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SOME REMARKS ON A THEOREM OF H. DABOUSSI

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Abstract: Several generalizations of a famous theorem of H. Daboussi are surveyed.

1. Notation

$\mathcal{A}$ = set of real valued additive arithmetical functions
$\mathcal{M}$ = set of complex valued multiplicative arithmetical functions
$\mathcal{M}_1 := \{f \in \mathcal{M} \mid \ |f(n)| \leq 1 \ (n \in \mathbb{N})\}$
$\mathcal{A}_q$ = set of q-additive functions
$\mathcal{M}_q$ = set of q-multiplicative functions
$e(x) := e^{2\pi i x}$
$\mathcal{L}^*$ = set of uniformly summable functions (introduced by K.-H. Indlekofer):
$f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{L}^*$ if

$$\lim_{y\to\infty} \sup_x \frac{1}{x} \sum_{\substack{|f(n)|\geq y \\ n\leq x}} |f(n)| = 0.$$

2. The theorem of Daboussi

H. Daboussi proved that for every irrational $\alpha(\in \mathbb{R})$, uniformly in $f \in \mathcal{M}_1$

$$\frac{1}{x}\sum_{n\le x} f(n)e(n\alpha) \to 0. \tag{2.1}$$

The proof is given in his paper [4] written jointly with H. Delange. Later Delange extended this result for $f \in \mathcal{L}^2$, i.e. for those $f \in \mathcal{M}$ for which

$$\frac{1}{x}\sum_{n\le x} |f(n)|^2 = O(1).$$

Indlekofer proved (2.1) for a wider class, namely for $f \in \mathcal{L}^*$.

Daboussi deduced his theorem by using the "large sieve inequality".

The speed of the convergence was treated by H. L. Montgomery and R. C. Voughan [18]. They proved that the left-hand side of (2.1) is less than a constant times of

$$\frac{x}{\log x} + \frac{x\log R}{\sqrt{R}},$$

where $2 \le r \le \sqrt{x}$, $|\alpha - \frac{r}{s}| \le \frac{R}{\Delta x}$, $R \le \Delta \le \frac{x}{R}$, $(r,s) = 1$. An immediate consequence of Daboussi's theorem is the following:

If α is an irrational number and $F \in \mathcal{A}$, then the sequence

$$\xi_n = \xi_n(F) = F(n) + \alpha n$$

is uniformly distributed mod 1, and even the discrepancy

$$D_N(\xi_1(F), \ldots, \xi_N(F))$$

tends to 0 uniformly as F runs over the class of additive functions.

Let $\mathcal{T}$ be the set of those $t : \mathbb{N} \to \mathbb{R}$ for which

$$\sup_{F\in\mathcal{A}} |D_N(\eta_1(F), \ldots, \eta_N(F)| \to 0,$$

where $\eta_n(F) = F(n) + t(n)$.

3. Generalization of Daboussi's theorem

I observed that (2.1) can be proved by using the Turán–Kubilius inequality instead of the large sieve inequality, and this method allows us to prove wide generalization of (2.1) ([13]).

Theorem 1. *Let $t : \mathbb{N} \to \mathbb{R}$. Let us assume that for every positive K there exists a finite set $\mathcal{P}_K$ of primes $p_1 < \ldots < p_R$ such that*

$$A_{\mathcal{P}_K} := \sum_{i=1}^{R} \frac{1}{p_i} > K, \tag{3.1}$$

and for the sequences

$$\eta_{i,j}(m) = t(p_i m) - t(p_j m)$$

the relation

$$\frac{1}{x} \sum_{m=1}^{[x]} e(\eta_{i,j}(m)) \to 0 \quad (x \to \infty) \tag{3.2}$$

holds, whenever $i \neq j$, $i, j \in \{1, \dots, R\}$. *Then there exists a sequence* ρ_x (> 0) *tending to zero such that*

$$\sup_{f \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{n<x} f(n) e(t(n)) \right| \leq \rho_x. \tag{3.3}$$

Theorem 2. *Let* $t : \mathbb{N} \to \mathbb{R}$, *and* $\mathcal{P}_K$ *be as in Th. 1. Assume that* $\eta_{i,j}(m)$ *are uniformly distributed modulo 1 for every* $i \neq j$, $i, j \in \{1, \dots, R\}$. *Then* $t \in \mathcal{T}$.

We give a proof of Th. 1. Let $c, c_1, c_2, \dots$ be absolute positive constants, $B, B_1, B_2, \dots$ be numbers, or functions which can be majorized by absolute constants. After fixing a K we put $\mathcal{P}_K = \mathcal{P}$, and

$$\omega_{\mathcal{P}}(n) = \sum_{\substack{p|n \\ p \in \mathcal{P}}} 1.$$

From the Turán–Kubilius inequality, we get immediately

$$\sum_{n \leq x} |\omega_{\mathcal{P}}(n) - A_{\mathcal{P}}| \leq c_1 x \sqrt{A_{\mathcal{P}}}. \tag{3.5}$$

Let

$$S(x) = S(x, f) = \sum_{n \leq x} f(n) e(t(n)) \tag{3.6}$$

$$H(x) = H(x, f) = \sum_{n \leq x} f(n) e(t(n)) \omega_{\mathcal{P}}(n). \tag{3.7}$$

From (3.5) we obtain that

$$|H(x) - A_{\mathcal{P}} S(x)| \le c_2 x \sqrt{A_{\mathcal{P}}}, \tag{3.8}$$

furthermore

$$H(x) = \sum_{\substack{pm \le x \\ p \in \mathcal{P}}} \sum f(pm) e(t(pm)). \tag{3.9}$$

For $(p, m) = 1$ we can write $f(pm) = f(p)f(m)$. The contribution of the pairs $p|m$ on the right-hand side of (3.9) can be majorized by $x \sum \frac{1}{P_i^2}$, consequently

$$H(x) = \sum_{m \le \frac{x}{p_1}} f(m) \sum_{p_i \le \frac{x}{m}} f(p_i) e(t(p_i m)) + B_1 x = \sum_{m \le \frac{x}{p_1}} f(m) \Sigma_m + B_1 x. \tag{3.10}$$

Since $(a + b)^2 \le 2(a^2 + b^2)$ for real a, b, by using the Cauchy-inequality, we get

$$|H(x)|^2 \le 2 \left\{ \left(\sum_{m \le \frac{x}{p_1}} |f(m)|^2 \right) \left(\sum_{m \le \frac{x}{p_1}} |\Sigma_m|^2 \right) \right\} + 2B_1^2 x^2 = 2UV + 2B_1^2 x^2. \tag{3.11}$$

We have $U \le x$. Furthermore,

$$V = \sum_{m \le \frac{x}{p_1}} \sum_{p_i, p_j \le \frac{x}{m}} f(p_i) \overline{f}(p_j) e(t(p_i m) - t(p_j m)). \tag{3.12}$$

The contribution of the terms $p_i = p_j$ on the right-hand side of (3.12) is

$$\sum_{i=1}^{R} \left[\frac{x}{p_i} \right] < x A_{\mathcal{P}}.$$

Consequently

$$V \le x A_{\mathcal{P}} + \sum_{\substack{p_i, p_j \in \mathcal{P} \\ i \ne j}} \left| \sum_{m \le \min\left(\frac{x}{p_i}, \frac{x}{p_j} \right)} e(\eta_{i,j}(m)) \right|. \tag{3.13}$$

Collecting our inequalities we get

$$\frac{|S(x)|^2 A_{\mathcal{P}}^2}{x^2} \le c_2 A_{\mathcal{P}} + \sum_{\substack{p_i, p_j \in \mathcal{P} \\ i \ne j}} \left| \frac{1}{x} \sum_{m \le \min\left(\frac{x}{p_i}, \frac{x}{p_j} \right)} e\left((t(p_i m) - t(p_j m) \right) \right|. \tag{3.14}$$

Let $B(x) = \sup_{f \in \mathcal{M}_1} |S(x,f)|$.

Since the right-hand side of (3.14) does not depend on f, therefore (3.14) holds for $B(x)$ instead of $S(x,f)$. Consequently

$$\limsup \left(\frac{B(x)}{x}\right)^2 \le \frac{c_3}{A_{\mathcal{P}}}. \tag{3.15}$$

Since $\mathcal{P} = \mathcal{P}_K$ can be chosen for an arbitrary K, and $A_{\mathcal{P}} > K$, therefore (3.15) equals to zero. The theorem is proved. $\diamondsuit$

Remarks. 1. Let $t(n) = \alpha_k n^k + \cdots + \alpha_1 n$ be a polynomial of n such that at least one of the coefficients $\alpha_1, \ldots, \alpha_k$ is irrational. Then the conditions of Ths. 1 and 2 hold.

2. If $t \in \mathcal{T}$, then $t(n)$ is uniformly distributed modulo one. The opposite assertion is not true. Let $\omega(n)$ be the number of prime divisors of n. It can be proved in several ways that $\alpha\omega(n)$ is uniformly distributed modulo 1 for every irrational α. Then $\alpha\omega(n) = u(n)$ can not be in $\mathcal{T}$, since for $F(n) = -\alpha\omega(n) \in \mathcal{A}$, $P(n) + u(n) = 0$ identically.

Th. 1 can be extended to functions of $f \in L^*$. See [10].

4. Generalization to q-multiplicative functions

It is clear that $e(\alpha n)$ is a q-multiplicative function of module 1.

In a paper written jointly with Indlekofer [11] we proved the following assertion:

Theorem 3. *Let $f \in L^*$, and $g \in \mathcal{M}_q$, $|g(n)| = 1$ $(n \in \mathbb{N})$. Assume that*

$$\limsup_x \frac{1}{x} \left|\sum_{n \le x} f(n) g(n)\right| > 0. \tag{4.1}$$

Then $g(n)$ can be written as $g(n) = e(\frac{r}{D}) h(n)$ with a suitable rational number $\frac{r}{D}$ and with a function $h \in \mathcal{M}_q$, $|h(n)| = 1$ $(n \in \mathbb{N})$ such that

$$\sum_{j=0}^{\infty} \sum_{c=0}^{q-1} Re(1 - h(cq^j)) < \infty. \tag{4.2}$$

If the Bohr–Fourier spectrum of f is empty, then

$$\frac{1}{x} \sum_{n \le x} f(n) g(n) \to 0$$

for each $g \in \mathcal{M}_q$, $|g(n)| = 1$ $(n \in \mathbb{N})$.

It is known from a theorem of Kim, that $\varphi \in \mathcal{A}_q$ is uniformly distributed modulo 1, if and only if either for every $k \in \mathbb{N}$ there exists such a j for which

$$\sum_{c=0}^{q-1} e(k\varphi(cq^j)) = 0,$$

or

$$\sum_{j=0}^{\infty}\sum_{c=0}^{q-1} ||\varphi(cq^j)||^2 = \infty.$$

Hence one can deduce that for $f \in \mathcal{A}_q$ the sequence $\varphi(nq^R)$ $(n \in \mathbb{N}_0)$ is uniformly distributed modulo 1 for every R, if and only if

$$\sum_{j=0}^{\infty}\sum_{c=0}^{q-1} ||\varphi(cq^j)||^2 = \infty.$$

From Th. 3 one gets easily:

Theorem 4. *Let* $\varphi \in \mathcal{A}_q$ *and* $\varphi(nq^R)$ $(n \in \mathbb{N}_0)$ *be uniformly distributed modulo* 1 *for every* $R \in \mathbb{N}_0$. *Then for each additive functions* $F(n)$, *the sequence* $F(n) + \varphi(nq^R)$ $(n \in \mathbb{N}_0)$ *is uniformly distributed modulo* 1 *for very* $R \in \mathbb{N}_0$.

5. The analogue of Daboussi's theorem for some special subsets of integers

Let $\mathcal{N}_k$ be the set of the integers the number of the prime powers of which is k. Let $N_k(x)$ be the size of $n \le x$, $n \in \mathcal{N}_k$. In our paper [11] we proved

Theorem 5. *Let* $0 < \delta(< 1)$ *be an arbitrary constant, and* α *be an irrational number. Then*

$$\lim_{x\to\infty} \sup_{\delta \le \frac{k}{\log\log x} < 2-\delta} \sup_{f\in\mathcal{M}_1} \frac{1}{N_k(x)} \left| \sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} f(m)e(m\alpha) \right| = 0.$$

The proof is similar to the proof of Th. 1. It depends on an important assertion due to Dupain, Hall, Tenenbaum [6], namely that

$$\sup_{\frac{k}{\log\log x} \le (2-\delta)} \frac{1}{N_k(x)} \left| \sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} e(m\alpha) \right| \to 0 \text{ as } x \to \infty.$$

6. Some other questions

Let

$$S(x, \alpha, X_p) := \sum_{\substack{p_1, p_2 < x \\ p_1 < p_2}} X_{p_1} X_{p_2} e(\alpha p_1 p_2)$$

where p_1, p_2 run over the set of primes,

$$\pi_2(x) = \sum_{\substack{p_1 p_2 < x \\ p_1 < p_2}} 1.$$

Conjecture 1. *If α is an irrational number, then*

$$\max_{|x_p| \leq 1} \frac{|S(x, \alpha, X_p)|}{\pi_2(x)} \to 0 \quad (x \to \infty). \tag{6.1}$$

In [14] I proved a weaker version of Conj. 1, namely

Condition$_\delta$. *Let α be an irrational number for which for all $x \geq x_0$ there exists a rational number $\frac{a}{q}$, $(a, q) = 1$, $x^{\frac{2}{3}+\delta} < q < x^{1-\delta}$, and for $\beta = \alpha - \frac{a}{q}$, $|\beta| \leq \frac{1}{q^2}$.*

Here δ is an arbitrary small positive number.

Theorem 6. *Let $\delta > 0$, and assume that Cond.$_\delta$ holds for α. Then* (6.1) *holds true.*

Huixue Lao [17] strengthened this theorem, proving that (6.1) holds true if the irrationality measure $\mu(\alpha)$ of α is finite.

Let $\mathcal{R}(\alpha)$ be the set of those positive real numbers μ for which

$$q^{\mu-1}||q\alpha|| > 1$$

for every q larger than a constant $\chi_0 = \chi_0(\mu)$. It is clear that $\mathcal{R}(\alpha)$ is a halfline. Then the irrationality measure of α is defined as

$$\mu(\alpha) = \inf_{\mu \in \mathcal{R}(\alpha)} \mu.$$

(If $\mathcal{R}(\alpha)$ is empty, then $\mu(\alpha)$ is defined to be $\mu(\alpha) = \infty$.) In our paper [12] written with K.-H. Indlekofer we proved the following assertion.

Let $\mathcal{M}_x = \{m_1 < m_2 < \ldots < m_t\}$ be a set of integers depending on the parameter x, and let

$$\nu(\mathcal{M}_x) := \sum_{j=1}^{t} \frac{1}{m_j}.$$

We shall assume that $m_t \leq x^{\delta x}$, where $\delta x \to 0$ $(x \to \infty)$. Let $\mathcal{P}$ be the whole set of primes,

$$\mathcal{B}_j = \left\{ m_j p | p \in \mathcal{P},\ \sqrt{x} \le p \le \frac{x}{m_j} \right\}, \quad \mathcal{H}_x = \bigcup_{j=1}^{t} \mathcal{B}_j,$$

$$\mathcal{B}_j(x) := \#(\mathcal{B}_j) = \pi\left(\frac{x}{m_j}\right) - \pi(\sqrt{x}) = (1+o_x(1))\frac{\pi(x)}{m_j}$$

uniformly in $j = 1, \dots, t$. Then

$$H(x) := \#(\mathcal{H}_x) = \sum_{j=1}^{t} \mathcal{B}_j(x) = (1+o_x(1))\nu(\mathcal{M}_x)\pi(x).$$

Let

$$S(x|\alpha) = \sum_{j=1}^{t} \sum_{m_j p \in \mathcal{B}_j} Y_{m_j} X_p e(m_j p \alpha),$$

where $|Y_{m_j}| \le 1$ $(j = 1, \dots, t)$, $|X_p| \le 1$ $(p \in \mathcal{P})$.

Theorem 7. *Assume that $\delta > 0$, Cond.$_\delta$ holds for α. Assume furthermore that $\nu(\mathcal{M}_x) \to \infty$ $(x \to \infty)$. Then*

$$\max_{\substack{Y_m,\, X_p \\ |Y_{m_j}| \le 1, |X_p| \le 1}} \frac{|S(x|\alpha)|}{H(x)} =: \Delta(x, \alpha) \to 0 \quad as \quad x \to \infty.$$

Theorem 8. *Assume that Cond.$_\delta$ holds for α. Let $\rho_x \downarrow 0$, $2 \le k \le$ $\le \rho_x \log\log x$. Let $\mathcal{P}_k = \{n | \omega(n) = k\}$, $P(n)$ be the largest prime factor of n, $\pi_k(x) = \#\{n \le x | n \in \mathcal{P}_k\}$. Let us write every $n \in \mathcal{P}_k$ in the form $n = mp$, $P(n) = p$. Assume that Y_m, X_p are defined for all $m \in \mathbb{N}$, $p \in \mathcal{P}$ which occur in the representation of $n = mp$, and let $|Y_m| \le 1$, $|X_p| \le 1$.*

Let

$$S_k(x|\alpha) := \sum_{\substack{mp \le x \\ \omega(mp) = k \\ p = P(mp)}} Y_m X_p e(mp\alpha).$$

Then

$$\lim_{x \to \infty} \max_{2 \le k \le \rho_x (\log\log x)} \sup_{Y_m,\, X_p} \frac{|S_k(x|\alpha)|}{\pi_k(x)} = 0.$$

Lao noted that Ths. 7 and 8 remain true under the weaker condition that α is of finite irrationality measure.

7. On the distribution of modulo 1 of the values of $F(n) + \alpha\sigma(n)$

In our paper [16] written jointly with J. M. De Koninck we proved the following assertion.

Theorem 9. *Let α be a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality*

$$||\alpha q|| > \frac{c}{q^\kappa}$$

holds for every positive integer q.

Let h be an integer valued multiplicative function such that $h(p) = Q(p)$ for every prime p, and $h(p^a) = \mathcal{O}(p^{ad})$ for some fixed number d for every prime p and every integer $a \geq 2$, where

$$Q(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$$

$$k \geq 1,\ a_k > 0,\ a_j \in \mathbb{Z}.$$

Then the function $t(n) = \alpha h(n)$ belongs to $\mathcal{T}$.

Remark. The above assertion is true for $t(n) = \sigma^k(n)$, $t(n) = \varphi^k(n)$, $(k = 1, 2, \ldots)$.

8. On an analogue of Daboussi's theorem related to the set of Gaussian integers

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, $\mathbb{Z}^*[i] = \mathbb{Z}[i] \setminus \{0\}$ be the multiplicative group of nonzero Gaussian integers.

Let χ be such an additive character on $\mathbb{Z}[i]$, for which $\chi(1) = e(A)$, $\chi(i) = e(B)$, and at least one of A and B is an irrational number.

Let W be the union of finitely many convex bounded domains in $\mathbb{C}$. In our paper [1] written jointly with N. L. Bassily and J. M. De Koninck we proved

Theorem 10. *Let $\mathcal{K}_1$ be the set g of multiplicative functions on $\mathbb{Z}^*[i]$ satisfying $|g(\alpha)| \leq 1 \quad (\alpha \in \mathbb{Z}^*[i])$. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \frac{1}{|xW|} \left| \sum_{\beta\in xW} g(\beta)\chi(\beta) \right| = 0.$$

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