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ON ALMOST PSEUDO SYMMETRIES OF SASAKIAN MANIFOLDS

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Abstract: The object of the present paper is to reveal the condition of existence of almost pseudo symmetric, almost pseudo Ricci symmetric and almost pseudo ϕ-symmetric Sasakian manifolds.

1. Introduction

Let M^n be a contact Riemannian manifold with a contact form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. Here the dimension n is odd. Then the manifold

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M^n is a *Sasakian manifold* if ξ is a Killing vector field and ϕ satisfies the condition

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X,$$

where ∇ denotes the Riemannian connection of g ([1], [6], [7]).

In a recent paper [5], the first and second authors introduced a type of non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ whose curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$(1.1)\quad \begin{aligned}(\nabla_X \tilde{R})(U,Y,Z,W) = {} & [A(X)+B(X)]\tilde{R}(U,Y,Z,W)+ \\ & +A(U)\tilde{R}(X,Y,Z,W)+A(Y)\tilde{R}(U,X,Z,W)+ \\ & +A(Z)\tilde{R}(U,Y,X,W)+A(W)\tilde{R}(U,Y,Z,X),\end{aligned}$$

where A, B are two non-zero 1-forms called the *associated* 1-*forms* defined by

$$(1.2)\qquad g(X,P) = A(X), \quad g(X,Q) = B(X),$$

for all vector fields X, Y, Z, U, W on M^n, ∇ denotes the operator of covariant differentiation with respect to the metric g, $\tilde{R}$ is defined by $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, where R is the curvature tensor of type $(1,3)$. Such a manifold was called *an almost pseudo symmetric manifold* and was denoted by $A(PS)_n$. If in particular $A = B$ in (1.1) then the manifold reduces to a pseudo symmetric manifold which is denoted by $(PS)_n$, introduced by M. C. Chaki [2]. In this connection we can mention the notion of weakly symmetric manifold introduced by Tamassy and Binh [9]. A non-flat Riemannian manifold of dimension > 2 is said to be weakly symmetric [9] if there exist 1-forms A, B, C, D and E, not simultaneously zero, such that the curvature tensor satisfies the condition

$$\begin{aligned}(\nabla_U \tilde{R})(X,Y,Z,W) = {} & A(U)\tilde{R}(X,Y,Z,W) + B(X)\tilde{R}(U,Y,Z,W) \\ & + C(Y)\tilde{R}(X,U,Z,W) + D(Z)\tilde{R}(X,Y,U,W) \\ & + E(W)\tilde{R}(X,Y,Z,U).\end{aligned}$$

It may be mentioned that almost pseudo symmetric manifold is not a particular case of a weakly symmetric manifolds. In a previous paper [5], it was proved the existence of an $A(PS)_n$ by the following:

Theorem. *Let* $(\mathbf{R}^4, g)$ *be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + (dx^4)^2,$$

$(i, j = 1, 2, 3, 4)$. *Then* $(\mathbf{R}^4, g)$ *is an* $A(PS)_4$ *with non-zero and non-constant scalar curvature.*

In a recent paper, M. C. Chaki and T. Kawaguchi [4] introduced another type of non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ whose Ricci tensor S of type $(0, 2)$ satisfies the condition

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(X, Y), \tag{1.3}$$

where A, B and ∇ have the meaning already stated. Such a manifold was called an *almost pseudo Ricci symmetric manifold* and an n-dimensional manifold of this kind was denoted by $A(PRS)_n$. If in particular $A = B$ then the manifold reduces to a pseudo Ricci symmetric manifold introduced by M. C. Chaki [3]. It can be mentioned that almost pseudo Ricci symmetric manifold is not a particular case of weakly Ricci symmetric manifold, introduced by Tamassy and Binh [10].

A Sasakian structure $M(\phi, \xi, \eta, g)$ is called *almost pseudo ϕ-symmetric* if there exist 1-forms A and B and a vector field P such that

$$\begin{aligned} \phi^2((\nabla_W R)(X, Y)Z) = {} & [A(W) + B(W)]R(X, Y)Z + \\ & + A(X)R(W, Y)Z + A(Y)R(X, W)Z + \\ & + A(Z)R(X, Y)W + g(R(X, Y)Z, W)P, \end{aligned} \tag{1.4}$$

where A and B is defined by (1.2). In this case if $A = B = 0$ and $P = 0$, almost pseudo ϕ-symmetry reduces to the ϕ-symmetry of T. Takahashi which is expressed by

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

though he requires this relation for vector fields W, X, Y, Z orthogonal to ξ only (see [13] p. 308 or, [11]). It is to be noted that almost pseudo ϕ-symmetry is not a particular case of weakly ϕ-symmetry of Tamassy and Binh [10].

In the present paper, the question whether an almost pseudo symmetric or, almost pseudo Ricci symmetric or, almost pseudo ϕ-symmetric manifold may be a Sasakian manifold is answered.

2. Preliminaries

Let R, S and r denote, respectively, the curvature tensor of type $(1, 3)$, the Ricci tensor of type (0,2) and the scalar curvature in a Sasakian manifold (M^n, g). It is known that in a Sasakian manifold M^n, the Riemannian metric may be chosen so that the following relations hold

([1], [7], [8]).

$$\phi(\xi)=0, \tag{2.1}$$

$$\eta(\xi)=1, \tag{2.2}$$

$$\phi^2X=-X+\eta(X)\xi, \tag{2.3}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.4}$$

$$g(\xi,X)=\eta(X), \tag{2.5}$$

$$\nabla_X\xi=-\phi X, \tag{2.6}$$

$$\eta(\phi X)=0, \tag{2.7}$$

$$S(X,\xi)=(n-1)\eta(X), \tag{2.8}$$

$$g(R(\xi,X)Y,\xi)=g(X,Y)-\eta(X)\eta(Y), \tag{2.9}$$

$$R(\xi,X)\xi=-X+\eta(X)\xi, \tag{2.10}$$

$$g(R(X,Y)\xi,Z)=g(X,Z)\eta(Y)-g(Y,Z)\eta(X) \tag{2.11}$$

and

$$(\nabla_X\phi)(Y)=R(\xi,X)Y \tag{2.12}$$

for any vector fields X,Y,Z on M^n.

The above results will be used in the next sections.

3. Sasakian $A(PS)_n$ $(n\geq 3)$

In this section, we suppose that an n-dimensional $A(PS)_n$ $(n\geq 3)$ is a Sasakian manifold.

Putting $U=W=\{e_i\}$, where $\{e_i\}$ $(i=1,2,\dots,n)$, is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i, 1\leq i\leq n$, we get from (1.1),

$$\begin{aligned}(\nabla_XS)(Y,Z)&=[A(X)+B(X)]S(Y,Z)+A(Y)S(X,Z)+\\&+A(Z)S(X,Y)+A(R(X,Y)Z)+A(R(X,Z)Y).\end{aligned} \tag{3.1}$$

Now taking account of the fact that $(\nabla_Xg)(Z,\xi)=0$ and using (2.5), (2.6) and (2.8) we get

$$(\nabla_XS)(\xi,Z)=-(n-1)g(Z,\phi X)+S(\phi X,Z). \tag{3.2}$$

In (3.1), we put $Y=\xi$ and in view of (3.2) and (2.8) we obtain

$$\begin{aligned}&-(n-1)g(Z,\phi X)+S(\phi X,Z)=\\&=(n-1)[A(X)+B(X)]\eta(Z)+A(\xi)S(X,Z)+\\&\quad+(n-1)A(Z)\eta(X)+A(R(X,\xi)Z)+A(R(X,Z)\xi).\end{aligned} \tag{3.3}$$

Putting $Z = \xi$ in (3.3) and taking into account (2.2), (2.6), (2.8) and (2.10) we have

$$(n+1)A(X) + (n-1)B(X) + 2(n-2)A(\xi)\eta(X) = 0. \tag{3.4}$$

Now since $n \geq 3$, if we put $X = \xi$ in (3.4), we get

$$3A(\xi) + B(\xi) = 0. \tag{3.5}$$

This implies that the vanishing of the 1-form $3A + B$ over the Killing vector field ξ of (M^n, g) is necessary in order that $M^n(\phi, \xi, \eta, g)$ be a Sasakian manifold. We now show that $3A + B = 0$ is also necessary for this.

Putting $X = \xi$ in (3.3) and taking into account (2.1), (2.2), (2.8) and (2.10) we have

$$(2n-1)A(\xi)\eta(Z) + (n-2)A(Z) + (n-1)B(\xi)\eta(Z) = 0. \tag{3.6}$$

Replacing Z by X in (3.6) and then adding (3.4), in view of (3.5), we get

$$(n-2)A(\xi)\eta(X) + (2n-1)A(X) + (n-1)B(X) = 0. \tag{3.7}$$

Finally, replacing Z by X in (3.6) and then adding (3.7), in view of (3.5), we obtain

$$(n-1)[3A(X) + B(X)] = 0, \tag{3.8}$$

for all X. This gives the following theorem:

Theorem 3.1. *There exists no almost pseudo symmetric Sasakian manifold if $3A + B$ is not everywhere zero.*

Thus our condition still allows the existence of a Sasakian structure on an almost pseudo symmetric manifold. If in particular $B = A$, then the manifold reduces to a $(PS)_n$ and from (3.8) we get $A = 0$ which is inadmissible by the definition of $(PS)_n$. Hence we have the following corollary:

Corollary 3.1. *There exists no proper pseudo symmetric Sasakian manifold.*

The above corollary has been proved by M. Tarafdar [12] in another way.

4. Sasakian $A(PRS)_n$ $(n \geq 3)$

In this section, we suppose that an n-dimensional $A(PRS)_n$ $(n \geq 3)$ is a Sasakian manifold. Now putting $Z = \xi$ in (1.3) we get

$$(\nabla_X S)(Y, \xi) = [A(X)+B(X)]S(Y, \xi)+A(Y)S(X, \xi)+A(\xi)S(X, Y). \tag{4.1}$$

In view of (3.2) [in deriving (3.2) we only used that the manifold is Sasakian and we did not exploit almost pseudo symmetry], and (2.8) the equation (4.1) can be written as

$$-(n-1)g(Y,\phi X)+S(\phi X,Y) = (n-1)[A(X)+B(X)]\eta(Y) + (n-1)A(Y)\eta(X)+A(\xi)S(X,Y). \tag{4.2}$$

Putting $X=\xi$ in (4.2) and using (2.1) we get

$$(n-1)[2A(\xi)\eta(Y)+A(Y)+B(\xi)\eta(Y)] = 0. \tag{4.3}$$

Next putting $Y=\xi$ in (4.3) we get

$$(n-1)[3A(\xi)+B(\xi)] = 0.$$

Now since $n \geq 3$ we get from above

$$3A(\xi)+B(\xi) = 0. \tag{4.4}$$

Again putting $Y=\xi$ in (4.2) and then using (2.5), (2.6) and (2.8) we get

$$(n-1)[A(X)+B(X)+2A(\xi)\eta(X)] = 0. \tag{4.5}$$

Now replacing Y by X in (4.3) and then adding with (4.5), in view of (4.4), we get

$$(n-1)[A(\xi)\eta(X)+2A(X)+B(X)] = 0. \tag{4.6}$$

Again replacing Y by X in (4.3) and then adding with (4.6), in view of (4.4), we obtain

$$(n-1)[3A(X)+B(X)] = 0, \tag{4.7}$$

for all X. Thus we have:

Theorem 4.1. *There exists no almost pseudo Ricci symmetric Sasakian manifold if $3A+B$ is not everywhere zero.*

If in particular $B=A$, then the manifold reduces to a $(PRS)_n$ and from (4.7) we get $A=0$ which is inadmissible by the definition of $(PRS)_n$. Hence we have the following corollary:

Corollary 4.1. *There exists no proper pseudo Ricci symmetric Sasakian manifold.*

The above corollary has already been proved by M. Tarafdar [12] in another way.

5. Almost pseudo ϕ-symmetric Sasakian manifolds

From (1.4) and (2.3) we have

$$\begin{aligned} &-(\nabla_X R)(W,Y)Z+\eta((\nabla_X R)(W,Y)Z)\xi = \\ &= [A(X)+B(X)]R(W,Y)Z+A(W)R(X,Y)Z+ \end{aligned} \tag{5.1}$$

$$+ A(Y)R(W,X)Z + A(Z)R(W,Y)X + g(R(W,Y)Z,X)P.$$

Putting on both sides of (5.1) $W = e_i$, multiplying by e_i and performing a summation over i, by using (2.5) and the symmetry and skew-symmetry properties of R we get

$$\begin{aligned} &-(\nabla_X S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_X R)(e_i,Y)Z)\eta(e_i) = \\ &= [A(X)+B(X)]S(Y,Z) + A(R(X,Y)Z) + \\ &\quad + A(Y)S(X,Z) + A(Z)S(Y,X) + A(R(X,Z)Y). \end{aligned} \tag{5.2}$$

If we put $Z = \xi$ in (5.2), then the second term of (5.2) vanishes (see the proof of Th. 6 in [10]). Replacing Z by ξ in (5.2) and using (2.8) and (3.2) and taking account of the second term is zero we have

$$\begin{aligned} (n-1)g(Y,\phi X) - S(\phi X,Y) = {} & (n-1)[A(X)+B(X)]\eta(Y) \\ & + A(R(X,Y)\xi) + (n-1)A(Y)\eta(X) + \\ & + A(\xi)S(Y,X) + A(R(X,\xi)Y). \end{aligned} \tag{5.3}$$

Now putting $Y = \xi$ in (5.3) and using (2.2), (2.8) and (2.10) we get (3.4) from which we obtain (3.5) and finally (3.8). Thus we have the following theorem:

Theorem 5.1. *There exists no almost pseudo ϕ-symmetric Sasakian manifold if $3A + B$ is not everywhere zero.*

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