Mathematica Pannonica
19/1 (2008), 107–115

DISCRETE DYNAMIC OLIGOPOLIES WITH INTERTEMPORAL DEMAND INTERACTIONS

Carl **Chiarella**

School of Finance and Economics, University of Technology, Sydney, Broadway P O Box 123, NSW 2007, Australia

Ferenc **Szidarovszky**

Systems and Industrial Engineering Department, The University of Arizona, Tucson, AZ 85721-0020, USA

Received: April 2008

MSC 2000: 91 A 20, 91 B 26

Keywords: Oligopoly, stability, discrete systems.

Abstract: Dynamic oligopolies are examined with discrete time scales under the assumption that the demand at each time period is affected by earlier demands and consumptions. This intertemporal demand interaction is modeled by introducing an additional variable in the inverse demand function. First, the dynamic model is formulated and then the asymptotic stability of the equilibrium is examined. It is proved that the equilibrium is asymptotically stable if the adjustment speeds are sufficiently small.

1. Introduction

In the literature of dynamic oligopolies very few researchers have considered changes in market demands. It was always assumed that the market demand function (or the inverse demand function) remains the same during the entire examined time interval, and all produced items are sold during a single time period. In real economies these assumptions are not realistic, since market saturation, taste and habit formation, etc. make the demand of each time period dependent on earlier consumptions and demands.

A very simple model of oligopolies with saturated market is introduced and discussed in [1], and a more realistic two stage model is shown in [2]. Intertemporal demand interaction has been considered by several authors in analyzing international trade ([3, 4]). In this paper we will extend the two-stage model of [2] in constructing an N-firm dynamic oligopoly with intertemporal demand interaction.

After the mathematical model is formulated, the local asymptotic behavior of the equilibrium will be examined. We will prove that under realistic conditions the equilibrium is always locally asymptotically stable if the adjustment speeds are sufficiently small.

2. Mathematical model

A single-product N-firm oligopoly is considered without product differentiation and with intertemporal demand interaction. Let x_k be the output of firm k, and let C_k be its cost function. If L_k is the capacity limit of firm k, then the feasible set of firm k is the closed interval $[0, L_k]$. At each time period the demand function of the market depends on the industry output and also on an accumulated effect of earlier consumptions, which is assumed to be described by a real variable Q. Let $S_k = \sum_{l \neq k} x_l$ be the output of the rest of the industry from the point of view of firm k, then its profit is given as

$$\pi_k = x_k f(x_k + S_k, Q) - C_k(x_k). \tag{1}$$

As it is usual in the theory of oligopoly we make the following assumptions. Functions f and C_k $(1 \leq k \leq N)$ are twice continuously differentiable and

(A) $f'_x < 0$; (B) $f'_x + x_k f''_{xx} \leq 0$; (C) $f'_x - C''_k < 0$

for all k and feasible values of x_k, S_k, and Q.

Notice first that under these conditions π_k is strictly concave in x_k, and since the feasible set of x_k is compact, there is a unique maximizer of the payoff function of firm k with fixed values of S_k and Q, which is called the best response function of this firm:

(2) $R_k(S_k, Q) =$

$$= \begin{cases} 0 & \text{if } f(S_k, Q) - C'_k(0) \leq 0 \\ L_k & \text{if } L_k f'_x(L_k + S_k, Q) + f(L_k + S_k, Q) - C'_k(L_k) \geq 0 \\ x^*_k & \text{otherwise} \end{cases}$$

where x^*_k is the unique solution of the monotonic equation

(3) $$f(x_k + S_k, Q) + x_k f'_x(x_k + S_k, Q) - C'_k(x_k) = 0$$

in the interval $(0, L_k)$. The left-hand side of this equation is the derivative of π_k with respect to x_k. The derivatives of the best response functions can be obtained by implicit differentiation. If in the first or second case strict inequality holds, then both R'_{ks} and R'_{kQ} are equal to zero. In the third case, from (3) we have

$$f'_x \cdot (1 + R'_{ks}) + R'_{ks} \cdot f'_x + x_k \cdot f''_{xx} \cdot (1 + R'_{ks}) - C''_k \cdot R'_{ks} = 0$$

and

$$f'_x \cdot R'_{kQ} + f'_Q + R'_{kQ} \cdot f'_x + x_k f''_{xx} \cdot R'_{kQ} + x_k f''_{xQ} - C''_k \cdot R'_{kQ} = 0$$

implying that

(4) $$r_k \equiv R'_{ks} = -\frac{f'_x + x_k f''_{xx}}{2f'_x + x_k f''_{xx} - C''_k}$$

and

(5) $$\overline{r_k} \equiv R'_{kQ} = -\frac{f'_Q + x_k f''_{xQ}}{2f'_x + x_k f''_{xx} - C''_k}.$$

Assumptions (B) and (C) imply that

(6) $$-1 < r_k \leq 0$$

and if in addition we assume that

(D) $$f'_Q + x_k f''_{xQ} \leq 0$$

for all k and feasible values of x_k, S_k, and Q, then

(7) $$\overline{r}_k \leq 0.$$

Assume discrete time scales, $t = 0, 1, 2, \ldots$ and assume in addition that the firms adjust their output adaptively, then

(8) $$x_k(t+1) = x_k(t) + \alpha_k \left(R_k\left(S_k(t), Q(t)\right) - x_k(t)\right) \quad (1 \leq k \leq N)$$

where $S_k(t) = \sum_{l \neq k} x_l(t)$ and α_k is a sign preserving differentiable function:

$$\alpha_k(\Delta) \begin{cases} < 0 & \text{if } \Delta < 0 \\ = 0 & \text{if } \Delta = 0 \\ > 0 & \text{if } \Delta > 0 \end{cases}.$$

We also assume that parameter Q changes according to the first order law:

$$Q(t+1) = H\left(\sum_{k=1}^{N} x_k(t), Q(t)\right), \tag{9}$$

where H is a real valued function on $\left[0, \sum_{k=1}^{N} L_k\right] \times R$. We also assume that H is continuously differentiable.

System (8)–(9) is an $(N+1)$-dimensional discrete system. A constant vector $\left(\overline{x}_1, \ldots, \overline{x}_N, \overline{Q}\right)$ is an equilibrium of the system if and only if for all k,

$$\overline{x}_k = R_k\left(\sum_{l \neq k} \overline{x}_l, \overline{Q}\right) \quad \text{and} \quad \overline{Q} = H\left(\sum_{k=1}^{N} \overline{x}_k, \overline{Q}\right).$$

The state of system (8)–(9) at time period t is the vector

$$\left(x_1(t), \ldots, x_N(t), Q(t)\right).$$

The asymptotic behavior of the equilibrium will be examined in the next section.

3. Asymptotic stability of the equilibrium

If at any time period t the state of the system is at the equilibrium, then the state will remain there for all future times. We say that the equilibrium is asymptotically stable if the initial state is selected sufficiently close to the equilibrium, then the state converges to the equilibrium as t tends to infinity. In this section we will derive sufficient conditions for the asymptotical stability of the equilibrium based on linearization.

Assume that $\left(\overline{x}_1, \ldots, \overline{x}_N, \overline{Q}\right)$ is an equilibrium, which is not on the boundary between the cases (2). In this case r_k and $\overline{r}_k$ exist for all k at the equilibrium. The Jacobian of the system (8)–(9) at the equilibrium is a constant matrix which has the special structure

$$\mathbf{J} = \begin{pmatrix} 1-a_1 & a_1 r_1 & \ldots & a_1 r_1 & a_1 \overline{r}_1 \\ a_2 r_2 & 1-a_2 & \ldots & a_2 r_2 & a_2 \overline{r}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N r_N & a_N r_N & \ldots & 1-a_N & a_N \overline{r}_N \\ h & h & \ldots & h & \overline{h} \end{pmatrix}$$

where $a_k = \alpha_k'(0), h = H_x'$ and $\overline{h} = H_Q'$ at the equilibrium. We assume that for all k, $0 < a_k \leq 1$ in order to guarantee that α_k is sign preserving and to avoid overshooting in the dynamic process (8). The eigenvalue equation of $\mathbf{J}$ has the form

$$(1 - a_k)u_k + a_k r_k \sum_{l \neq k} u_l + a_k \overline{r}_k v = \lambda u_k \quad (1 \leq k \leq N) \tag{10}$$

$$h \sum_{k=1}^{N} u_k + \overline{h} v = \lambda v. \tag{11}$$

From (11) we have

$$hU = (\lambda - \overline{h})v, \tag{12}$$

where $U = \sum_{k=1}^{N} u_k$.

Assume first that $h = 0$. Then the eigenvalues of $\mathbf{J}$ are $\overline{h}$ and the eigenvalues of matrix

$$\begin{pmatrix} 1 - a_1 & a_1 r_1 & \dots & a_1 r_1 \\ a_2 r_2 & 1 - a_2 & \dots & a_2 r_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_N r_N & a_N r_N & \dots & 1 - a_N \end{pmatrix}$$

which is the Jacobian of discrete single-product oligopolies without intertemporal demand interaction. It is known (see for example [5]) that the eigenvalues are inside the unit circle if and only if

$$-1 < \overline{h} < 1, \tag{13}$$

$$a_k(1 + r_k) < 2 \tag{14}$$

for all k, and

$$\sum_{k=1}^{N} \frac{r_k a_k}{2 - a_k(1 + r_k)} > -1. \tag{15}$$

Notice that under conditions $a_k \in (0, 1]$ and (6), relation (14) always holds.

Assume next that $h \neq 0$. Then from (12) we have

$$U = \frac{\lambda - \overline{h}}{h} v, \tag{16}$$

and from (10),

$$a_k r_k U + a_k \overline{r}_k v = (\lambda - (1 - a_k(1 + r_k)))\, u_k.$$

Since $1 - a_k(1 + r_k)$ is inside the unit circle, we may assume that λ differs from this value. Then

$$u_k = \frac{a_k r_k U + a_k \overline{r}_k v}{\lambda - (1 - a_k(1 + r_k))} = \frac{a_k r_k(\lambda - \overline{h}) + a_k \overline{r}_k h}{[\lambda - (1 - a_k(1 + r_k))]h} v. \tag{17}$$

By adding this equation for $k = 1, 2, \ldots, N$ and using (16) again we have

$$\left(\sum_{k=1}^{N} \frac{a_k r_k(\lambda - \overline{h}) + a_k \overline{r}_k h}{\lambda - (1 - a_k(1 + r_k))} - (\lambda - \overline{h}) \right) v = 0.$$

If $v = 0$, then from (17), $u_k = 0$ for all k, which is impossible, since eigenvectors must be nonzero. Therefore $v \neq 0$, and we have the following equation:

$$\sum_{k=1}^{N} \frac{a_k r_k(\lambda - \overline{h}) + a_k \overline{r}_k h}{\lambda - (1 - a_k(1 + r_k))} = \lambda - \overline{h}. \tag{18}$$

The above derivation implies the following result.

Theorem 3.1. *The equilibrium is locally asymptotically stable, if all roots of equation* (18) *are inside the unit circle.*

Notice first that the left-hand side is a rational function of λ with derivative

$$\sum_{k=1}^{N} \frac{-a_k r_k(1 - a_k(1 + r_k)) + a_k(r_k \overline{h} - \overline{r}_k h)}{(\lambda - (1 - a_k(1 + r_k)))^2}. \tag{19}$$

The numerator of the general term of (19) is nonpositive, if

$$a_k r_k(1 + r_k) \leq r_k(1 - \overline{h}) + \overline{r}_k h.$$

Since the left-hand side is nonpositve, this is true for all a_k, if

$$r_k(1 - \overline{h}) + \overline{r}_k h \geq 0. \tag{20}$$

The first term is negative under condition (13), so the value of h has to be negative with sufficiently large absolute value. If (20) is violated, then the value of a_k must not be too small:

$$a_k \geq \frac{r_k(1 - \overline{h}) + \overline{r}_k h}{r_k(1 + r_k)}. \tag{21}$$

Let K be the set of the k values such that the numerator is nonzero. Then the poles of the left-hand side are the values $1 - a_k(1 + r_k)$ $(k \in K)$, which are inside the unit circle. Since (19) is nonpositive, there is a root before the smallest pole, one after the largest pole, and one between each pair of consecutive poles. Hence we have the following result.

Theorem 3.2. *All roots of equation* (18) *are real and they are between* -1 *and* $+1$ *if in addition to* (20) *or* (21),

$$\sum_{k=1}^{N} \frac{r_k(1-\overline{h}) + \overline{r}_k h}{1+r_k} < 1 - \overline{h}$$

and

$$\sum_{k=1}^{N} \frac{a_k r_k(-1-\overline{h}) + a_k \overline{r}_k h}{-2 + a_k(1+r_k)} > -1 - \overline{h}.$$

If the left-hand side of (18) is not strictly decreasing, then there is no guarantee that the roots are real. In this case no simple stability condition can be given in general. However if the firms are identical, then it is still possible. By assuming symmetric firms, equation (18) becomes

$$Nar(\lambda - \overline{h}) + Na\overline{r}h = \left(\lambda - \overline{h}\right)(\lambda - 1 + a(1+r)),$$

which can be written as

$$\lambda^2 + \lambda\left[-1 - \overline{h} + a(1 + (1-N)r)\right] + \\ + \left[\overline{h} - \overline{h}a(1 + (1-N)r) - Na\overline{r}h\right] = 0. \tag{22}$$

By introducing the notation $z = a(1 + (1-N)r) > 0$, the discriminant becomes

$$D = (-1 + \overline{h} + z)^2 + 4Na\overline{r}h,$$

which can have negative value if $h > 0$ and at least one of the quantities $h, |\overline{r}|$ and N is sufficiently large. Hence the apperance of complex roots is a possibility. All roots are inside the unit circle if and only of

$$\overline{h}(1-z) - Na\overline{r}h < 1,$$

$$-1 - \overline{h} + z + \overline{h}(1-z) - Na\overline{r}h + 1 > 0$$

and

$$1 + \overline{h} - z + \overline{h}(1-z) - Na\overline{r}h + 1 > 0.$$

These relations can be rewritten as follows:

$$\overline{h}z + aN\overline{r}h > \overline{h} - 1 \tag{23}$$

$$z(1-\overline{h}) - aN\overline{r}h > 0 \tag{24}$$

and

$$z(1+\overline{h}) + aN\overline{r}h < 2(1+\overline{h}). \tag{25}$$

We can summarize these conditions as follows:

Theorem 3.3. *The equilibrium is locally asymptotically stable if*

$$\overline{h}(1-z) - 1 < aN\overline{r}h < \min\left\{z(1-\overline{h}); (2-z)(1+\overline{h})\right\}. \tag{26}$$

This condition clearly has feasible solution if

$$\overline{h} - 1 < z < \overline{h} + 3.$$

In this case (26) holds if the value of $\overline{r}$ is bounded from both sides.

We can have an easy interpretation of these conditions in the saturation model introduced in [1]. In that case $h = 1$ and $\overline{h} = \alpha \in (0, 1)$.

Relation (23) can be rewritten as

$$a \cdot [\alpha(1 + (1 - N)r) + N\overline{r}] > a - 1,$$

where the right-hand side is negative. Therefore this holds if the multiplier of a is nonnegative. Otherwise it can be written as

$$a < \frac{\alpha - 1}{\alpha(1 + (1 - N)r) + N\overline{r}}$$

with positive right-hand side. So the value of a has to be sufficiently small. Inequality (24) has the form

$$a \cdot [(1 + (1 - N)r)(1 - \alpha) - N\overline{r}] > 0$$

which always hold, since the multiplier of a is positive. Relation (25) can be rewritten as

$$a \cdot [(1 + (1 - N)r)(1 + \alpha) + N\overline{r}] < 2(1 + \alpha),$$

where the right-hand side is positive. So this condition is clearly satisfied if the multiplier of a is nonpositive. Otherwise it can be rewritten as

$$a < \frac{2(1 + \alpha)}{(1 + (1 - N)r)(1 + \alpha) + N\overline{r}}.$$

In summary, if a is sufficiently small, then the equilibrium is locally asymptotically stable.

In the more general case, we cannot get analytic results, however in particular cases we can use computer methods to locate the eigenvalues and generate output trajectories starting from a large variety of initial states to see the asymptotic behavior of the equilibrium.

4. Conclusions

We have investigated nonlinear dynamic oligopolies with discrete time scales and intertemporal demand interaction. The effect of the consumptions at earlier time periods is modeled by a real valued parameter which was also a variable of the inverse demand function. It was also assumed that this additional variable followed a simple dynamic rule.

The driving dynamic equations were first introduced, and then the local asymptotic stability of the equilibrium was examined. General stability conditions were derived, and the special case of symmetric firm was analysed. We have also demonstrated that the general conditions can be easily interpreted in the case of a special saturation model, when the stability conditions are satisfied if the adjustment speeds of the firms are sufficiently small.

References

[1] OKUGUCHI, K. and SZIDAROVSZKY, F.: The Theory of Oligopoly with Multi-product Firms (2nd Edition), Springer-Verlag, Berlin/New York, 1999.

[2] OKUGUCHI, K. and SZIDAROVSZKY, F.: Oligopoly with Intertemporal Demand Interaction, *Journal of Economic Research* **8** (2003), 51–61.

[3] FROOT, K. and KLEMPERER, P.: Exchange Rate Pass-through when Market Share Matters, *American Economic Review* **79** (1989), 637–653.

[4] TIVIG, T.: Exchange Rate Pass-through in Two-period Duopoly, *International Journal of Industrial Organization* **14** (1996), 631–645.

[5] BISCHI, G.-I., CHIARELLA, C., KOPEL, M. and SZIDAROVSZKY, F.: Nonlinear Oligopolies: Stability and Bifurcation, Springer-Verlag, Berlin/New York (in press).