we have: $\top(Q_{k+1}) \circ \top(Q_{k+1}) \left[\frac{1}{k}\right] = \left\{\frac{1}{k}\right\}$ hence $\top(Q_{k+1}) \circ \top(Q_{k+1}) \left[\frac{1}{k}\right] \subset \top(Q_n) \left[\frac{1}{k}\right]$.

a) Let $A = \{\frac{1}{3}, \frac{1}{6}, \dots, \frac{1}{3n}, \dots\}$, let us see that

$$Q_m^+ \circ Q_m^+[A] \not\subset Q_1^+[A],\ \forall m \in \mathbb{N}$$

with

$$Q_1^+[A] = \mathcal{P}_0\left(\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{3n-1}, \frac{1}{3n}, \dots\right\}\right) \cup \emptyset.$$

We have:

$$\begin{cases} \left(A, \left\{\frac{1}{3m-1}\right\}\right) \in Q_m^+ \\ \left(\left\{\frac{1}{3m-1}\right\}, \left\{\frac{1}{3m-2}\right\}\right) \in Q_m^+ \end{cases}.$$

Therefore $\left\{\frac{1}{3m-2}\right\} \in Q_m^+ \circ Q_m^+[A]$ $\forall m \in \mathbb{N}$, but $\left\{\frac{1}{3m-2}\right\} \notin Q_1^+[A]$.

b) Let $A = \{\frac{1}{3}, \frac{1}{6}, \dots, \frac{1}{3n}, \dots\}$, let we us see that

$$Q_m^- \circ Q_m^-[A] \not\subset Q_1^-[A],\ \forall m \in \mathbb{N}$$

with

$$Q_1^-[A] = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{3n-1}, \frac{1}{3n}, \dots\right\}.$$

Consider

$$B_m = \left\{\frac{1}{3k} : 1 \le k < m\right\} \cup \left\{\frac{1}{3k-2} : k \ge m\right\}$$

and

$$C_m = \left\{\frac{1}{3k} : 1 \le k < m\right\} \cup \left\{\frac{1}{3m-1}, \frac{1}{3m+2}, \frac{1}{3m+5}, \dots\right\}.$$

We have

$$\begin{cases} (A, C_m) \in Q_m^- \\ (C_m, B_m) \in Q_m^- \end{cases},$$

hence $B_m \in Q_m^- \circ Q_m^-[A]$, $\forall m \in \mathbb{N}$, but $B_m \notin Q_1^-[A]$.

c) Let $A = \{\frac{1}{3}, \frac{1}{6}, \dots, \frac{1}{3n}, \dots\}$, then by a) and b) we have

$$Q_1^*[A] = (Q_1^+ \cap Q_1^-)[A] \subset Q_1^+[A] \cap Q_1^-[A] =$$
$$= \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{3n-1}, \frac{1}{3n}, \dots\right\}$$

Let C_m and B_m the sets defined in b). We have also

$$\begin{cases}(A, C_m) \in Q_m^+ \cap Q_m^- \\ (C_m, B_m) \in Q_m^+ \cap Q_m^-\end{cases},$$

hence $B_m \in (Q_m^+ \cap Q_m^-) \circ (Q_m^+ \cap Q_m^-)[A]$, $\forall m \in \mathbb{N}$, but $B_m \notin Q_1^*[A]$. $\Diamond$

Taking into account Rem. 2.1 it is easy to prove the following:

Proposition 2.5. *Let $\mathcal{Q}$ and $\mathcal{P}$ be quasi-uniformity on X. Then:*

(1) $\mathcal{P}^* \vee \mathcal{Q}^* \subset (\mathcal{P} \vee \mathcal{Q})^*$.

(2) $(\mathcal{P} \wedge \mathcal{Q})^* \subset \mathcal{P}^* \wedge \mathcal{Q}^*$.

(3) *If the set $\{P \cup Q \mid P \in \mathcal{P},\ Q \in \mathcal{Q}\}$ is a quasi-uniform base of $\mathcal{Q} \wedge \mathcal{P}$, then*

(a) $\{(P \cup Q)^+ \mid P^+ \in \mathcal{P}^+,\ Q^+ \in \mathcal{Q}^+\} = \{P^+ \cup Q^+ \mid P^+ \in \mathcal{P}^+, Q^+ \in \mathcal{Q}^+\}$ *and both are quasi-uniform bases. Consequently,* $\mathcal{Q}^+ \wedge \mathcal{P}^+ = (\mathcal{Q} \wedge \mathcal{P})^+$.

(b) $\{(P \cup Q)^- \mid P^- \in \mathcal{P}^-, Q^- \in \mathcal{Q}^-\} = \{P^- \cup Q^- \mid P^- \in \mathcal{P}^-, Q^- \in \mathcal{Q}^-\}$ *and both are quasi-uniform bases. Consequently,* $\mathcal{Q}^- \wedge \mathcal{P}^- = (\mathcal{Q} \wedge \mathcal{P})^-$.

(c) $\{(P \cup Q)^* \mid P^* \in \mathcal{P}^*, Q^* \in \mathcal{Q}^*\} = \{P^* \cup Q^* \mid P^* \in \mathcal{P}^*, Q^* \in \mathcal{Q}^*\}$ *are quasi-uniform bases and* $\mathcal{Q}^* \wedge \mathcal{P}^* = (\mathcal{Q} \wedge \mathcal{P})^*$.

(4) *In particular, we have*

(a) $(\mathcal{Q}^*)^{\vee} \subset (\mathcal{Q}^{\vee})^*$.

(b) $(\mathcal{Q}_{\wedge})^* \subset (\mathcal{Q}^*)_{\wedge}$.

(c) *When $\{\top(Q) \cup Q \mid Q \in \mathcal{Q}\}$ is base of $\mathcal{Q}_{\wedge}$ then* $(\mathcal{Q}^*)_{\wedge} = (\mathcal{Q}_{\wedge})^*$.

The following proposition shows that the local symmetry is preserved for singletons.

Proposition 2.6. *Let $(X, \mathcal{Q})$ be a weakly locally symmetric quasi-uniform space. Then:*

a) $(\mathcal{P}_0(X), \mathcal{Q}^-)$ *is weakly locally symmetric at* $\{x\}$, $\forall x \in X$;

b) $(\mathcal{P}_0(X), \mathcal{Q}^+)$ *is weakly locally symmetric at* $\{x\}$, $\forall x \in X$;

c) $(\mathcal{P}_0(X), \mathcal{Q}^*)$ *is weakly locally symmetric at* $\{x\}$, $\forall x \in X$.

Proof. a) Fix $\mathfrak{Q} \in \mathcal{Q}^-$. There exists $Q \in \mathcal{Q}$ such that $Q^- \subset \mathfrak{Q}$. For a $x \in X$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S[x] \subset Q[x]$.

Let $B \in S^-[\{x\}]$, then there is a $b \in B$ such that

$$(x, b) \in S \text{ hence } (x, b) \in Q.$$

Therefore $(\{x\}, B) \in Q^-$ and so $B \in Q^-[\{x\}]$.

b) Is analogous to a).

c) Follows from a) and b) because the supremum of a family of weakly locally symmetric quasi-uniformities is weakly locally symmetric. $\Diamond$

Proposition 2.7. *Let $(X, \mathcal{Q})$ be a locally symmetric quasi-uniform space. We have:*

a) $(\mathcal{P}_0(X), \mathcal{Q}^-)$ *is locally symmetric at* $\{x\}$, $\forall x \in X$.

b) $(\mathcal{P}_0(X), \mathcal{Q}^+)$ *is locally symmetric at* $\{x\}$, $\forall x \in X$.

c) $(\mathcal{P}_0(X), \mathcal{Q}^*)$ *is locally symmetric at* $\{x\}$, $\forall x \in X$.

Proof. a) Fix $\mathfrak{Q} \in \mathcal{Q}^-$. There exists $Q \in \mathcal{Q}$ such that $Q^- \subset \mathfrak{Q}$. For a $x \in X$ there is a symmetric entourage $S \in \mathcal{Q}$ such that $S \circ S[x] \subset Q[x]$.

Let $B \in S^- \circ S^-[\{x\}]$, then there is a $C \subset X$ such that

$$(\{x\}, C) \in S^- \text{ and } (C, B) \in S^-.$$

Then for each $c \in C$ there is a $b \in B$ such that

$$(x, c) \in S \text{ and } (c, b) \in S.$$

Hence, there is $b \in B$ such that $(x, b) \in S \circ S$ then $(x, b) \in Q$.

Therefore $(\{x\}, B) \in Q^-$ and so $B \in Q^-[\{x\}]$.

b) Is analogous to a).

c) Follows from a) and b) because the supremum of family of locally symmetric quasi-uniformities is weakly locally symmetric. $\Diamond$

2.1. Hyperspaces with algebraic structures

If $(X, +, \theta)$ is a monoid, then $\mathcal{P}_0(X)$ is a monoide as well with respect to the internal operation

$$\begin{array}{rccl} + : & \mathcal{P}_0(X) \times \mathcal{P}_0(X) & \to & \mathcal{P}_0(X) \\ & (A, B) & \mapsto & A + B \end{array}$$

and the neutral element $\{\theta\}$.

Theorem 2.8. *Let $(X, +, \theta, \mathcal{Q})$ be a quasi-uniform monoid, then $(\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^-)$, $(\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^+)$ and $(\mathcal{P}_0(X), +, \{\theta\}, \mathcal{Q}^*)$ are quasi-uniform monoids.*

Proof. Fix $\mathfrak{Q} \in \mathcal{Q}^+$. There exists $Q \in \mathcal{Q}$ such that $Q^+ \subset \mathfrak{Q}$. Since $+$ is uniformly continuous, there is a entourage P such that $P + P \subset Q$. Observe that:

- if $(A_1, B_1) \in Q^+$ then $B_1 \subset P[A_1]$;
- if $(A_2, B_2) \in Q^+$ then $B_2 \subset P[A_2]$.

Then

$$B_1 + B_2 \subset P[A_1] + P[A_2] \subset P[A_1 + A_2] \subset Q[B_1 + B_2].$$

Hence

$$P^+ + P^+ \subset Q^+.$$

In the same way it is easy to see that $+$ is also uniformly continuous with respect to $\mathcal{Q}^-$.

Since $+$ is uniformly continuous with respect $\mathcal{Q}^+$ and $\mathcal{Q}^-$, by Prop. 1.1 it is also uniformly continuous with respect to $\mathcal{Q}^*$. $\Diamond$

Let $(X,+,\theta,m)$ be a conoid. The external operation m can be extended to $\mathcal{P}_0(X)$ in a natural manner:

$$\begin{array}{rccc} m: & \mathcal{P}_0(X)\times\mathbb{R}_+ & \rightarrow & \mathcal{P}_0(X) \\ & (A,\alpha) & \mapsto & A\cdot\alpha \end{array}$$

The structure $(\mathcal{P}_0(X),+,\{\theta\},m)$ may not be a conoid, because, in general, property $A.3$ may fail.

Denote $\mathcal{P}_c(X)$ be the collection of all convex members of $\mathcal{P}_0(X)$. By Rem. 1.3(2) the structure $(\mathcal{P}_c(X),+,\{\theta\},m)$ is a *conoid.* This is an important example of conoid. Observe that, since $X+X=X$, this conoid is not cancellative provided $X\neq\{\theta\}$.

Let $\mathcal{Q}$ be a quasi-uniformity in a conoid $(X,+,\theta,m)$. We denote $\mathcal{Q}_c^+$, $\mathcal{Q}^-{}_c$ and $\mathcal{Q}_c^*$ the induced quasi-uniformities on $\mathcal{P}_c(X)$ by the quasi-uniformities $\mathcal{Q}^+$, $\mathcal{Q}^-$ and $\mathcal{Q}^*$.

The following result is a particular case of Th. 2.8.

Corollary 2.9. *Let $(X, +, \theta, m, \mathcal{Q})$ be a quasi-uniform conoid, then $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^-)$, $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^+)$ and $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^*)$ are quasi-uniform conoids.*

Proposition 2.10. *Let $(X,+,\theta,m)$ be a conoid, and $\mathcal{Q}$ be a quasi-uniformity on X.*

a) *If $\mathcal{Q}$ is locally convex, then $\mathcal{Q}_c^-$, $\mathcal{Q}_c^+$ and $\mathcal{Q}_c^+$ are locally convex.*

b) *If $\mathcal{Q}$ is locally balanced, then $\mathcal{Q}_c^-$, $\mathcal{Q}_c^+$ and $\mathcal{Q}_c^+$ are locally balanced.*

Proof. a) Fix $\mathfrak{P}\in\mathcal{Q}_c^+$. There exists a convex $P\in\mathcal{Q}$ such that $P^+\subset\mathfrak{P}$. Fix $(A_1,B_1),(A_2,B_2)\in P^+$, we have that $B_1\subset P[A_1]$ and $B_2\subset P[A_2]$.

For each $b_1\in B_1, b_2\in B_2$ there is a $a_1\in A_1, a_2\in A_2$ such that

$$(a_1,b_1)\in P \text{ and } (a_2,b_2)\in P,$$

since P is a convex entourage then

$$(a_1\cdot\alpha+a_2\cdot\beta, b_1\cdot\alpha+b_2\cdot\beta)\in P \text{ with } \alpha+\beta=1.$$

Therefore $b_1\cdot\alpha+b_2\cdot\beta\in P[a_1\cdot\alpha+a_2\cdot\beta]\Rightarrow B_1\cdot\alpha+B_2\cdot\beta\in P[A_1\cdot\alpha+A_2\cdot\beta]$. Then

$$(A_1,B_1)\cdot\alpha+(A_2,B_2)\cdot\beta\in P^+ \text{ with } \alpha+\beta=1.$$

In a similar way we can prove that the lower quasi-uniformity $\mathcal{Q}_c^-$, is locally convex too.

Since $\mathcal{Q}_c^* = \mathcal{Q}_c^+ \vee \mathcal{Q}_c^-$, then $\mathcal{Q}_c^*$ has also a base consisting of convex sets.

b) Now we will prove that if P is a balanced entourage then P^+ is also balanced. Let $(A,B) \in P^+$, then

$$B \subset P[A] \Rightarrow \ \forall b \in B\ \exists a \in A \text{ such that}$$
$$(a,b) \in P \Rightarrow (a \cdot t, b \cdot t) \in P,\ \forall t \in [0,1],$$

hence $B \cdot t \subset P[A \cdot t]$ with $t \in [0,1]$.

In a similar way we can prove that the lower quasi-uniformity is locally balanced too.

Since $\mathcal{Q}_c^* = \mathcal{Q}_c^+ \vee \mathcal{Q}_c^-$, then $\mathcal{Q}_c^*$ has also a base consisting of balanced sets. $\Diamond$

In the following propositions we study the stability of the partial continuity of the action on the hyperspace $\mathcal{P}_c(X)$.

We begin with the maps $m_\alpha : \mathcal{P}_c(X) \to \mathcal{P}_c(X)$.

Proposition 2.11. *Let $(X,+,\theta,m)$ be a conoid and $\mathcal{Q}$ be a quasi-uniformity for which m is $C_{r,\theta}$. Then m is $C_{r,\{\theta\}}$ in the conoids $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^-)$, $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^+)$ and $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^*)$.*

Proof. Fix $Q \in \mathcal{Q}$ and $\alpha \in \mathbb{R}_+$. Since m_α is $\tau_{\mathcal{Q}}$- continuous at θ, there is a $P \in \mathcal{Q}$ such that $P[\theta] \cdot \alpha \subset Q[\theta]$. Let $B \subset P^-[\{\theta\}]$, then there is $b \in B$ such that

$$(\theta, b) \in P \Rightarrow (\theta, b \cdot \alpha) \in Q \Rightarrow \{\theta\} \subset \top(Q)[b \cdot \alpha].$$

Thus $B \cdot \alpha \in Q^-[\{\theta\}]$.

In the same way we can prove that m_α is $\tau_{\mathcal{Q}_c^+}$- continuous at $\{\theta\}$, and using the previous results and Prop. 1.1 we can conclude that m is also $C_{r,\{\theta\}}$ in $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^*)$. $\Diamond$

Proposition 2.12. *Let $(X,+,\theta,m)$ be a conoid and $\mathcal{Q}$ be a quasi-uniformity for which m is UC_r. Then m is UC_r in the conoids $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^-)$, $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^+)$ and $(\mathcal{P}_c(X),+,\{\theta\},m,\mathcal{Q}_c^*)$.*

Proof. Fix $Q \in \mathcal{Q}$ and $\alpha \in \mathbb{R}_+$. Since m_α is $\mathcal{Q}$-uniformly continuous, there is a entourage P such that $P \cdot \alpha \subset Q$.

If $B \subset P[A]$ then for each $b \in B$ there is a $a \in A$ such that

$$(a,b) \in P \Rightarrow (a \cdot \alpha, b \cdot \alpha) \in Q \Rightarrow b \cdot \alpha \subset Q[a \cdot \alpha],$$

then

$$b \cdot \alpha \subset \bigcup_{a \in A} Q[a \cdot \alpha] = Q[A \cdot \alpha].$$

Hence $B \cdot \alpha \subset Q[A \cdot \alpha]$. Thus $P^+ \cdot \alpha \subset Q^+$.

The case $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$ is analogous, and using the previous results and Prop. 1.1, we can prove that m is UC_r in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$. $\diamondsuit$

Now we study the maps $m_A : \mathbb{R}_+ \to \mathcal{P}_c(X)$, $A \in \mathcal{P}_c(X)$.

Proposition 2.13. *Let $(X, +, \theta, m)$ be a conoid and $\mathcal{Q}$ a quasi-uniformity on X. If m is $C_{\ell,0}$ then*

a) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$.*

b) *If $(X, \mathcal{Q})$ is a locally balanced, precompact quasi-uniform space, then:*

i) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)$;*

ii) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$.*

Proof. a) Let A be a non-empty convex subset of X, and fix $Q \in \mathcal{Q}$. Let $x \in A$. As m_x is $\tau_{\mathcal{Q}}$-continuous at 0, there is $\varepsilon > 0$ such that $(\theta, x \cdot t) \in Q$, $\forall t \in [0, \varepsilon[$. Then

$$\{\theta\} \subset \top(Q)[A \cdot t],\ \forall t \in [0, \varepsilon[,$$

hence

$$A \cdot t \in Q^-[\{\theta\}],\ \forall t \in [0, \varepsilon[.$$

b) i) Let A be a convex subset of X. Fix $P \in \mathcal{Q}$. There is a balanced entourage Q such that $Q \circ Q \subset P$. Since $(X, \mathcal{Q})$ is precompact, there is a finite subset $F = \{x_1, x_2, \dots, x_n\} \subset X$ such that $A \subset \bigcup_{i=1}^n Q[x_i]$.

Since for $i \le n$ the map m_{x_i} is continuous, there is $\varepsilon_{x_i} \in]0, 1[$ such that

$$(\theta, x_i \cdot t) \in Q, \forall t \in [0, \varepsilon_{x_i}[.$$

Put $\varepsilon = \min\{\varepsilon_{x_i} \mid 1 \le i \le n\}$.

For all $x \in A$, there is $i \le n$ such that $(x_i, x) \in Q$. Since $\mathcal{Q}$ is balanced,

$$(x_i \cdot t, x \cdot t) \in Q, \forall t \in [0, \varepsilon] \subset [0, 1].$$

Since m_{x_i} is continuous, $(\theta, x_i \cdot t) \in Q, \forall t \in [0, \varepsilon] \subset [0, \varepsilon_{x_i}]$. Thus

$$\forall x \in A,\ \forall t \in [0, \varepsilon], \quad (\theta, x \cdot t) \in Q \circ Q \subset P,$$

and so, $A \cdot t \subset P[\{\theta\}]$ and $A \cdot t \in P^+[\{\theta\}]$.

ii) This item is a consequence of the last statements and Prop. 1.1. $\diamondsuit$

Proposition 2.14. *Let $(X, +, \theta, m, \mathcal{Q})$ be a uniform conoid.*

a) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$ if and only if m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$.*

b) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)$ if and only if m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)$.*

c) *m is $C_{\ell,0}$ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$ if and only if m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$.*

Proof. The statements follow from Prop. 1.6(e). ♢

Corollary 2.15. *Let $(X, +, \theta, m, \mathcal{Q})$ be a uniform conoid. If m is $C_{\ell,0}$, then*

a) *m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$.*

b) *If $(X, \mathcal{Q})$ is a locally balanced, precompact quasi-uniform space, then:*

i) *m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)$;*

ii) *m is UC_ℓ in $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$.*

Proof. The statements follows from Props. 2.13 and 2.14. ♢

At last we study the joint continuity of the action

$$m : \mathcal{P}_c(X) \times \mathbb{R}_+ \to \mathcal{P}_c(X).$$

Proposition 2.16. *Let $(X, +, \theta, m)$ be a conoid and $\mathcal{Q}$ a quasi-uniformity on X for which m is $JC_{(\theta,0)}$. Then m is $JC_{(\{\theta\},0)}$ in the conoids $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^-)$, $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^+)$ and $(\mathcal{P}_c(X), +, \{\theta\}, m, \mathcal{Q}_c^*)$.*

Proof. Fix $Q \in \mathcal{Q}$. Since m is continuous at $(\theta, 0)$, there are $P \in \mathcal{Q}$ and $\varepsilon > 0$ such that

$$P[\theta] \cdot t \subset Q[\theta], \quad \forall t \in [0, \varepsilon[.$$

Let $B \subset P^-[\{\theta\}]$. There is $b \in B$ such that

$$(\theta, b) \in P \Rightarrow (\theta, b \cdot t) \in Q \Rightarrow \{\theta\} \subset \top(Q)[b \cdot t], \ \forall t \in [0, \varepsilon[.$$

Thus

$$B \cdot t \subset Q^-[\{\theta\}], \ \forall t \in [0, \varepsilon[.$$

The others cases are analogous. ♢

Open questions 2.17. Let $(X, +, m, \mathcal{Q})$ be a quasi-uniform conoid.

(1) If m is C_r in $(X, +, m, \mathcal{Q})$ can we say that m is C_r $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^-)$, $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^+)$ or $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^*)$?

(2) If m is JC in $(X, +, m, \mathcal{Q})$ can we say that m is JC in $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^-)$, $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^+)$ or $(\mathcal{P}_c(X), +, m, \mathcal{Q}_c^*)$?

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