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GENERALIZED HERONIAN MEANS

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Abstract: Two families of generalized Heronian means of several variables are introduced and studied. Emphasis is on Schur-concavity, superadditivity and concavity. Several inequalities involving means under discussion are obtained. In particular, the Ky Fan type inequalities are proven. The latter provide refinements of several known inequalities.

1. Introduction, notation, and definitions

Let a and b be positive numbers. Classical Heronian mean $\mathrm{He}(a,b)$ of a and b is defined as

$$\mathrm{He}(a,b)=\frac{a+b+\sqrt{ab}}{3}\,.$$

Clearly

$$\mathrm{He}(a,b)=\frac{2A+G}{3}\,, \tag{1.1}$$

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where $A = (a+b)/2$ and $G = \sqrt{ab}$ are the arithmetic and geometric means of a and b. Using (1.1) as a prototype, W. Janous [7] has introduced a one-parameter family of bivariate means denoted by $H_\omega(a,b)$ $(\omega \ge 0)$ and defined as

$$H_\omega(a,b) \equiv H_\omega = \frac{2A + \omega G}{2+\omega}\,. \tag{1.2}$$

It is easy to see that H_ω interpolates the inequality of arithmetic and geometric means:

$$G \le H_\omega \le A. \tag{1.3}$$

If numbers a and b are not equal, then equalities hold in (1.3) if either $\omega = \infty$ or $\omega = 0$. This mean has been studied extensively in the recent paper [6]. Therein the authors have introduced a generalization of H_ω to an arbitrary number of variables.

Motivated by the research reported in [6], we introduce two three-parameter families of multivariate means which include those defined in [6] as a special case.

For later use let us introduce more notation. The symbol $\mathbb{R}_>$ will stand for the positive semi-axis. In what follows we will always assume that a vector $x = (x_1, \ldots, x_n)$ is a member of $\mathbb{R}^n_>$ unless otherwise stated. Also, we will use a family of multivariate means $\{\phi_1(x), \phi_2(x), \ldots\}$ that are comparable means which satisfy

$$\phi_i(x) \ge \phi_j(x) \tag{1.4}$$

for all $1 \le i < j$. Also, let

$$\mu = \frac{n}{n+\omega}\,, \quad \nu = 1 - \mu\,, \tag{1.5}$$

where $\omega \ge 0$. Clearly μ and ν are nonnegative numbers whose sum is equal to 1.

We shall define now two families of the generalized Heronian means. Throughout the sequel they are denoted by $H_{i,j}$ and $\mathcal{H}_{i,j}$ $(1 \le i < j)$ and called the generalized Heronian means of the first and second kind, respectively. We define

$$H_{i,j}(\omega; x) = \mu\,\phi_i(x) + \nu\,\phi_j(x) \tag{1.6}$$

and

$$\mathcal{H}_{i,j}(\omega;x) = \phi_i(x)^{\mu}\phi_j(x)^{\nu}, \tag{1.7}$$

where the underlying means ϕ_i and ϕ_j satisfy the comparability condition (1.4) and the weights μ and ν are defined in (1.5). Since some of the results of this paper apply to means of both kinds, the symbol $\Lambda_{i,j}(\omega;x)$ will stand for $H_{i,j}(\omega;x)$ and $\mathcal{H}_{i,j}(\omega;x)$.

In what follows we shall employ standard notation for the unweighted arithmetic, geometric and harmonic means of several variables x:

$$A_n(x) \equiv A_n = \frac{1}{n}\sum_{i=1}^{n} x_i\,, \tag{1.8}$$

$$G_n(x) \equiv G_n = \prod_{i=1}^{n} x_i^{\frac{1}{n}}, \tag{1.9}$$

$$H_n(x) \equiv H_n = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}. \tag{1.10}$$

On several occasions we will assume that $\phi_1 = A_n$, $\phi_2 = G_n$ and $\phi_3 = H_n$. Thus

$$H_{1,2}(\omega;x) = \mu\, A_n(x) + \nu\, G_n(x), \quad \mathcal{H}_{1,2}(\omega;x) = A_n(x)^{\mu} G_n(x)^{\nu},$$
$$H_{1,3}(\omega;x) = \mu\, A_n(x) + \nu\, H_n(x), \quad \mathcal{H}_{1,3}(\omega;x) = A_n(x)^{\mu} H_n(x)^{\nu},$$

etc. Mean $H_{1,2}$ is studied in [6].

Another mean used in this paper is a special case of the multivariate logarithmic mean introduced in [10, (2.2)] and defined as

$$L_n(x) \equiv L_n = (n-1)!\int_{E_{n-1}} \prod_{k=1}^{n} x_k^{u_k}\, du, \tag{1.11}$$

where

$$E_{n-1} = \big\{(u_1,\ldots,u_{n-1}) : u_i \geq 0,\ 1 \leq i \leq n-1,\ u_1+\ldots+u_{n-1} \leq 1\big\}$$

is the Euclidean simplex, $u_n = 1-(u_1+\ldots+u_{n-1})$ and $du = du_1\ldots du_{n-1}$. If $x_i \neq x_j$ for all $i \neq j$, then

$$L_n(x) = (n-1)!\sum_{k=1}^{n}\left[x_k \Big/ \prod_{\substack{l=1\\ l\neq k}}^{n} \ln\left(\frac{x_k}{x_l}\right)\right]$$

(see [10, p. 899]). It has been shown in [10, Th. 1] that this mean interpolates the inequality of arithmetic and geometric means, i.e.,

$$G_n(x) \le L_n(x) \le A_n(x) \tag{1.12}$$

with equalities in (1.12) if and only if $x_1 = \ldots = x_n$.

This paper is devoted to the study of two families of means defined in (1.6) and (1.7) and is organized as follows. In Sec. 2 we give four lemmas which will be utilized in the remaining part of this paper. The next section deals with the properties and inequalities satisfied by the generalized Heronian means. We shall prove, among other things, that $\Lambda_{i,j}(\omega; x)$ is a logarithmically convex (log-convex) function of ω and is Schur-concave in its variables x. Superadditivity of $\Lambda_{i,j}(\omega; x)$ and variables x is also established. In Sec. 4 we shall prove several Ky Fan type inequalities involving means under discussion.

2. Lemmas

Let D denote an interval in $\mathbb{R}$ with nonempty interior.

Lemma 2.1. *Let $f : D \to \mathbb{R}_>$ be a log-convex function. If $u, v \in D$ $(u \le v)$ and if $\lambda > 0$ is such that $u + \lambda$, $v + \lambda \in D$, then*

$$\frac{f(u+\lambda)}{f(v+\lambda)} \le \frac{f(u)}{f(v)} \tag{2.1}$$

and

$$\frac{f(\alpha u)}{f(\alpha v)} \le \left[\frac{f(u)}{f(v)}\right]^{\alpha} \tag{2.2}$$

where the last inequality is valid provided $u \ge 0$ and $\alpha \ge 1$. The inequality (2.2) is reversed if $0 < \alpha \le 1$.

Proof. We shall prove the inequality (2.1) only because (2.2) is established in [11, Th. 2.1]. There is nothing to prove when $u = v$. Assume that $u < v$. Logarithmic convexity of f implies the inequality

$$\frac{\ln f(v) - \ln f(u)}{v - u} \le \frac{\ln f(v+\lambda) - \ln f(u+\lambda)}{(v+\lambda) - (u+\lambda)}\,.$$

Hence the assertion follows. $\Diamond$

A slight modification of the proof of the inequality (1) in [5, p. 26] gives the following result.

Lemma 2.2. *Let x_k and y_k $(1 \le k \le n)$ be positive numbers. If $\lambda_i \ge 0$ for $1 \le i \le n$ and if $\lambda_1 + \ldots + \lambda_n = 1$, then the inequality*

$$\prod_{k=1}^{n} (x_k + y_k)^{\lambda_k} \ge \prod_{k=1}^{n} x_k^{\lambda_k} + \prod_{k=1}^{n} y_k^{\lambda_k} \tag{2.3}$$

is valid with equality if $y_k = cx_k$ *where* $c > 0$ *or if* $x_k = x$ *and* $y_k = 1 - x$ $(0 < x < 1)$. *Inequality* (2.3) *with* $\lambda_k = 1/n$, $1 \le k \le n$ *is also called the Minkowski inequality* (*see* [5, p. 26]).

The remaining two lemmas will be utilized in Sec. 4 in proofs of the Ky Fan type inequalities for the means under discussion.

Lemma 2.3. *Let* a, a', b *and* b' *be positive numbers and let* $\alpha \ge 0$, $\beta \ge 0$ *with* $\alpha + \beta = 1$. *If*

$$\frac{b}{b'} \le \frac{a}{a'} \tag{2.4}$$

and

$$b' \le a', \tag{2.5}$$

then

$$\frac{b}{b'} \le \frac{a^\alpha b^\beta}{a'^\alpha b'^\beta} \le \frac{\alpha a + \beta b}{\alpha a' + \beta b'} \le \frac{a}{a'}\,. \tag{2.6}$$

Proof. The first inequality in (2.6) follows easily from (2.4). We have

$$\frac{b}{b'} = \left(\frac{b}{b'}\right)^\alpha \left(\frac{b}{b'}\right)^\beta \le \left(\frac{a}{a'}\right)^\alpha \left(\frac{b}{b'}\right)^\beta.$$

For the proof of the second inequality in (2.6) we use (2.4) and (2.5) to obtain

$$(ab' - a'b)(b' - a') \le 0$$

or what is the same as

$$a'^2 b + ab'^2 \le aa'b' + a'bb'.$$

Multiplying both sides by $\alpha\beta = \alpha(1-\alpha) = \beta(1-\beta)$ we obtain

$$\alpha\beta(a'^2 b + ab'^2) \le \alpha(1-\alpha)aa'b' + \beta(1-\beta)a'bb'$$

which also can be written as

$$(\alpha ab' + \beta a'b)(\alpha a' + \beta b') \le a'b'(\alpha a + \beta b).$$

Dividing both sides by $(\alpha a' + \beta b')a'b'$ we obtain

$$\alpha\frac{a}{a'} + \beta\frac{b}{b'} \le \frac{\alpha a + \beta b}{\alpha a' + \beta b'}\,.$$

Application of the inequality for the arithmetic and geometric means gives the desired result. In order to prove the third inequality in (2.6) we use the following one [14, (2.3)]

$$\frac{a+b}{a'+b'} \le \frac{a}{a'}$$

which is valid provided the numbers a, a', b and b' satisfy (2.4). To complete the proof we let above $a := \alpha a$, $a' := \alpha a'$, $b := \beta b$ and $b' := \beta b'$. $\Diamond$

The next result is known [14, Lemma 2.1].

Lemma 2.4. *Let a, a', b and b' be positive numbers.*

(i) *If $b \le a$ and $b/b' \le a/a' \le 1$ or if $a \le b$ and $1 \le b/b' \le a/a'$, then*

$$\frac{1}{b'} - \frac{1}{b} \le \frac{1}{a'} - \frac{1}{a}\,. \tag{2.7}$$

(ii) *If the numbers a, a', b and b' satisfy the inequalities* (2.4) *and* (2.5)*, then*

$$bb' \le aa'. \tag{2.8}$$

3. Properties of generalized Heronian means

In this section we shall often write $\Lambda_{i,j}$ or $\Lambda_{i,j}(\omega)$ instead of $\Lambda_{i,j}(\omega; x)$ ($\omega \ge 0$, $x \in \mathbb{R}^n_>$) when no confusion would arise. Similarly, we will write ϕ_i for $\phi_i(x)$ and ϕ_j for $\phi_j(x)$.

Proposition 3.1. *Let $1 \le i < j < k$. Then*

$$\phi_j \le \mathcal{H}_{i,j} \le H_{i,j} \le \phi_i \tag{3.1}$$

and

$$\Lambda_{j,k} \le \Lambda_{i,k} \le \Lambda_{i,j}\,. \tag{3.2}$$

Proof. Inequalities (3.1) follow immediately from (1.4)–(1.7) and the inequality of the weighted arithmetic and geometric means. For the proof of (3.2) it suffices to use (1.6), (1.7) and the assumption that $\phi_i \ge \phi_j \ge$ $\ge \phi_k$. $\Diamond$

For particular means ϕ_i and ϕ_j the second inequality in (3.1) can be refined. For instance, Seiffert's mean $P(x)$ ($x \in \mathbb{R}^2_>$) which is defined as

$$P(x) = \begin{cases} \dfrac{x_1 - x_2}{2 \arcsin \frac{x_1 - x_2}{x_1 + x_2}}\,, & x_1 \ne x_2 \\ x_1\,, & x_1 = x_2 \end{cases}$$

(see [16]) satisfies

$$\left[A^2(x) G(x)\right]^{\frac{1}{3}} \le P(x) \le \left[2A(x) + G(x)\right]/3$$

(see [13, (2.8) and (3.10)]). Using (1.7) and (1.6) we see that the first and third members in the last inequality are equal to $\mathcal{H}_{1,2}(1; x)$ and $H_{1,2}(1; x)$, respectively.

We shall utilize the following.

Proposition 3.2. *The function $\omega \to \Lambda_{i,j}(\omega)$ is nonincreasing and log-convex on the nonnegative semi-axis.*

Proof. Let $f(\omega) = H_{i,j}(\omega)$. Logarithmic differentiation together with the use of (1.6) and (1.5) gives

$$\frac{f'(\omega)}{f(\omega)} = -\frac{n(\phi_i - \phi_j)}{(n+\omega)(n\,\phi_i + \omega\,\phi_j)}\,. \tag{3.3}$$

Hence

$$\left[\frac{f'(\omega)}{f(\omega)}\right]' = \frac{n(\phi_i - \phi_j)[n\,\phi_i + \omega\,\phi_j + (n+\omega)\phi_j]}{\big[(n+\omega)(n\,\phi_i + \omega\,\phi_j)\big]^2}\,. \tag{3.4}$$

Making use of (1.4) we see that the right sides of (3.3) and (3.4) are nonpositive and nonnegative, respectively. Hence the assertion follows for the means $H_{i,j}$. Now let $f(\omega) = \mathcal{H}_{i,j}(\omega)$. Using (1.7) we obtain

$$\frac{f'(\omega)}{f(\omega)} = -\frac{n}{n+\omega}\ln\left(\frac{\phi_i}{\phi_j}\right) \le 0$$

and

$$\left[\frac{f'(\omega)}{f(\omega)}\right]' = \frac{2n}{(n+\omega)^3}\ln\left(\frac{\phi_i}{\phi_j}\right) \ge 0,$$

where the inequalities follow from (1.4). The proof is complete. $\Diamond$

We are in a position to prove the following.

Theorem 3.3. *Let $\alpha \ge 1$, $\lambda \ge 0$, $0 \le \omega_1 \le \omega_2$ and let $\omega \ge 0$. Then for $1 \le i < j$*

$$\frac{\Lambda_{i,j}(\alpha\omega_1 + \lambda)}{\Lambda_{i,j}(\alpha\omega_2 + \lambda)} \le \frac{\Lambda_{i,j}(\alpha\,\omega_1)}{\Lambda_{i,j}(\alpha\,\omega_2)} \le \left[\frac{\Lambda_{i,j}(\omega_1)}{\Lambda_{i,j}(\omega_2)}\right]^\alpha \tag{3.5}$$

and

$$\left[\frac{\Lambda_{i,j}(\omega)}{\phi_i}\right]^\alpha \le \frac{\Lambda_{i,j}(\alpha\,\omega)}{\phi_i}\,. \tag{3.6}$$

Proof. Let $f(\omega) = \Lambda_{i,j}(\omega)$. For the proof of the first inequality in (3.5) we use the inequality (2.1) with $u = \alpha\omega_1$ and $v = \alpha\omega_2$. The second inequality in (3.5) follows from (2.2) with $u = \omega_1$ and $v = \omega_2$. Inequality (3.6) follows from (2.2) by letting $u = 0$ and $v = \omega$. Taking into account that $f(0) = \Lambda_{i,j}(0) = \phi_i$ (see (1.6), (1.7), and (1.5)) we obtain the desired result. $\Diamond$

Corollary 3.4. *Let $\omega \geq \alpha \geq 0$. Then the function*

$$\omega \to \frac{\Lambda_{i,j}(\omega-\alpha)}{\Lambda_{i,j}(\omega)}$$

is nonincreasing on its domain.

Proof. In the first inequality in (3.5) put $\alpha\,\omega_1 := \omega - \alpha$ and $\alpha\,\omega_2 := \omega$ to obtain the assertion. $\diamondsuit$

A special case of the last result when $\Lambda_{i,j} = H_{1,2}$ and $\alpha = 1$ is obtained in [6, Th. 5.1, case (ii)].

We will now deal with Schur-concavity of the generalized Heronian means. For the reader's convenience let us recall the definition of the Schur-concave functions. Let D be an interval with nonempty interior and let $f : D^n \to \mathbb{R}$ $(n \geq 2)$. Function f is said to be Schur-concave on D^n if $f(x) \geq f(y)$ for all n-tuples x and y in D^n such that $x \prec y$. The relationship of majorization $\prec$ means that

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}\,,$$

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are nonincreasing rearrangements of x and y, respectively (see, e.g., [8]). A well-known result states that the function $f \in C^1(D^n)$ is Schur-concave if and only if f is symmetric in its variables and the inequality

$$(x_k - x_l)\left(\frac{\partial f(x)}{\partial x_k} - \frac{\partial f(x)}{\partial x_l}\right) \leq 0 \tag{3.7}$$

holds for all $1 \leq k,\ l \leq n$ (see [8]).

For later use let us introduce the operator

$$\Delta_{k,l} = (x_k - x_l)\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_l}\right). \tag{3.8}$$

Theorem 3.5. *If both means ϕ_i and ϕ_j are symmetric and Schur-concave, then the generalized Heronian mean $\Lambda_{i,j}$ is also Schur-concave.*

Proof. Let $\Lambda_{i,j} = H_{i,j}$. Using (1.6) we obtain

$$\Delta_{k,l}\Lambda_{i,j} = \mu(\Delta_{k,l}\phi_i) + \nu(\Delta_{k,l}\phi_j) \leq 0$$

where the last inequality holds true provided

$$\Delta_{k,l}\phi_i \le 0 \text{ and } \Delta_{k,l}\phi_j \le 0. \tag{3.9}$$

The assertion now follows. Let now $\Lambda_{i,j} = \mathcal{H}_{i,j}$. Application of (3.8) to (1.7) gives

$$\Delta_{k,l}\mathcal{H}_{i,j} = \phi_i^{\mu-1}\phi_j^{\nu-1}\big[(\mu\,\phi_j)\Delta_{k,l}\phi_i + (\nu\,\phi_i)\Delta_{k,l}\phi_j\big] \le 0,$$

where, again, the last inequality is holds true if the conditions (3.9) are satisfied. The proof is complete. $\Diamond$

Corollary 3.6. *The generalized Heronian means $\Lambda_{1,2}$, $\Lambda_{1,3}$, and $\Lambda_{2,3}$ are Schur-concave functions of their variables.*

Proof. Using (1.8)–(1.10) one easily obtains

$$\frac{\partial}{\partial x_k}A_n(x) = \frac{1}{n}\,,\ \ \frac{\partial}{\partial x_k}G_n(x) = \frac{G_n(x)}{nx_k}\,,\ \ \frac{\partial}{\partial x_k}H_n(x) = \frac{H_n^2(x)}{nx_k^2}\,.$$

This and (3.8) imply

$$\begin{aligned}
\Delta_{k,l}A_n(x) &= 0,\\
\Delta_{k,l}G_n(x) &= -\frac{(x_k - x_l)^2}{nx_kx_l}G_n(x) \le 0,\\
\Delta_{k,l}H_n(x) &= -\frac{(x_k + x_l)(x_k - x_l)^2}{n(x_kx_l)^2}H_n^2(x) \le 0.
\end{aligned}$$

The assertion now follows from Th. 3.5. $\Diamond$

We close this section with a result about superadditivity of the mean $\Lambda_{i,j}$. Now let D stand for a nonempty subset of $\mathbb{R}^n$ ($n \ge 1$). Recall that the function $f : D \to \mathbb{R}$ is said to be superadditive if

$$f(x+y) \ge f(x) + f(y) \tag{3.10}$$

holds for all $x, y, x+y \in D$.

An important result states that if f is homogeneous of degree 1 in its variables, i.e., if $f(\lambda x) = \lambda f(x)$ for $\lambda > 0$, then f is superadditive if and only if f is a concave function (see [15]).

We are in a position to prove the following.

Theorem 3.7. *If the means ϕ_i and ϕ_j are superadditive functions of their variables, then so is the generalized Heronian mean $\Lambda_{i,j}$.*

Proof. Superadditivity of the mean $H_{i,j}$ follows immediately from (2.3). For the proof of superadditivity of $\mathcal{H}_{i,j}$ we shall employ the inequality

$$(a+b)^\alpha(c+d)^\beta \ge a^\alpha c^\beta + b^\alpha d^\beta \tag{3.11}$$

($a, b, c, d > 0,\ \alpha, \beta \ge 0,\ \alpha + \beta = 1$) which follows (2.3). Using (1.7), (3.10) and (3.11) with $\alpha = \mu$ and $\beta = \nu$, we obtain

$$\begin{aligned}\mathcal{H}_{i,j}(\omega; x+y) &= \phi_i(x+y)^{\mu}\phi_j(x+y)^{\nu}\\ &\geq \big[\phi_i(x)+\phi_i(y)\big]^{\mu}\big[\phi_j(x)+\phi_j(y)\big]^{\nu}\\ &= \mathcal{H}_{i,j}(\omega; x)+\mathcal{H}_{i,j}(\omega; y)\end{aligned}$$

($x, y \in \mathbb{R}^n_{>}$). The proof is complete. ◊

Corollary 3.8. *The means $\Lambda_{1,2}$, $\Lambda_{1,3}$, and $\Lambda_{2,3}$ are superadditive and concave functions of their variables.*

Proof. It follows from (1.1), Lemma 2.2 and [15], respectively, that the means A_n, G_n and H_n are superadditive. Since the last three means are homogeneous of degree 1, the assertion follows from Th. 3.7. ◊

Application of Lemma 2.2 to (1.11) shows that the logarithmic mean L_n is superadditive. Also, it follows from (1.11) that this mean is homogeneous of degree 1. Thus the generalized Heronian mean $\Lambda_{i,j}$ where $(\phi_i, \phi_j) = (A_n, L_n)$ or (L_n, G_n) or (L_n, H_n) is superadditive and concave in its variables.

4. Ky Fan inequality and related inequalities

This section is devoted to the study of Ky Fan and Ky Fan type inequalities for the generalized Heronian means. To this end we will always assume that $x \in I^n$ where $I = (0, 1/2]$. Also, we will write x' for $1 - x = (1 - x_1, \ldots, 1 - x_n)$.

The following result is well known.

$$\frac{H_n(x)}{H_n(x')} \leq \frac{G_n(x)}{G_n(x')} \leq \frac{A_n(x)}{A_n(x')}, \tag{4.1}$$

where the first inequality in (4.1) was established by W.-L. Wang and P.-F. Wang in [17] while the second one is due to Ky Fan (see, e.g., [5, p. 5]). Many results about inequalities of the form (4.1) and related inequalities have been obtained by numerous researchers. The interested reader is referred to [1]–[4], [12, 14] and the references therein.

The first result of this section reads as follows.

Theorem 4.1. *Let $1 \leq i < j$. If the means ϕ_i and ϕ_j satisfy the Ky Fan inequality*

$$\frac{\phi_j(x)}{\phi_j(x')} \leq \frac{\phi_i(x)}{\phi_i(x')}\,, \tag{4.2}$$

then

$$\frac{\phi_j(x)}{\phi_j(x')} \le \frac{\mathcal{H}_{i,j}(\omega;x)}{\mathcal{H}_{i,j}(\omega;x')} \le \frac{H_{i,j}(\omega;x)}{H_{i,j}(\omega;x')} \le \frac{\phi_i(x)}{\phi_i(x')} \tag{4.3}$$

holds for every $\omega \ge 0$ *and the inequality*

$$\frac{\Lambda_{i,j}(\omega_2;x)}{\Lambda_{i,j}(\omega_2;x')} \le \frac{\Lambda_{i,j}(\omega_1;x)}{\Lambda_{i,j}(\omega_1;x')} \tag{4.4}$$

is valid provided $0 \le \omega_1 \le \omega_2$.

Proof. Inequalities (4.3) follow immediately from Lemma 2.3 with $a = \phi_i(x)$, $a' = \phi_i(x')$, $b = \phi_j(x)$, $b' = \phi_j(x')$, $\alpha = \mu$ and $\beta = \nu$, where μ and ν are defined in (1.5). For the proof of (4.4) when $\Lambda_{i,j} = H_{i,j}$ we let

$$f(\omega) = \frac{H_{i,j}(\omega;x)}{H_{i,j}(\omega;x')}\,.$$

Differentiating with respect to ω and using (1.6) and (1.5) we obtain

$$f'(\omega) = \frac{n\big[\phi_j(x)\phi_i(x') - \phi_i(x)\phi_j(x')\big]}{(n+\omega)^2\big[H_{i,j}(\omega;x')\big]^2}\,.$$

This in conjunction with (4.2) gives $f'[(\omega) \le 0$ for every $\omega \ge 0$. Thus the function $f(\omega)$ is nonincreasing and the assertion follows. Assume now that $\Lambda_{i,j} = \mathcal{H}_{i,j}$ and define

$$g(\omega) = \frac{\mathcal{H}_{i,j}(\omega;x)}{\mathcal{H}_{i,j}(\omega;x')}\,.$$

Logarithmic differentiation together with the use of (1.7) and (1.5) gives

$$\frac{g'(\omega)}{g(\omega)} = \frac{n}{(n+\omega)^2}\ln\left[\frac{\phi_i(x')\phi_j(x)}{\phi_i(x)\phi_j(x')}\right] \le 0,$$

where the last inequality is the consequence of (4.2). This shows that the function $g(\omega)$ is nonincreasing and (4.4) follows in the case under discussion. This completes the proof. $\Diamond$

Let $1 \le i < j < k$. The following inequalities

$$\Lambda_{j,k}(\omega;x) \le \Lambda_{i,k}(\omega;x) \le \Lambda_{i,j}(\omega;x) \tag{4.5}$$

($\omega \ge 0$, $x \in \mathbb{R}^n_>$) are immediate consequence of (1.4), (1.6), and (1.7). The Ky Fan analogue of (4.5) for the generalized Heronian means of the second kind is obtained in the following.

Theorem 4.2. *Let the means* ϕ_i, ϕ_j *and* ϕ_k ($1 \le i < j < k$) *satisfy the Ky Fan inequalities*

$$\frac{\phi_k(x)}{\phi_k(x')} \le \frac{\phi_j(x)}{\phi_j(x')} \le \frac{\phi_i(x)}{\phi_i(x')}\,. \tag{4.6}$$

Then for every $\omega \geq 0$

$$\frac{\mathcal{H}_{j,k}(\omega;x)}{\mathcal{H}_{j,k}(\omega;x')} \leq \frac{\mathcal{H}_{i,k}(\omega;x)}{\mathcal{H}_{i,k}(\omega;x')} \leq \frac{\mathcal{H}_{i,j}(\omega;x)}{\mathcal{H}_{i,j}(\omega;x')}\,. \tag{4.7}$$

Proof. Inequalities (4.7) follow from (4.6) and (1.7). $\Diamond$

The remaining part of this section deals with the Ky Fan type inequalities for the means under discussion. In particular, inequalities for the differences, reciprocals and products of the generalized Heronian means are obtained. They provide generalizations and refinements of certain inequalities which appear in mathematical literature.

The following inequalities

$$A_n(x') - A_n(x) \leq G_n(x') - G_n(x), \tag{4.8}$$

$$A_n(x') - A_n(x) \leq H_n(x') - H_n(x) \tag{4.9}$$

and

$$\frac{1}{H_n'(x)} - \frac{1}{H_n(x)} \leq \frac{1}{G_n(x')} - \frac{1}{G_n(x)} \leq \frac{1}{A_n(x')} - \frac{1}{A_n(x)} \tag{4.10}$$

have been established by H. Alzer in [1], [3], and [2], respectively.

Our next result reads as follows.

Theorem 4.3. *Let the means* ϕ_i *and* ϕ_j $(1 \leq i < j)$ *satisfy the Ky Fan type inequality*

$$\phi_i(x') - \phi_i(x) \leq \phi_j(x') - \phi_j(x). \tag{4.11}$$

Then for every $\omega \geq 0$ *the inequalities*

$$\phi_i(x') - \phi_i(x) \leq H_{i,j}(\omega;x') - H_{i,j}(\omega;x) \leq \phi_j(x') - \phi_j(x) \tag{4.12}$$

and

$$\frac{1}{\phi_j(x')} - \frac{1}{\phi_j(x)} \leq \frac{1}{H_{i,j}(\omega;x')} - \frac{1}{H_{i,j}(\omega;x)} \leq \frac{1}{\phi_i(x')} - \frac{1}{\phi_i(x)} \tag{4.13}$$

hold true.

Proof. For the proof of (4.12) we use (1.6) to obtain

$$H_{i,j}(\omega;x') - H_{i,j}(\omega;x) = \mu\big[\phi_i(x') - \phi_i(x)\big] + \nu\big[\phi_j(x') - \phi_j(x)\big]\,,$$

where μ and ν are defined in (1.5). Since the right side in the last equality is the weighted mean of two nonnegative quantities, the assertion follows. In order to establish the first inequality in (4.13) we apply Lemma 2.4

with $a = H_{i,j}(\omega; x)$, $a' = H_{i,j}(\omega; x')$, $b = \phi_j(x)$ and $b' = \phi_j(x')$. Using the first and third members of (3.1) and (4.3) we see that, with a, a', b and b' as defined earlier, the assumptions of Lemma 2.4 are satisfied and the desired inequality is established. The second inequality in (4.13) can be established in an analogous manner. We omit further details. $\Diamond$

Refinements of (4.8) and (4.9) can be obtained using (4.12) with $(\phi_i, \phi_j) = (A_n, G_n)$ and $(\phi_i, \phi_j) = (A_n, H_n)$. Similarly, using (4.13) with $(\phi_i, \phi_j) = (G_n, H_n)$ and $(\phi_i, \phi_j) = (A_n, G_n)$ one obtains refinements of (4.10).

We close this section with the proof of the multiplicative version of inequalities (4.3).

Theorem 4.4. *Let the means ϕ_i and ϕ_j $(1 \le i < j)$ satisfy the Ky Fan inequality* (4.2). *Then for every $\omega \ge 0$*

$$\begin{aligned}\phi_j(x)\phi_j(x') &\le \mathcal{H}_{i,j}(\omega; x)\mathcal{H}_{i,j}(\omega; x') \\ &\le H_{ij}(\omega; x)H_{i,j}(\omega; x') \le \phi_i(x)\phi_i(x')\,.\end{aligned} \tag{4.14}$$

Proof. The first inequality in (4.14) can be established using part (ii) of Lemma 2.4 with $b = \phi_j(x)$, $b' = \phi_j(x')$, $a = \mathcal{H}_{i,j}(\omega; x)$ and $a' = \mathcal{H}_{i,j}(\omega; x')$. The first inequalities in (3.1) and (4.3) show that the assumptions of Lemma 2.4 are satisfied. The assertion now follows from (2.8). The remaining two inequalities in (4.14) can be established in a similar way. We omit further details. $\Diamond$

Corollary 4.5. *For every $\omega \ge 0$ the following inequalities*

$$\begin{aligned}H_n(x)H_n(x') &\le \mathcal{H}_{2,3}(\omega; x)\mathcal{H}_{2,3}(\omega; x') \\ &\le H_{2,3}(\omega; x)H_{2,3}(\omega; x') \le G_n(x)G_n(x')\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}G_n(x)G_n(x') &\le \mathcal{H}_{1,2}(\omega; x)\mathcal{H}_{1,2}(\omega; x') \\ &\le H_{1,2}(\omega; x)H_{1,2}(\omega; x') \le A_n(x)A_n(x')\end{aligned} \tag{4.16}$$

hold true.

Proof. For the proof of (4.15) we apply (4.14) with $\phi_2 = G_n$ and $\phi_3 = H_n$. Since the harmonic and geometric means satisfy the Ky Fan inequality (see (4.1)) the assertion follows. Inequalities (4.16) can be established in a similar manner. $\Diamond$

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