Mathematica Pannonica
19/1 (2008), 23–35

ENVELOPES OF ABSOLUTE QUADRATIC SYSTEMS OF CONICS OF THE HYPERBOLIC PLANE I

E. **Jurkin**

Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

O. **Röschel**

Institute for Geometry, Graz University of Technology, Kopernikusgasse 24, Graz, Austria

Received: March 2008

MSC 2000: 51 N 25, 51 M 09, 51 M 15, 51 N 15

Keywords: Hyperbolic plane, entirely circular curves of order 4, quadratic systems of conics, envelopes of special quadratic systems of conics.

Abstract: A curve in the hyperbolic plane $\mathcal{H}_2$ (we use the Cayley–Klein-model) is called *entirely (completely) circular* if it possesses an isotropic asymptote at each intersection point with the absolute. Any planar curve of fourth order can be generated as an envelope of a quadratic set of conics. If one of these tangent conics coincides with the absolute conic $\mathcal{A}$ we refer to it as an *absolute quadratic systems of conics*. Most of the envelopes of these absolute quadratic systems of conics turn out to be entirely circular. In this paper we characterize absolute quadratic systems of conics which have entirely circular envelopes $\mathcal{C}_4$. On such an envelope $\mathcal{C}_4$ a triple point may occur. We unambiguously decide which absolute quadratic systems of conics of $\mathcal{H}_2$ with absolute conic $\mathcal{A}$ deliver triple points of the envelope $\mathcal{C}_4$ on $\mathcal{A}$. In a second part of the paper we will give a complete list of types of non-degenerate entirely circular curves of order 4 generated as envelopes of absolute quadratic systems of conics in the hyperbolic plane $\mathcal{H}_2$.

E-mail addresses: ejurkin@rgn.hr, roeschel@tugraz.at

1. Introduction

We will use the projective Cayley–Klein-model of the hyperbolic plane $\mathcal{H}_2$. Points of the projective plane $\mathrm{PG}(2,\mathbb{R})$ are described by projective coordinates $\vec{x} = (x_0 : x_1 : x_2)^T \neq (0 : 0 : 0)^T$. The absolute conic $\mathcal{A}$ of $\mathcal{H}_2$ is represented by

$$\mathcal{A} \cdots \vec{x}^\top \mathcal{A}\, \vec{x} = x_0^2 - x_1^2 - x_2^2 = 0. \tag{1}$$

Points $\vec{x}$ of the absolute conic $\mathcal{A}$ are called *absolute points* (see [2], [3] and [10]).

An absolute point $\vec{y}$ on a C^1-curve $\mathcal{C}$ of the hyperbolic plane $\mathcal{H}_2$ is called a *circular point* of $\mathcal{C}$ if the curve has a common tangent with the absolute conic $\mathcal{A}$ in this point. The tangent in this point is an *isotropic asymptote* of $\mathcal{C}$. According to [15] an algebraic curve $\mathcal{C}$ is called *entirely (completely) circular*, if there exists at least one common tangent to $\mathcal{C}$ and $\mathcal{A}$ at any intersection point $\mathcal{C} \cap \mathcal{A}$ (there it possesses an isotropic asymptote).[1]

For the isotropic plane entirely circular curves of order 4 have been studied and classified by D. Palman in [8], [9] and H. Sachs [11], p. 179, [12].

We are interested in special algebraic curves of order 4. For them circularity is an algebraic property. Therefore our considerations will include complex points as well.[2]

Absolute points of an entirely circular quartic curve $\mathcal{C}_4$ of $\mathcal{H}_2$ are points with a common tangent with the absolute conic $\mathcal{A}$. They can be simple, double or triple points of the curve. Depending on the types of the multiple joint points and their combinations on the curve there are seven types of such entirely circular curves of degree 4: (2+2+2+2), (3+3+2), (4+2+2), (4+4), (5+3), (6+2) and (8), where the numbers give the multiplicity of the single intersections of $\mathcal{C}_4$ and the absolute conic $\mathcal{A}$.

[1] If this point is a singular point on $\mathcal{C}$ only one branch of $\mathcal{C}$ has to be tangent to the absolute $\mathcal{A}$.

[2] For these considerations we make use of the complex extension of $\mathrm{PG}(2,\mathbb{R})$.

2. Planar curves of order 4 as envelopes of quadratic systems of conics

According to G. Kohn [5] every quartic curve $\mathcal{C}_4$ can be generated[3] as the envelope of quadratic systems of conics (see G. Kohn and G. Loria [7]).

Let A_0, A_1, A_2 be linearly independent symmetric 3×3-matrices and weights $\omega_0, \omega_1, \omega_2 \in \mathbb{R} - \{0\}$. They define a quadratic system $\mathcal{Q}$ which can be parametrized via

$$Q(\lambda) := \omega_0 A_0 + 2\lambda\omega_1 A_1 + \lambda^2\omega_2 A_2 \quad \lambda \in \mathbb{R} \cup \{\infty\}. \tag{2}$$

Any conic $Q(\lambda)$[4] of this system is represented by an equation

$$\vec{x}^\top Q(\lambda)\vec{x} = 0. \tag{3}$$

The set $Q(\lambda)$ of these conics is called *quadratic system of conics* with the *basic matrices* (conics) A_0, A_1, A_2 and weights $\omega_0, \omega_1, \omega_2$.

Remark. Proportional symmetric matrices A_i and $\omega_i A_i$ ($\omega_i \in \mathbb{R} - \{0\}$) define the same conic in $\mathrm{PG}(2,\mathbb{R})$, but the corresponding quadratic systems of conics defined by (2) are different.

Some calculation delivers the following equation of the envelope $\mathcal{C}_4$ of this quadratic system of conics

$$\mathrm{F}(\vec{x}) \equiv \omega_0\omega_2(\vec{x}^\top A_0\vec{x})(\vec{x}^\top A_2\vec{x}) - \omega_1^2(\vec{x}^\top A_1\vec{x})^2 = 0. \tag{4}$$

In general it is an algebraic curve of order 4. In special cases it can be degenerate. In the following we will concentrate on the non-degenerate cases.

Remarks. 1. The envelope $\mathcal{C}_4$ (4) in algebraic sense intersects any conic of the system $Q(\lambda) = \omega_0 A_0 + 2\lambda\omega_1 A_1 + \lambda^2\omega_2 A_2$ in four points – each with intersection multiplicity ≥ 2. These points are the intersections of the two conics $Q(\lambda)$ and $\frac{d}{d\lambda}Q(\lambda)$.

2. This fact holds particularly for the basic conics A_0 and A_2. The common points of A_0 (A_2) and $\mathcal{C}_4$ are exactly the intersections of A_0 (or A_2) and A_1. According to (4) any point of $\mathcal{C}_4 \cap A_0$ has twice the multiplicity of its multiplicity on the intersection $A_0 \cap A_1$.

3. Reparametrisation of the quadratic system facilitates the use of any conic of the system $Q(\lambda)$ as basic conic A_0.

[3]In general in 63 different ways!
[4]We will denote the conics by the same letters as the matrices.

Another way to determine points on the curve $\mathcal{C}_4$ (4) is the following:

The space of conics of the real projective plane $\mathrm{PG}(2,\mathbb{R})$ can be interpreted as a 5-dimensional projective space $\mathrm{PG}(5,\mathbb{R})$. Then a quadratic system of conics defines a conic $\mathcal{Q}$ in $\mathrm{PG}(5,\mathbb{R})$. It contains the points A_0 and A_2 with tangents $\mathcal{T}_0 := [A_0, A_1]$ and $\mathcal{T}_2 := [A_1, A_2]$. The other tangents of this conic $\mathcal{Q}$ intersect the tangents $\mathcal{T}_0$ and $\mathcal{T}_2$ in projectively linked pairs of points. This projectivity will be denoted by π.

In $\mathrm{PG}(2,\mathbb{R})$ this gives a parametrisation of π interlinking pairs of the conics of the two pencils $\mathcal{T}_0(\lambda)$ and $\mathcal{T}_2(\lambda)$ by the same parameter $\lambda \in \mathbb{R} \cup \infty$:

(5) $\mathcal{T}_0(\lambda) := \lambda\omega_0 A_0 + \omega_1 A_1$ and $\mathcal{T}_2(\lambda) := \lambda\omega_1 A_1 + \omega_2 A_2 \quad \lambda \in \mathbb{R} \cup \infty.$

The points of intersection $\mathcal{T}_0(\lambda) \cap \mathcal{T}_2(\lambda)$ define the envelope $\mathcal{C}_4$, too.[5]

Remark. We can define this projectivity by 3 pairs of conics of the two pencils: It maps the conics A_0 to A_1, A_1 to A_2 and $(\omega_0 A_0 + \omega_1 A_1)$ and to $(\omega_1 A_1 + \omega_2 A_2)$.

3. Absolute quadratic systems of conics in the hyperbolic plane

An interesting class of Euclidean entirely circular curves are bicircular curves of order 4 in the Euclidean plane. They can be generated as envelopes of quadratic systems of circles (see G. Loria [6], p. 112). In the hyperbolic plane the situation is different from the Euclidean one: The corresponding envelope of a quadratic system of hyperbolic circles splits into the absolute conic $\mathcal{A}$ and some remaining part of order 2.

In order to get entirely circular curves of order 4 in the hyperbolic plane as envelopes of special quadratic systems of conics we define:

Definition 1. A quadratic system $Q(\lambda)$ of conics of the hyperbolic plane $\mathcal{H}_2$ is called *absolute* iff it contains the absolute conic $\mathcal{A}$.[6]

According to the remarks in Sec. 2 we have: The points $\mathcal{A} \cap \mathcal{C}_4$ have even multiplicity. So we cannot generate those entirely circular curves of order 4 as envelopes of absolute quadratic systems of conics which intersect $\mathcal{A}$ in one or more points with odd multiplicity. Thus we cannot get the whole class of entirely circular curves of degree 4 of the hyperbolic plane as envelopes of absolute quadric systems of conics.

[5] See G. Kohn [5].

[6] There exists $\lambda^* \in \mathbb{R} \cup \{\infty\}$ with $A(\lambda^*) \sim \mathcal{A}$.

On the other hand we will demonstrate that there are envelopes $\mathcal{C}_4$ of absolute quadratic systems of conics in the hyperbolic plane $\mathcal{H}_2$ which are not entirely circular.

The goal of this paper is to characterize those absolute quadratic systems of conics which define a non-degenerate entirely circular envelope $\mathcal{C}_4$ of order 4.

In order to get *standard representations of absolute quadratic systems of conics* in $\mathcal{H}_2$ we can transform the basic conic A_0 into the absolute conic $\mathcal{A}$ with weight $\omega_0 := 1$. As $A_0 = \mathcal{A}$ is regular, the pencil $\mathcal{T}_0$ spanned by $[A_0 = \mathcal{A}, A_1]$ contains at least one singular conic. Changing the parameterisation of this pencil allows to consider the *basic conic* A_1 *as a singular conic* in $\mathcal{H}_2$ – for the following we assume the symmetric matrix A_1 to be singular (with arbitrary $\omega_1 \in \mathbb{R} - \{0\}$). The quotient $\omega_2 : \omega_1^2$ then is the only geometric shape parameter of $\mathcal{Q}$.

4. Absolute quadratic systems of conics and circular points on their envelopes

According to our considerations the envelope $\mathcal{C}_4$ of an absolute quadratic system $Q(\lambda)$ of conics in general will intersect the absolute $\mathcal{A}$ in points with even multiplicity ≥ 2. These points are exactly the points of intersection $\vec{y} \in \mathcal{A} \cap A_1$ ($\vec{y}$ can be a complex point). We will look at the tangential behavior of the envelope $\mathcal{C}_4$ in this point $\vec{y}$.

Let us start with such an absolute basic point $\vec{y} \in \mathcal{A} \cap A_1$ of the pencil of conics spanned by $[A_0 = \mathcal{A}, A_1]$. $\vec{y}$ is a point on the envelope $\mathcal{C}_4$ and we have

$$\vec{y}^\top A_0 \vec{y} = 0 \text{ and } \vec{y}^\top A_1 \vec{y} = 0. \tag{6}$$

The tangential behavior in this point on $\mathcal{C}_4$ is usually being studied by observing the intersections of $\mathcal{C}_4$ with arbitrary straight lines through $\vec{y}$. Such a line $\mathcal{L}$ will be spanned by $\vec{y}$ and a further point $\vec{z} \neq \rho\vec{y}$ with $\rho \in \mathbb{R}$. It can be parametrized by

$$\mathcal{L} \cdots \vec{y} + t\ \vec{z} \text{ with } t \in \mathbb{R}. \tag{7}$$

We compute the intersections of $\mathcal{L}$ and $\mathcal{C}_4$: They belong to the zeros of the following polynomial of degree 4[7] in t

$$(8)\quad p(t) := F(\vec{y}+t\vec{z}) = \sum_{i=0}^{4} t^i F_i(\vec{y},\vec{z}) \quad \text{with}$$

$$\begin{aligned}
F_0(\vec{y},\vec{z}) &= 0,\\
F_1(\vec{y},\vec{z}) &= 2\,\omega_2(\vec{y}^\top A_0\vec{z})(\vec{y}^\top A_2\vec{y}),\\
F_2(\vec{y},\vec{z}) &= 4\,\omega_2(\vec{y}^\top A_0\vec{z})(\vec{y}^\top A_2\vec{z}) + \omega_2(\vec{z}^\top A_0\vec{z})(\vec{y}^\top A_2\vec{y}) - 4\,\omega_1^2(\vec{y}^\top A_1\vec{z})^2,\\
F_3(\vec{y},\vec{z}) &= 2\,\omega_2(\vec{y}^\top A_0\vec{z})(\vec{z}^\top A_2\vec{z}) + 2\,\omega_2(\vec{z}^\top A_0\vec{z})(\vec{y}^\top A_2\vec{z}) -\\
&\quad - 4\,\omega_1^2(\vec{y}^\top A_1\vec{z})(\vec{z}^\top A_1\vec{z}),\\
F_4(\vec{y},\vec{z}) &= \omega_2(\vec{z}^\top A_0\vec{z})(\vec{z}^\top A_2\vec{z}) - \omega_1^2(\vec{z}^\top A_1\vec{z})^2.
\end{aligned}$$

Now we will discuss the following two cases: Either the point $\vec{y} \in \in A_0 = \mathcal{A} \cap A_1$ does or does not belong to the third basic conic A_2 of the absolute quadratic system (Cases B and A, resp.). In order to gain $\vec{y}$ as a circular point on $\mathcal{C}_4$ we have to compare the tangent of the envelope $\mathcal{C}_4$ at $\vec{y}$ with the tangent of $\mathcal{A}$ at this point. The tangent to $\mathcal{A}$ at $\vec{y} \in \mathcal{A}$ contains the points $\vec{z}$ of $\mathcal{H}_2$ with the equation

$$(9)\qquad \vec{y}^\top A_0\vec{z} = 0 \ \ (\text{with } \vec{y}^\top A_0\vec{y} = 0).$$

Case A. The absolute point $\vec{y}$ on $A_0 = \mathcal{A}$ and A_1 does not belong to A_2. This case is characterized by $\vec{y}^\top A_2\vec{y} \neq 0$. Equation $F_1(\vec{y},\vec{z}) = 0$ from (8) characterizes points $\vec{z}$ on the tangent of $\mathcal{C}_4$ at $\vec{y}$. This condition reduces to

$$(10)\qquad \vec{y}^\top A_0\vec{z} = 0$$

which shows that the tangent of $\mathcal{C}_4$ at $\vec{y}$ is tangent of $A_0 = \mathcal{A}$ at $\vec{y}$, too and the point $\vec{y} \in \mathcal{A} \cap A_1$ is circular on $\mathcal{C}_4$. If Case A holds for any basic point $\vec{y}$ of the pencil $[A_0 = \mathcal{A}, A_1]$ this implies the following

Theorem 1. *If the 3 basic conics $A_0 = \mathcal{A}$, A_1 and A_2 of an absolute quadratic system $\mathcal{Q}$ of conics of the hyperbolic plane $\mathcal{H}_2$ do not have any (real or complex) common basic point the envelope $\mathcal{C}_4$ of this absolute quadratic system is an entirely circular curve of the hyperbolic plane $\mathcal{H}_2$.*

Fig. 1 displays such a curve with the basic conics $A_0 = \mathcal{A} \cdots x_0^2 - - x_1^2 - x_2^2 = 0$, $A_1 \cdots x_1^2 - x_2^2 = 0$ and $A_2 \cdots 4x_0^2 - x_1^2 - x_2^2 = 0$ with

[7] In order to simplify we use the condition (6).

$\omega_1 = -1$ and $\omega_2 = -2$ in an *auxiliary Euclidean view*: We put $(x_0 : : x_1 : x_2) := (1 : x : y)$ $(x_0 \neq 0)$ for Cartesian coordinates (x, y).[8] The envelope's equation is $2(x_0^2 - x_1^2 - x_2^2)(4x_0^2 - x_1^2 - x_2^2) + (x_1^2 - x_2^2)^2 = 0$.

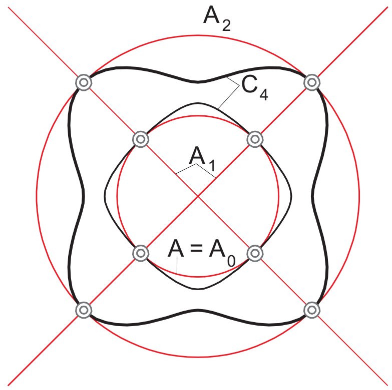

Figure 1. An entirely circular curve $\mathcal{C}_4$ of order 4 according to Th. 1.

Remark. The conics A_0 and A_2 can be exchanged. Therefore the tangents to $\mathcal{C}_4$ in the points of $A_1 \cap A_2$ are also tangents of A_2 (if $A_0 = \mathcal{A}$ does not contain these points of intersection).

Case B. The absolute point $\vec{y}$ on $A_0 = \mathcal{A}$ and A_1 also belongs to A_2. Additionally to (6) this case is characterized by $\vec{y}^\top A_2 \vec{y} = 0$. Then (8) yields $F_1(\vec{y}, \vec{z}) = 0 \ \forall \vec{z}$. Therefore the point $\vec{y}$ is *a singular point on the envelope* $\mathcal{C}_4$ in Case B. The tangents to $\mathcal{C}_4$ at $\vec{y}$ in general are given by the following quadratic equation for the points $\vec{z}$

$$0 = F_2(\vec{y}, \vec{z}) = 4\ \omega_2(\vec{y}^\top A_0 \vec{z})(\vec{y}^\top A_2 \vec{z}) - 4\ \omega_1^2(\vec{y}^\top A_1 \vec{z})^2. \tag{11}$$

Remark. In Case A we have got circularity at any intersection of $\mathcal{A}$ and A_1.[9] Fig. 2 displays an interesting example of Case B demonstrating that *there are examples of envelopes* $\mathcal{C}_4$ *of absolute quadratic systems* $\mathcal{Q}$ *of conics in the hyperbolic plane* $\mathcal{H}_2$ *which do not have circularity at all absolute points.* This is why we will try to characterize those absolute quadratic systems $\mathcal{Q}$ of conics which have envelopes $\mathcal{C}_4$ with circularity also at singular points on the absolute $\mathcal{A}$.

[8] Note that different values of ω_1 and ω_2 (both $\neq 0$) determine different envelopes, but all share their points of contact on A_0 and A_2.

[9] Under the assumption that the conic A_2 does not pass through any of the basic points of the pencil $[\mathcal{A}, A_1]$.

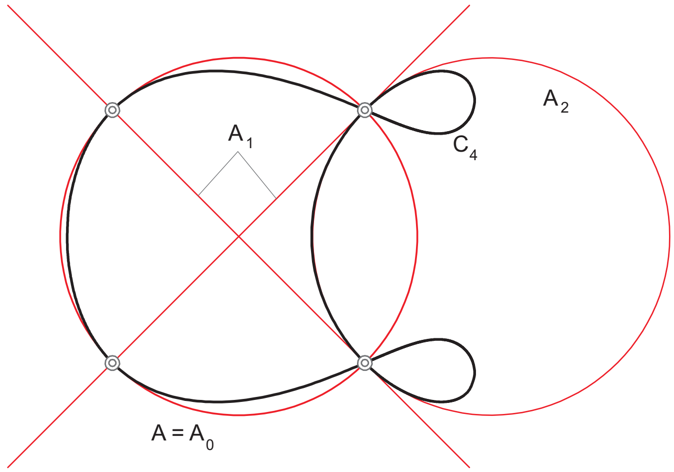

Figure 2. Real double points on the envelope $\mathcal{C}_4$ of an absolute quadratic system of conics based on the absolute conic $\mathcal{A}$ and the conics A_1 and A_2.

Now we have to discuss two subcases:

Case B1. We do not have $F_2(\vec{y},\vec{z}) = 0$ for all $\vec{z}$. Then the absolute point $\vec{y}$ on $\mathcal{A} = A_0, A_1$ and A_2 is a singularity (of first kind – see R. J. Walker [16], p. 52) on the envelope $\mathcal{C}_4$. In order to gain $\vec{y}$ as a circular point on $\mathcal{C}_4$ one of the tangents in this singular point has to coincide with the tangent to $A_0 = \mathcal{A}$ at $\vec{y}$. The equation $F_2(\vec{y},\vec{z}) = 0$ has to contain the factor $\vec{y}^\top A_0\vec{z}$ which according to (11) implies the condition

$$\vec{y}^\top A_1\vec{z} = 0. \tag{12}$$

Together with (6) this condition characterizes $\vec{y}$ as *an intersection of $A_0 = \mathcal{A}$ and A_1 with multiplicity* ≥ 2. This is a necessary condition to gain circularity of $\mathcal{C}_4$ at $\vec{y}$ in the Case B1. We can state

Theorem 2. *If the* 3 *basic conics $A_0 = \mathcal{A}$, A_1 and A_2 of an absolute quadratic system $\mathcal{Q}$ of conics of the hyperbolic plane $\mathcal{H}_2$ have a (real or complex) common basic point $\vec{y}$ this point is a singularity on the envelope C_4. Is this singularity of first kind we have: C_4 has an isotropic tangent at $\vec{y}$ iff $\mathcal{A}$ and A_1 intersect in $\vec{y}$ with multiplicity* ≥ 2.

Remark. Fig. 2 can also be interpreted as an example for this case. If we exchange A_0 for A_2 the conic $A_2 = \mathcal{A}$ is the absolute conic of $\mathcal{H}_2$ and is tangent to A_1 in two real points. These two points are double points on $\mathcal{C}_4$ – each of them has one branch osculating A_2 in the double point. In this interpretation (with $A_2 = \mathcal{A}$) the envelope $\mathcal{C}_4$ is an entirely circular curve of order 4 in $\mathcal{H}_2$. We have one interesting example for the case $(4+4)$.

Case B2. We have $F_2(\vec{y},\vec{z}) \equiv 0$ for all $\vec{z}$. The absolute point $\vec{y}$ on $\mathcal{A} = A_0, A_1$ and A_2 will be a singularity of the second kind on $\mathcal{C}_4$. On this curve $\mathcal{C}_4$ this point has to be a *triple point* (see R. J. Walker [16], p. 52). A triple point on an algebraic curve of order 4 has to be real. Therefore we can transform this point by some hyperbolic displacement into $\vec{y} = (1:1:0)^\top$.

First we want to *geometrically interpret the condition*

$$F_2(\vec{y},\vec{z}) = 4\,\omega_2(\vec{y}^\top A_0\vec{z})(\vec{y}^\top A_2\vec{z}) - 4\,\omega_1^2(\vec{y}^\top A_1\vec{z})^2 \equiv 0 \quad \forall \vec{z} \tag{13}$$

for the absolute quadratic system of conics. The conics A_1 and A_2 contain the real point $\vec{y} = (1:1:0)^\top$. Their corresponding symmetric matrices can be written as

$$A_i = \begin{pmatrix} a_i & b_i & c_i \\ b_i & -a_i - 2b_i & e_i \\ c_i & e_i & f_i \end{pmatrix} \tag{14}$$

with $a_i, b_i, c_i, e_i, f_i \in \mathbb{R}$ $(i = 1, 2)$. The condition (13) yields

$$\begin{aligned} 0 = {} & (z_0 - z_1)\omega_2[(a_2+b_2)z_0 - (a_2+b_2)z_1 + (c_2+e_2)z_2] - \\ & - \omega_1^2[(a_1+b_1)z_0 - (a_1+b_1)z_1 + (c_1+e_1)z_2]^2 \end{aligned} \tag{15}$$

$\forall\ (z_0 : z_1 : z_2) \neq (0:0:0)$. Apart from (15) we can easily get

$$c_1 + e_1 = 0,\ c_2 + e_2 = 0 \tag{16}$$

and

$$\omega_2(a_2+b_2) = \omega_1^2(a_1+b_1)^2. \tag{17}$$

From the condition (16) follows that the 3 basic conics $A_0 = \mathcal{A}, A_1$ and A_2 *have to be pairwise intersecting with multiplicity* ≥ 2 *in the point* $\vec{y}$.[10]

The additional condition (17) also affects the definition of the quadratic system $\mathcal{Q}$ of conics from the 3 given basic conics $\mathcal{A} = A_0, A_1$ and A_2 and the weights ω_1, ω_2.

If A_1 and A_2 are given according to Case B2 this condition can be used to change the weights ω_1, ω_2 such that the envelope $\mathcal{C}_4$ has a triple point in $\vec{y} \in \mathcal{A}$.

In order to give a geometric interpretation we go back to the generation of $\mathcal{C}_4$ by projectively linked pencils (5) of conics. According to (16)

[10] If the conics A_1 and A_2 were regular then all 3 basic conics had the same tangent at the point $\vec{y}$.

each pencil $\mathcal{T}_0 = [A_0, A_1]$ and $\mathcal{T}_2 = [A_1, A_2]$ contains at least one degenerate conic with a singular point in $\vec{y}$.[11] Some short calculation yields: In the first and the second pencil $\mathcal{T}_0(\lambda)$ and $\mathcal{T}_2(\lambda^*)$ the corresponding symmetric matrices are given by λ and λ^* with

$$\begin{gathered} \lambda + \omega_1(a_1 + b_1) = 0 \quad \text{and} \\ \omega_1\lambda^*(a_1 + b_1) + \omega_2(a_2 + b_2) = 0. \end{gathered} \tag{18}$$

Again we arrive at two subcases: 1. If $a_1 + b_1 = 0$ the conic A_1 is the singular conic of the pencil $\mathcal{T}_0$ with singular point $\vec{y}$. Then (17) gives $a_2 + b_2 = 0$. Therefore the conic A_2 must also have $\vec{y}$ as a singular point.

2. If $a_1 + b_1 \neq 0$ the conic A_1 does not have $\vec{y}$ as singular point. Then the condition (17) characterizes $\mathcal{T}_0(\lambda)$ and $\mathcal{T}_2(\lambda^*)$ with λ and λ^* from (18) as conics linked via the projectivity π (5).

Both cases are described in

Theorem 3. *Given 3 basic conics $A_0 = \mathcal{A}, A_1$ and A_2 of an absolute quadratic system $\mathcal{Q}$ of conics of the hyperbolic plane $\mathcal{H}_2$ with a (real or complex) common basic point $\vec{y} \in \mathcal{A}$ and common tangent at $\vec{y}$ (intersection with multiplicity ≥ 2). Its envelope $\mathcal{C}_4$ shall be a non-degenerate algebraic curve of order 4. This absolute quadratic system $\mathcal{Q}$ of conics induces a projectivity between the two pencils of conics $\mathcal{T}_0 = [\mathcal{A} = A_0, A_1]$ and $\mathcal{T}_2 = [A_1, A_2]$ denoted by π. The point $\vec{y}$ is a triple point of the envelope $\mathcal{C}_4$ iff the projectivity π maps the singular conic with singular point $\vec{y}$ from the pencil $\mathcal{T}_0$ to a singular conic with singular point $\vec{y}$ from $\mathcal{T}_2$.*

Now we are interested in the tangents at this triple point $\vec{y} \in \mathcal{C}_4$: With (16) and (17) we have

$$\begin{aligned} \vec{y}^\top A_0 \vec{z} &= z_0 - z_1, \\ \vec{y}^\top A_1 \vec{z} &= (a_1 + b_1)(z_0 - z_1), \\ \omega_2 \vec{y}^\top A_2 \vec{z} &= \omega_1^2 (a_1 + b_1)^2 (z_0 - z_1) \end{aligned} \tag{19}$$

and

$$\begin{aligned} &F_3(\vec{y}, \vec{z}) = \\ &= 2(z_0 - z_1)[\omega_2(\vec{z}^\top A_2 \vec{z}) + \omega_1^2(a_1 + b_1)^2(\vec{z}^\top A_0 \vec{z}) - 2\,\omega_1^2(a_1 + b_1)(\vec{z}^\top A_1 \vec{z})]. \end{aligned} \tag{20}$$

[11] We stated A_1 to be singular. But there can be two singular conics in the pencil $[A_0, A_1]$. Only one of them has the desired property.

$F_3(\vec{y}, \vec{z}) = 0$ in (20) is the equation of the tangents to $\mathcal{C}_4$ in the triple point $\vec{y}$ on the absolute $\mathcal{A}$. One of them is $z_0 - z_1 = 0$ which coincides with the tangent of $\mathcal{C}_4$ in the triple point $\vec{y}$. Therefore in all cases the *triple point $\vec{y}$ is a circular point* on $\mathcal{C}_4$.

Theorem 4. *Given 3 basic conics $A_0 = \mathcal{A}$, A_1 and A_2 of an absolute quadratic system $\mathcal{Q}$ of conics of the hyperbolic plane with a (real or complex) common absolute basic point $\vec{y} \in \mathcal{A}$. If the envelope $\mathcal{C}_4$ of this quadratic system $\mathcal{Q}$ of conics is a non-degenerate curve of degree 4 and $\vec{y}$ is a triple point on $\mathcal{C}_4 \cap \mathcal{A}$ (characterization in Th. 3) one of the tangents to $\mathcal{C}_4$ in $\vec{y}$ will be tangent to $\mathcal{A}$, too. The triple point $\vec{y}$ is a circular point on $\mathcal{C}_4$.*

Fig. 3 displays the interesting case of a triple point for $A_0 \cdots x_0^2 - x_1^2 - x_2^2 = 0$, $A_1 \cdots x_2^2 = 0$ and $A_2 \cdots (x_0 - x_1)^2 - x_2^2 = 0$ with $\omega_1 = 1, \omega_2 = 0.75$. According to Th. 4 the envelope $\mathcal{C}_4$ is an entirely circular curve of order 4 and belongs to the type $(4 + 4)$.

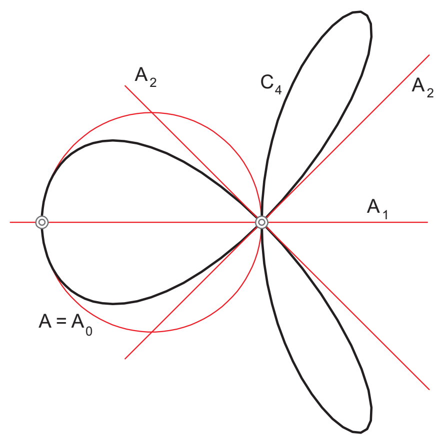

Figure 3. One absolute triple point on the envelope $\mathcal{C}_4$ of an absolute quadratic system of conics.

Fig. 4 addresses another interesting case of a triple point for $A_0 \cdots x_0^2 - x_1^2 - x_2^2 = 0$, $A_1 \cdots (x_0 - x_1)^2 = 0$ and $A_2 \cdots (x_0 - x_1)^2 - x_2^2 = 0$ with $\omega_1 = -1, \omega_2 = -0.4$. The envelope $\mathcal{C}_4$ is an entirely circular curve of order 4 and belongs to the type (8).

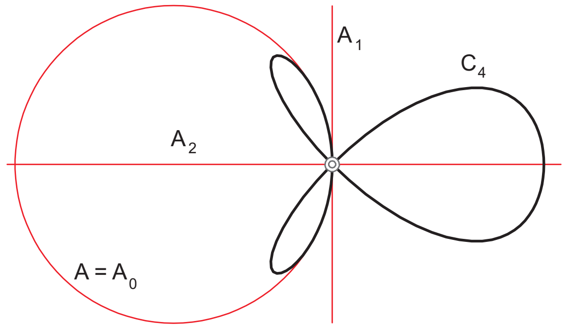

Figure 4. One absolute triple point on the envelope $\mathcal{C}_4$ of an absolute quadratic system of conics.

According to our Ths. 1–4 we can characterize such absolute quadratic systems $\mathcal{Q}$ of conics of the hyperbolic plane $\mathcal{H}_2$ with entirely circular envelopes $\mathcal{C}_4$ of order 4. The following theorem is an immediate consequence of our previous results:

Theorem 5. *Given an absolute quadratic system $\mathcal{Q}$ of conics $Q(\lambda) :=$ $:= \mathcal{A} + 2\ \omega_1\lambda A_1 + \omega_2\lambda^2 A_2$ of the hyperbolic plane $\mathcal{H}_2$ with absolute $\mathcal{A}$. Its non-degenerate envelope $\mathcal{C}_4$ of order 4 is an entirely circular curve in the hyperbolic plane $\mathcal{H}_2$ iff we have: In any absolute point $\vec{y}$ (real or complex) belonging to $\mathcal{A}, A_1$ and A_2 at the same time the conics $\mathcal{A}$ and A_1 intersect with multiplicity ≥ 2.*

5. Conclusions

These results provide us with a comprehensive criterion to decide wether an absolute quadratic system of the hyperbolic plane $\mathcal{H}_2$ has an entirely circular envelope or not.

A further part of this paper will contain a complete list of absolute quadratic systems $\mathcal{Q}$ of conics which possess entirely circular envelopes $\mathcal{C}_4$. These curves will be generated by the help of the projectivity π (5) linking the pencils $\mathcal{T}_0$ and $\mathcal{T}_2$.

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