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CHARACTERISATION RESULTS FOR SHIFT RADIX SYSTEMS

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Abstract: For $\mathbf{r} \in \mathbb{R}^d$ define the function $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way:

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{a} = (a_1, \dots, a_d) \mapsto (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor).$$

$\tau_{\mathbf{r}}$ is called a shift radix system (SRS) if $\forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$. In this paper we deal with new results concerning the characterisation of the set $\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^d \, | \tau_{\mathbf{r}} \text{ is an SRS} \right\}$, especially for $d = 2$. For this purpose we adapt and generalise several results and methods presented in earlier papers.

1. Introduction

Let $\mathbf{r} \in \mathbb{R}^d$ and

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d,$$
$$\mathbf{a} = (a_1, \dots, a_d) \mapsto (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor).$$

$\tau_{\mathbf{r}}$ is called a shift radix system, short SRS, if

$$\forall \mathbf{a} \in \mathbb{Z}^d \, \exists k \in \mathbb{N} : \tau_{\mathbf{r}}^k = \mathbf{0}.$$

Further define the sets

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \, \middle| \forall \mathbf{x} \in \mathbb{Z}^d \, \exists n, l \in \mathbb{N} : \tau_{\mathbf{r}}^k(\mathbf{x}) = \tau_{\mathbf{r}}^{k+l}(\mathbf{x}) \; \forall k \geq n \right\} \text{ and}$$
$$\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^d \, | \tau_{\mathbf{r}} \text{ is an SRS} \right\}.$$

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Shift radix systems were first introduced by Akiyama et al. in [1]. There the basic properties have been analysed, as well as the close relation to β-expansions [1, 8, 10] and canonical number systems (CNS) [1, 9] were studied. It is easy to see that $\mathcal{D}_d \supset \mathcal{D}_d^0$. The set $\mathcal{D}_d$ is, except for the boundary, easy to describe. For an $\mathbf{r} = \{r_1, \dots, r_d\} \in \mathbb{R}^d$ let

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & -r_{d-1} & -r_d \end{pmatrix}$$

and $\mathcal{E}_d := \left\{\mathbf{s} \in \mathbb{R}^d | \rho(R(\mathbf{s})) < 1\right\}$, where $\rho(\cdot)$ denotes the spectral radius. Then we have $\mathcal{E}_d \subset \mathcal{D}_d \subset \overline{\mathcal{E}_d}$. Note that $\overline{\mathcal{E}_d}$ is compact and thus $\mathcal{D}_d$ and $\mathcal{D}_d^0$ are bounded. For a more detailed treatment of $\mathcal{D}_d$, see [1, Sec. 4] and, for $d = 2$, see [3, Sec. 2] or [2]. According to [3, Corollary 2.5] we have $\mathcal{D}_2^0 \subset \mathcal{E}_2$. $\mathcal{D}_d^0$ can be gained by cutting out polyhedra from $\mathcal{D}_d$ in the following way: An element $\mathbf{r}$ of $\mathcal{D}_d$ does not belong to $\mathcal{D}_d^0$ if there exists a sequence $a_0, \dots, a_{l-1}, l \geq d$ of integers, not all equal to zero, such that for the points

$$\begin{aligned} \mathbf{v}_0 &= (a_0, \dots, a_{d-1}), \\ \mathbf{v}_1 &= (a_1, \dots, a_d), \\ &\vdots \\ \mathbf{v}_{l-1} &= (a_{l-1}, a_l, \dots, a_{l+d-2}), \end{aligned} \tag{1.1}$$

where the indices of a have to be taken modulo l, we have that $\tau_{\mathbf{r}}(\mathbf{v}_0) = \mathbf{v}_1, \dots, \tau_{\mathbf{r}}(\mathbf{v}_{l-2}) = \mathbf{v}_{l-1}, \tau_{\mathbf{r}}(\mathbf{v}_{l-1}) = \mathbf{v}_0$. According to [1] we write $(a_0, \dots, a_{d-1}); a_d, \dots, a_{l-1}$ for the chain of mappings $\mathbf{v}_0 \mapsto \mathbf{v}_1 \mapsto \cdots \mapsto \mathbf{v}_{l-1} \mapsto \mathbf{v}_0$ and say that it is a period (of length l) of $\tau_{\mathbf{r}}$ and generally a period of $\mathcal{D}_d$. Observe that for a period the order of elements is important, but it is invariant concerning rotations. Further the sequences $a_0, \dots, a_{l-1}$ and $a_0, \dots, a_{l-1}, a_0, \dots, \dots, a_{l-1}$ describe the same period. In this manner we can enlarge the length of a sequence to ensure that it is bigger than d. For a given period $\pi = (a_0, \dots, a_{d-1}); a_d, \dots, a_{l-1}$ we set $P(\pi) := \{\mathbf{r} \in \mathbb{R}^d \,|\, \pi \text{ is a period of } \tau_{\mathbf{r}}\}$. By the definition of $\tau_{\mathbf{r}}$ this set consists of all the points $\mathbf{r} = (r_1, \dots, r_d)$ satisfying the system of inequalities

$$\begin{array}{ccccccccccc} 0 \leq & a_0 r_1 & + & a_1 r_2 & + & \cdots & + & a_{d-1} r_d & + & a_d & < 1, \\ \vdots & \vdots & & \vdots & & & & \vdots & & \vdots & \vdots \\ 0 \leq & a_{l-1} r_1 & + & a_l r_2 & + & \cdots & + & a_{l+d-2} r_d & + & a_{l+d-1} & < 1 \end{array} \tag{1.2}$$

(again with the indices of a taken modulo l), which describes a polyhedron. This polyhedron is not necessarily a d-dimensional figure, in many cases it is degenerated or even equal to the empty set. According to [1] we call π a non-degenerated period, if $P(\pi)$ is a non-degenerated d-dimensional polyhedron and we call it empty, if $P(\pi)$ is the empty set. By the Lifting Theorem [1, Th. 6.2] for the non-degenerated period $\pi = (a_0, \dots, a_{d-1}); a_d, \dots, a_{l-1}$ the lift $l(\pi) = (a_0, \dots, a_d); a_{d+1}, \dots, a_{l-1}$ to a higher dimension is also non-degenerated.

Now it is easy to see that

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \text{ period}} P(\pi),$$

which is the representation of $\mathcal{D}_d^0$ by cutting out polyhedra from $\mathcal{D}_d$. We will refer to these polyhedra as cutout polyhedra. The difficulty of the characterisation of $\mathcal{D}_d^0$ is now clear: the set of all periods is a priori infinite and we will see in Sec. 4 that infinitely many of them are not empty. Therefore this representation is only a theoretical one.

The paper is organised as follows: in Sec. 2 we deal with an algorithm that was originally presented by Brunotte in [4] (see also [1]). With its aid we can construct a finite set Π_Q of periods π with $P(\pi) \cap Q \neq \emptyset$ for a closed convex set $Q \subset \mathcal{E}_d$, such that

$$Q \cap \mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \in \Pi_Q} P(\pi).$$

We will use this algorithm to extend the existing analysis of $\mathcal{D}_2^0$ given in [3]. But note that it can only be used for areas away from the boundary of $\mathcal{D}_2$. For sets near this boundary other ways are needed. For characterising an area near the upper boundary of $\mathcal{D}_2$ we will improve a method which was presented by Akiyama et al. in [3]. This is also stated in Sec. 2.

A small area near the point $(1, 1)$ will be treated in Sec. 3. The set

$$P := \left\{ (1 - T, 1 + \delta T) \mid 0 < T < \frac{1}{30}, 0 \leq \delta \leq 1 \right\}$$

cannot be investigated with the above mentioned algorithm, because it "touches" the right boundary of $\mathcal{D}_2$. We will look directly at the orbits of $\tau_{\mathbf{r}}$ and find out that P belongs to $\mathcal{D}_2^0$.

Fig. 1 shows the set $\mathcal{D}_2$. The black parts are cutouts, the white areas are already known to be subsets of $\mathcal{D}_2^0$. In Sections 2 and 3 all the light grey areas are investigated in the mentioned theorems. The sets where cutouts are found are shown in the subscribed figures. Only the

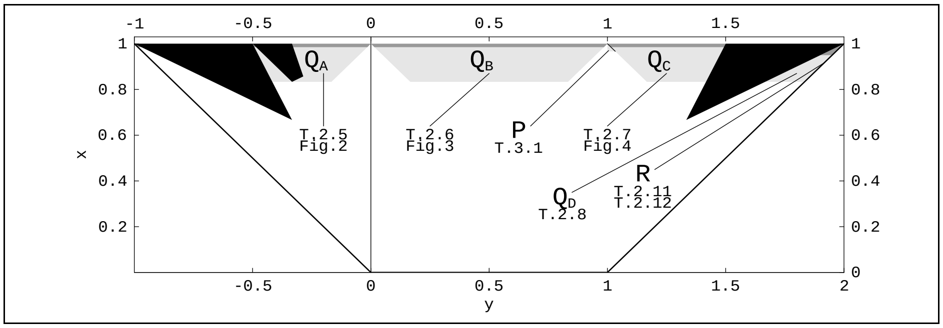

Figure 1. An overview of $\mathcal{D}_2$

thin dark grey regions near the upper boundary of the trapezian in Fig. 1 are still not analysed.

As we have seen, $\mathcal{D}_2^0$ can be explained as set $\mathcal{D}_2$ with a number of polygons cut out. In this representation we walk through all possible periods, which are of course infinitely many. But not all of them give nonempty polygons. In [1, Sec. 6] was already shown that there are no upper bounds neither for the length nor for the size of the entries of a period π to ensure that $P(\pi) \neq \emptyset$. Sec. 4 is dedicated to infinite families of non-degenerated periods. We will complete the analysis of such a family, which was already described in [1] and we will present a new one. With the aid of this result we will conclude that the point $K_d' = (0, \ldots, 0, 1, 1) \in \mathbb{R}^d$ is a critical point in the sense of [1, Def. 7.1], i.e., any neighbourhood of K_d' cannot be described by using only finitely many cutout polyhedra.

2. Algorithmic solutions

2.1. The set of witnesses $\boldsymbol{\mathcal{V}(Q)}$ and the graph $\boldsymbol{G(\mathcal{V}, Q)}$. For $Q \subset \mathbb{R}^d, V \subset \mathbb{Z}^d, \mathbf{x} \in \mathbb{Z}^d$ let

$$\tau_Q(\mathbf{x}) = \{\tau_{\mathbf{r}}(\mathbf{x}) \,|\, \mathbf{r} \in Q\} \text{ and}$$
$$\tau_Q(V) = \{\tau_{\mathbf{r}}(\mathbf{v}) \,|\, \mathbf{r} \in Q, \mathbf{v} \in V\}.$$

Definition 2.1 (cf. [1]). For a closed convex set $Q \subset \operatorname{int} \mathcal{D}_d$, the smallest set $\mathcal{V}(Q) \subset \mathbb{Z}^d$ with the properties

(V1) $\pm(\delta_{1i}, \delta_{2i} \ldots, \delta_{di}) \in \mathcal{V}(Q), i = 1, \ldots, d,$

(V2) $\mathbf{x} \in \mathcal{V}(Q) \Rightarrow \tau_Q(\mathbf{x}) \cup -\tau_Q(-\mathbf{x}) \subset \mathcal{V}(Q),$

where δ_{ij} denotes the Kronecker delta, is called the *set of witnesses* of the set Q. Additionally, for a finite set $\mathcal{W} \subset \mathbb{Z}^d$ and a closed convex set $Q \subset \mathcal{D}_d$, we define $G(\mathcal{W}, Q) = V \times E$ to be the smallest directed graph with vertices $V \subset \mathbb{Z}^d$ and edges $E \subset \mathbb{Z}^d \times \mathbb{Z}^d$, with

(G1) $\mathcal{W} \subseteq V$,

(G2) $\mathbf{x} \in V \Rightarrow \tau_Q(\mathbf{x}) \subset V$,

(G3) $E = \{(\mathbf{x}, \tau_{\mathbf{r}}(\mathbf{x})) | \mathbf{x} \in V, \mathbf{r} \in Q\}$.

The original idea of a set of witnesses comes from Brunotte [4], who defined an analogue in context with canonical number systems. The term *set of witnesses* for SRS was used for the first time by Akiyama et al. [1]. The present definition is modified in that effect, that in [1] any set which fulfils (V1)–(V2) is called a set of witnesses.

What are the requirements to ensure that $\mathcal{V}(Q)$ is finite? Let $A \in \mathbb{R}^{d\times d}$. For a $\delta > \rho(A)$ choose a vector norm $\|\cdot\|_{A,\delta}$ with

$$\forall \mathbf{x} \in \mathbb{R}^d : \|A\mathbf{x}\|_{A,\delta} \leq \delta \|\mathbf{x}\|_{A,\delta}.$$

Denote also by $\|\cdot\|_{A,\delta}$ a compatible matrix norm, i.e.

$$\forall \mathbf{x} \in \mathbb{R}^d \, \forall B \in \mathbb{R}^{d\times d} : \|B\mathbf{x}\|_{A,\delta} \leq \|B\|_{A,\delta} \|\mathbf{x}\|_{A,\delta}.$$

Such a norm always exists and is explicitly constructed for instance in [6, Eq. (3.2)]. Choose a vector $\mathbf{r} \in \mathcal{E}_d$ and a δ with $1 > \delta > \rho(R(\mathbf{r}))$. Further, for $\varepsilon > 0$, set

$$U_\varepsilon(\mathbf{r}, \delta) := \left\{ \mathbf{s} \in \mathbb{R}^d \,\middle|\, \|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta} < \varepsilon \right\}.$$

Lemma 2.2. *$\mathcal{V}(Q)$ is finite for $Q \subset U_\varepsilon(\mathbf{r}, \delta)$ with $\varepsilon < 1 - \delta$.*

Proof. It suffices to prove that there is a finite set $\mathcal{V}'(Q)$ that satisfies (V1) and (V2) in Def. 2.1. Then $\mathcal{V}(Q) \subseteq \mathcal{V}'(Q)$, which implies that $\mathcal{V}(Q)$ is finite, too. Let

$$N = \max_{i=1,\ldots,d} \| \pm (\delta_{1i}, \ldots, \delta_{di})\|_{R(\mathbf{r}),\delta}.$$

We will show that the finite set $\mathcal{V}'(Q) := \{\mathbf{x} \in \mathbb{Z}^d | \|\mathbf{x}\|_{R(\mathbf{r}),\delta} \leq \frac{N}{1-\delta-\varepsilon}\}$ fulfils the requirements. Because $N < \frac{N}{1-\delta-\varepsilon}$ it includes the canonical base vectors ((V1) is satisfied). For $\mathbf{x} \in \mathcal{V}'(Q), \mathbf{s} \in Q$ the functions $\tau_{\mathbf{s}}(\mathbf{x})$ and $-\tau_{\mathbf{s}}(-\mathbf{x})$ can be written as $R(\mathbf{s})\mathbf{x} + (0, \ldots, 0, \nu)$ with $|\nu| < 1$. Thus

$$\begin{aligned}
&\|R(\mathbf{s})\,\mathbf{x} + (0, \ldots, 0, \nu)\|_{R(\mathbf{r}),\delta} \leq \\
&\quad \leq \|(R(\mathbf{s}) - R(\mathbf{r}) + R(\mathbf{r}))\,\mathbf{x}\|_{R(\mathbf{r}),\delta} + \|(0, \ldots, 0, \nu)\|_{R(\mathbf{r}),\delta} < \\
&\quad < \|R(\mathbf{s}) - R(\mathbf{r})\|_{R(\mathbf{r}),\delta}\, \|\mathbf{x}\|_{R(\mathbf{r}),\delta} + \|R(\mathbf{r})\mathbf{x}\|_{R(\mathbf{r}),\delta} + N \leq \\
&\quad \leq \varepsilon\, \|\mathbf{x}\|_{R(\mathbf{r}),\delta} + \delta\, \|\mathbf{x}\|_{R(\mathbf{r}),\delta} + N \leq \frac{N(\varepsilon + \delta)}{1 - \varepsilon - \delta} + N = \frac{N}{1 - \varepsilon - \delta}.
\end{aligned}$$

This proves that $\tau_Q(\mathcal{V}'(Q)) \cup -\tau_Q(-\mathcal{V}'(Q)) \subseteq \mathcal{V}'(Q)$, satisfying (V2). $\Diamond$

The lemma shows that $\mathcal{V}(Q)$ is finite for sufficiently small Q. For our further proceeding it is sufficient to know this. We will avoid to calculate the maximal size of some set Q by using norms we do not know explicitly. Algorithm 1 shows how a calculation of the set of witnesses could look like. The algorithm starts with

Algorithm 1 Calculation of $\mathcal{V}(Q)$.

Input: Q, p

Output: $\mathcal{V}$ set of witnesses of the set Q

$\mathcal{V} \leftarrow \{\pm(\delta_{1j}, \dots, \delta_{dj}) | j = 1, \dots, d\}$
$M \leftarrow \emptyset$
while $\mathcal{V} \neq M$ **do**
  if $\#\mathcal{V} > p$ **then**
    Return(Overflow)
  end if
  $N \leftarrow \mathcal{V} \setminus M$
  $M \leftarrow \mathcal{V}$
  for all $(x_1, \dots, x_d) \in N$ **do**
    $i \leftarrow \min_{(r_1,\dots,r_d)\in Q} \lfloor -\sum_{k=1}^{d} x_k r_k \rfloor$
    $j \leftarrow \max_{(r_1,\dots,r_d)\in Q} -\lfloor \sum_{k=1}^{d} x_k r_k \rfloor$
    $\mathcal{V} \leftarrow \mathcal{V} \cup \{(x_2, \dots, x_d, k) | k = i, \dots, j\}$
  end for
end while

$$\mathcal{V}_0(Q) := \{\pm(\delta_{1i}, \delta_{2i} \dots, \delta_{di}) | i = 1, \dots, d\} \tag{2.1}$$

and calculates inductively $\mathcal{V}_1(Q), \mathcal{V}_2(Q), \dots$ by applying the rule

$$\mathcal{V}_{i+1}(Q) := \tau_Q(\mathcal{V}_i(Q)) \cup -\tau_Q(-\mathcal{V}_i(Q)) \cup \mathcal{V}_i(Q). \tag{2.2}$$

For all i we have $\mathcal{V}_i(Q) \subseteq \mathcal{V}(Q)$. Hence, for a finite set $\mathcal{V}(Q)$, there is a $j \in \mathbb{N}$ such that $\mathcal{V}_{j+1}(Q) = \mathcal{V}_j(Q) = \mathcal{V}(Q)$. To avoid problems with the possible infiniteness of the set, we use an additional input parameter p. If the size of the set of witnesses exceeds p, the process stops and the algorithm returns an overflow. We will deal with this later. At the moment it is more interesting, how the minima and maxima can be calculated. This depends on the way, how the algorithm is implemented. `Mathematica`®, for instance, offers the possibility to minimise or maximise a given function under some conditions. Hence it suffices to let `Mathematica`® search for the maximum and minimum of the function $\mathbf{rx}$ under the condition $\mathbf{r} \in Q$. Without this possibility, we have other methods to get the maximum and minimum. Define $Q_{\mathbf{x}} \subset Q$ to be the set of those $\mathbf{r}$, where $\mathbf{rx}$ is extreme. Because $\mathbf{rx}$ is linear and Q is closed, we have $Q_{\mathbf{x}} \subset \partial Q$ for each $\mathbf{x}$. The easiest case is, when Q is a polygon. Then $Q_{\mathbf{x}}$ consists of its vertices. But also for nonpolygonial Q with differentiable curves as boundaries, it should be no problem to calculate $Q_{\mathbf{x}}$. With the usage of $Q_{\mathbf{x}}$, the rule of calculating $\mathcal{V}_{i+1}(Q)$ from $\mathcal{V}_i(Q)$ changes to

$$\begin{aligned}&\mathcal{V}_{i+1}(Q) := \\ &:= \bigcup_{\mathbf{x}\in\mathcal{V}_i(Q)} \left\{(x_2,\dots,x_d,j) \,\middle|\, j = \min_{\mathbf{r}\in Q_{\mathbf{x}}} \lfloor -\mathbf{rx}\rfloor,\dots,\max_{\mathbf{r}\in Q_{\mathbf{x}}} -\lfloor \mathbf{rx}\rfloor\right\} \cup \mathcal{V}_i(Q),\end{aligned} \tag{2.3}$$

where $\mathbf{x} = (x_1,\dots,x_d)$. The construction of the graph $G(\mathcal{W},Q) = V \times E$ for a set $\mathcal{W} \subset \mathbb{Z}^d$ runs analogously, at least the calculation of the set of vertices V. For this purpose we have to modify algorithm 1 only little. At first replace all $\mathcal{V}$ by V because we have the set of vertices V as output. Besides the input of the additional parameter $\mathcal{W}$ replace line 1 by

$V \leftarrow \mathcal{W}$

and line 10 by

$i \leftarrow \min_{(r_1,\dots,r_d)\in Q} -\lfloor \sum_{k=1}^{d} x_k r_k \rfloor$.

Again the algorithm builds up V inductively by starting with

$$V_0 = \mathcal{W} \tag{2.4}$$

and observing the rule

$$V_{i+1} = \bigcup_{\mathbf{x}\in V_i} \left\{(x_2,\dots,x_d,j) \,\middle|\, j = \min_{\mathbf{r}\in Q_{\mathbf{x}}} -\lfloor \mathbf{rx}\rfloor,\dots,\max_{\mathbf{r}\in Q_{\mathbf{x}}} -\lfloor \mathbf{rx}\rfloor\right\} \cup V_i \tag{2.5}$$

with $\mathbf{x}$ denoting the vector $(x_1,\dots,x_d)$. As soon as $V_{i+1} = V_i$ we set $V = V_i$. Of course, all remarks about the extremes of the function $\mathbf{rx}$ for $\mathbf{r} \in Q$ are also valid in this context. For each vertex it is not hard to get the list of outgoing edges. We omit an explicit calculation here. It depends on the later process whether it is needed or not.

Let us make a few remarks on the finiteness of $G(\mathcal{W},Q)$. It is an easy exercise to prove it in an analogous way as in Lemma 2.2 for a sufficiently small closed convex set $Q \subset \mathcal{E}_d$ and a finite set $\mathcal{W}$. As in Def. 2.1 Condition (V2) obviously is stronger than Condition (G2), we can expect that there are weaker requirements for $G(\mathcal{W},Q)$ to be finite.

In the further proceeding we are going to calculate this graph with $Q \subset \subset \mathcal{D}_2$ and $Q \cap \partial\mathcal{D}_2 \neq \emptyset$ and (fortunately) it is finite there. It is an interesting but up to now unsolved question what the exact conditions are. However, it is not obsolete to keep a (generously) bound p while calculating $G(\mathcal{V}, Q)$ to ensure that the algorithm stops even if the graph were infinite.

The cyclic structure of $G(\mathcal{W}, Q)$ is very important. In order to get an uniform nomenclature, we give a short summary of basic graph theoretical definitions. For a graph with set of vertices V and set of edges $E \subset V \times V$, we call a sequence $v_0 \to v_2 \to \ldots \to v_{l-1} \to v_l$ with $v_i \in V, i = 0 \ldots, l$ and $(v_i, v_{i+1}) \in E, i = 0, \ldots, l-1$ a path. A path with $v_l = v_0$ is said to be a closed path. A closed path is a cycle if all v_i are distinct (except v_0 and v_l). A closed path which is no cycle obviously includes (at least two) cycles. For a closed path $\mathbf{v}_0 \to \mathbf{v}_2 \to \to \ldots \to \mathbf{v}_{l-1} \to \mathbf{v}_0$ of $G(\mathcal{W}, Q)$ we may ask if there is an $\mathbf{r}$ with $\tau_\mathbf{r} : : \mathbf{v}_0 \mapsto \mathbf{v}_2 \mapsto \ldots \mapsto \mathbf{v}_{l-1} \mapsto \mathbf{v}_0$, hence if the closed path induces a period of length l of $\tau_\mathbf{r}$ for some $\mathbf{r}$. It is easy to see that this is only possible for cycles otherwise we would have edges $\mathbf{v}_m \to \mathbf{v}_{m+1}$ and $\mathbf{v}_n \to \mathbf{v}_{n+1}$ for some $m, n \leq l-1$ with $\mathbf{v}_m = \mathbf{v}_n$ and $\mathbf{v}_{m+1} \neq \mathbf{v}_{n+1}$. A function $\tau_\mathbf{r}$ which had a period deduced from this closed path would have to map $\mathbf{v}_m = \mathbf{v}_n$ onto the two different points $\mathbf{v}_{m+1}$ and $\mathbf{v}_{n+1}$, which is impossible. Thus only cycles can induce nonempty periods. The set of all these $\mathbf{r}$ is the solution of the system of inequalities (1.2) induced by the cycle. In many cases $G(\mathcal{W}, Q)$ contains only the trivial cycle, i.e., the self-loop $\mathbf{0} \to \mathbf{0}$.

Lemma 2.3. *Suppose the graph $G(\mathcal{W}, Q) = V \times E$ has n cycles without the self-loop $\mathbf{0} \to \mathbf{0}$. These cycles induce n periods $\pi_1, \ldots, \pi_n$. Then*

$$\forall \mathbf{r} \in \left(Q \setminus \bigcup_{i=1}^{n} P(\pi_i) \right) \forall \mathbf{x} \in V (\supseteq \mathcal{W})\, \exists k \in \mathbb{N} : \tau_\mathbf{r}^k(\mathbf{x}) = \mathbf{0}.$$

In particular, if $G(\mathcal{W}, Q)$ contains no cycle except the trivial one then

$$\forall \mathbf{r} \in Q\, \forall \mathbf{x} \in V\, \exists k \in \mathbb{N} : \tau_\mathbf{r}^k(\mathbf{x}) = \mathbf{0}.$$

Proof. From the definition of $G(\mathcal{W}, Q)$ and the monotonicity of the floor-function it is easy to see that $\forall \mathbf{r} \in Q, \forall \mathbf{x} \in V, \forall k \in \mathbb{N} : \tau_\mathbf{r}^k(\mathbf{x}) \in V$ and $\forall \mathbf{r} \in Q, \forall \mathbf{x} \in V : (\mathbf{x}, \tau_\mathbf{r}(\mathbf{x})) \in E$. Hence each $\mathbf{r}$ with $\tau_\mathbf{r}$ having a period with elements in V produces a cycle in $G(\mathcal{W}, Q)$. Therefore, if we remove these $\mathbf{r}$ from Q, the remaining part has no cycles and so fulfils the stated condition. $\Diamond$

The lemma is similar to [3, Lemma 4.7]. Note that if we find such a period π with nonempty set $P(\pi)$, the sets $P(\pi)$ and D_d^0 are disjoint. Then π is a period of $\mathcal{D}_d^0$ as it is described in the introduction. With the above lemma we can find some areas within a convex set, which are not in $\mathcal{D}_d^0$, but we cannot be sure that the rest is. But the lemma provides a base for other methods.

2.2. Brunotte's Theorem. The next theorem is an algorithm based on Brunotte. For proofs, exact background and adaption for SRS, see [4] and [1, Sec. 5]. Here we will present only the important facts and state them in a somewhat different way. This helps us to generalise the existing results in order to give an improved analysis of $\mathcal{D}_2^0$.

Theorem 2.4. *Let $Q \subset \mathcal{D}_d^0$ be a sufficiently small, closed, convex set. Then $G(\mathcal{V}(Q), Q) = V \times E$ is finite. Furthermore we have $Q \subset \mathcal{D}_d^0$ if for all $\mathbf{r} \in Q, \mathbf{x} \in \mathcal{V}(Q)$ there exists a $k \in \mathbb{N}$ with $\tau_{\mathbf{r}}^k = \mathbf{0}$. Otherwise*

$$\mathcal{D}_d^0 \supset Q \setminus \bigcup_{\pi \in \Pi_Q} P(\pi),$$

where Π_Q is the set of all periods which are described by the nontrivial cycles of $G(\mathcal{V}(Q), Q)$.

Proof. Obviously $V = \mathcal{V}(Q)$ and, according to Lemma 2.2, $\mathcal{V}(Q)$ is finite for sufficiently small sets Q. The remaining part of the proof runs exactly as in [1, Th. 5.2]. $\Diamond$

[1, Th. 5.2] treated only the case that Q is a polyhedron. Indeed, it is easier to handle this case, but as mentioned above we do not have to restrict on it. The theorem can be used to determine whether a given area is in $\mathcal{D}_d^0$ or not. If it is not, we get a set of periods $\Pi(Q)$, whose corresponding cutout polyhedra completely describe $\mathcal{D}_d^0 \cap Q$ in the sense that

$$\mathcal{D}_d^0 \supset Q \setminus \bigcup_{\pi \in \Pi_Q} P(\pi).$$

Note that Π can contain periods whose cutout polyhedra are empty or do not intersect with Q. Further Π in general does not contain all periods that have polyhedra intersecting with Q. Algorithm 2 shows a scheme of a corresponding algorithm. We bypass the problem of the eventually infiniteness of the set of witnesses by giving a bound for its size. If it gets bigger the calculation stops and Q is divided into two parts and the algorithm is applied on each of them separately. The only way of division to keep convexity is by a line. We know that the set of witnesses is finite for sufficiently small sets. We only have to pay attention on the choice

Algorithm 2 Search for the periods describing $Q \cap \mathcal{D}_d^0$ (recursively).

Input: Q

Output: Π_Q list of cycles

```
1:  p ← suitable bound
2:  calculate V(Q) (algorithm 1 with boundary p)
3:  if ¬(overflow) then
4:      E ← set of edges of G(V(Q), Q)
5:      Π_Q ← Π_Q ∪ all cycles of G(V(Q), Q)
6:  else
7:      Split Q into sets Q_1, Q_2
8:      Search for the periods describing Q_1 ∩ D_d^0 (algorithm 2)
9:      Search for the periods describing Q_2 ∩ D_d^0 (algorithm 2)
10: end if
```

of p. Since it is difficult to extract from Lemma 2.2 the exact size of the set of witnesses, it turned out to be the best to increase p whenever Q is split. Using this way, we can ensure that eventually the procedure will terminate. Concrete values for p depend on the location of Q relatively to the boundary of D_d^0 (trial and error).

We have to make a few remarks on finding cycles in a directed graph. In general our graph will have only few edges compared to the number of vertices. Cycles can only occur within the strongly connected components. They can be found with the aid of an algorithm of Tarjan [12]. Its requirements in time and space is linear to the size of V and E. Once the strongly connected components are found, we can extract the cycles from each such component.

Each cycle we find forms a system of inequalities which describes a polyhedron. We may also find cycles, where the corresponding polyhedron is the empty set. Thus we need an algorithm, which identifies these empty cycles and transforms the other ones into polyhedra. Such an algorithm was implemented by Fukuda and is available, beside other algorithms concerning polyhedral computation, as *cdd/cdd+* on the homepage [5]. For our purpose the version working with rational arithmetics is needed. The program is based on the Double Description Method of Motzkin at al. [7]. For further information consult the documentation, which also can be found at [5].

We now consider the two-dimensional case and $\mathcal{D}_2^0$. Write $\mathrm{Br}(Q)$ for

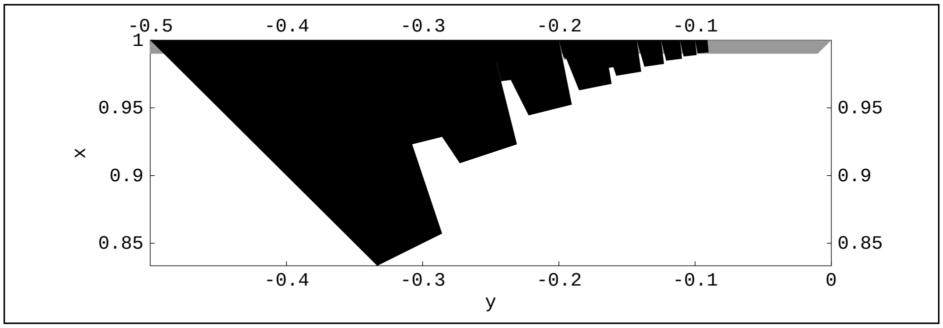

Figure 2. Cutouts of the set Q_A

the application of algorithm 2 on the set Q. As environment define $m_1 := \max_{(x,y)\in Q}\{x\} - \min_{(x,y)\in Q}\{x\}$, $m_2 := \max_{(x,y)\in Q}\{y\} - \min_{(x,y)\in Q}\{y\}$, $p := \frac{20}{\max(m_1,m_2)}$ and the split of the set Q into Q_1 and Q_2 by

$$Q_1 = \begin{cases} \left\{(x,y)\in Q \,\middle|\, x \leq \frac{\max_{(x,y)\in Q}\{x\}+\min_{(x,y)\in Q}\{x\}}{2}\right\} & m_1 > m_2 \\ \left\{(x,y)\in Q \,\middle|\, y \leq \frac{\max_{(x,y)\in Q}\{y\}+\min_{(x,y)\in Q}\{y\}}{2}\right\} & \text{otherwise} \end{cases},$$

$$Q_2 = \begin{cases} \left\{(x,y)\in Q \,\middle|\, x \geq \frac{\max_{(x,y)\in Q}\{x\}+\min_{(x,y)\in Q}\{x\}}{2}\right\} & m_1 > m_2 \\ \left\{(x,y)\in Q \,\middle|\, y \geq \frac{\max_{(x,y)\in Q}\{y\}+\min_{(x,y)\in Q}\{y\}}{2}\right\} & \text{otherwise} \end{cases}.$$

Hence we halve the set in x-direction when its x-expanse is bigger than its y-expanse, otherwise we halve the set in y-direction.

With this we have the following results:

Theorem 2.5. $\mathrm{Br}(Q_A)$ *terminates for*

$$Q_A := \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \frac{5}{6} \leq x \leq \frac{99}{100} \wedge -\frac{x}{2} \leq y \leq x-1\right\}$$

yielding 402 *nonempty periods.*

Theorem 2.6. $\mathrm{Br}(Q_B)$ *terminates for*

$$Q_B := \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \frac{5}{6} \leq x \leq \frac{99}{100} \wedge -x+1 \leq y \leq x\right\}$$

yielding 1010 *nonempty periods.*

Theorem 2.7. $\mathrm{Br}(Q_C)$ *terminates for*

$$Q_C := \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \frac{5}{6} \leq x \leq \frac{99}{100} \wedge -x+2 \leq y \leq 1+\frac{x}{2}\right\}$$

yielding 787 *nonempty periods.*

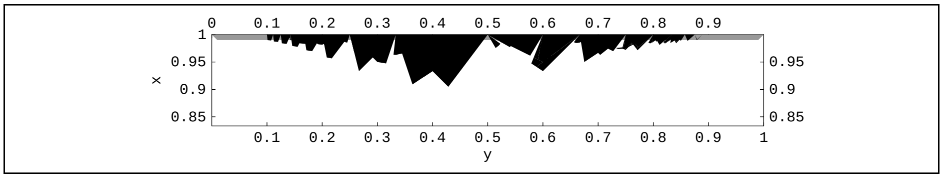

Figure 3. Cutouts of the set Q_B

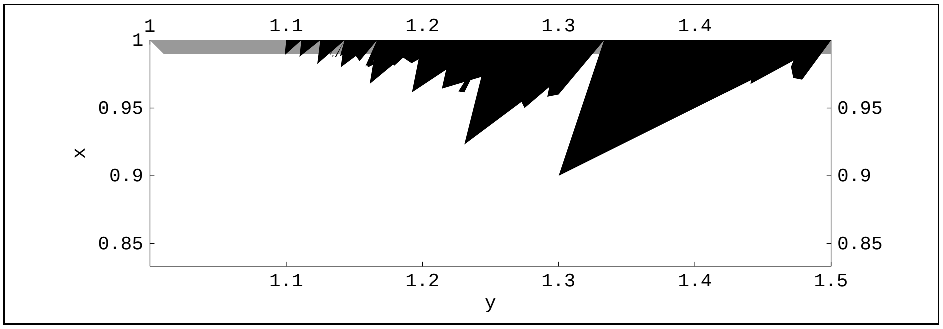

Figure 4. Cutouts of the set Q_C

The sets Q_A, Q_B, Q_A have a minimum distance of 0.01 from the boundary of $\mathcal{D}_2$. Figures 2–4 show all found cutout polyhedra located in the sets in black. To save space, the axes have been reversed. The white spaces are in $\mathcal{D}_2^0$, the dark grey ones are still unknown. Note that only those parts of the polygons are shown that intersect with $\mathcal{D}_2$, e.g. the parts where $x \leq 1$ and that there may be cutouts totally overlaid by others. For computational processing, the lists of all periods are available in the internet, as well as an implementation in Mathematica® of the presented algorithms (2 dimensional case) [11].

We will use $\mathrm{Br}(Q)$ for the last time with modified $p := \frac{200}{\max(d_1,d_2)}$.

Theorem 2.8. *Let* $x_0 = \frac{5}{6}$, $x_1 = \frac{17}{20}$, $x_2 = \frac{9}{10}$, $x_3 = \frac{19}{20}$ *and*

$$Q_{D_i} := \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x \geq \frac{y^2}{4} \wedge y \geq 2x \wedge x_{i-1} \leq x \leq x_i \right\}.$$

Then, for $i=1,2,3$ *the algorithm* $\mathrm{Br}(Q_{D_i})$ *terminates yielding that* $Q_D := \bigcup_{i=1}^{3} Q_{D_i} \subset \mathcal{D}_2^0$.

The choice of Q_D is not arbitrary, we will understand it in the further context.

2.3. A lemma of Akiyama et al. In [3], Akiyama et al. presented a way of analysing an area near the upper boundary of $\mathcal{D}_2$. Let

$$R := \left\{ \mathbf{r} \in \mathbb{R}^2 \middle| x > 0 \wedge y < x + 1 \wedge x < \frac{y^2}{4} \right\}.$$

Further, for $\kappa \in (0, 1)$ and $q \in \mathbb{N}$, define

$$R_\kappa := \left\{ \mathbf{r} \in R \middle| x < \kappa y - \kappa^2 \right\}$$

and

$$A_{\kappa,q} := \left\{ (a, b) \in \mathbb{Z}^2 \middle| |a| < q, -\frac{\kappa}{1 - \kappa^2} - q + 1 < b < \frac{1}{1 - \kappa^2} + q - 1 \right\}.$$

Following [3, Sec. 4.1] we have

Lemma 2.9. *Let γ_q be the positive root of the polynomial $qt^3 + qt^2 - qt - q + 1$ and $0 < \kappa \leq \gamma_q < 1$. Then, for $\mathbf{r} \in R_\kappa$,*

$$\mathbf{r} \in \mathcal{D}_2^0 \Leftrightarrow \forall \mathbf{x} \in A_{\kappa,q} \exists k : \tau_{\mathbf{r}}^k(\mathbf{x}) = \mathbf{0}.$$

For the proof see [3, Lemma 4.3]. In view of Lemma 2.3 this motivates the following

Lemma 2.10. *Let γ_q be the positive root of the polynomial $qt^3 + qt^2 - qt - q + 1$, $0 < \kappa \leq \gamma_q < 1$ and let Π_Q be the set of all periods induced by the nontrivial cycles of the graph $G(A_{\kappa,q}, Q)$. Then*

$$(Q \cap \bigcup_{0<\iota\leq\kappa} R_\iota) \setminus \bigcup_{\pi \in \Pi_Q} P(\pi) \subset \mathcal{D}_2^0.$$

Proof. It follows immediately from Lemma 2.3 and Lemma 2.9 that

$$(R_\iota \cap Q) \setminus \bigcup_{\pi \in \Pi_Q} P(\pi) \subset \mathcal{D}_2^0$$

for $\iota = \kappa$. Because $A_\iota \subset A_\kappa$ for $\iota < \kappa$ the statement is true for all $\iota \leq \kappa$. ◊

This lemma provides a powerful tool for determining which areas of R are contained in $\mathcal{D}_2^0$. Because of the concavity of R it may happen that Q is not fully contained in R. Whenever parts of the line $y = x + 1$ are included in Q, $G(A_{\kappa,q}, Q)$ will contain a lot of cycles of the form $(a, -a)^T \to (-a, a)^T \to (a, -a)^T$. These cycles induce the period $\rho = (a, -a)$ that corresponds exactly to the line $y = x + 1$ which we already know not to be part of $\mathcal{D}_2^0$ (see [3, Lemma 2.3]). Based on this considerations we can specify Algorithm 3. Line 4 is only of importance if Q contains parts of the line $y = x + 1$. Referring to the remarks on the finiteness of the graph, its size should be bounded during the calculation in Line 1. We omitted such a bound here, because the graphs we need are all finite. Denote the application of algorithm 3 with parameters Q and q by $\mathrm{Ak}(Q, q)$.

Algorithm 3 Search for all periods within an area $Q \cap \bigcup_{0<\iota<\gamma_q} R_\iota$.

Input: Q, q
Output: Π_Q list of cycles
calculate the set of vertices V of $G(A_{\gamma_q,q}, Q)$ (modified algorithm 1)
calculate the set of edges E of $G(A_{\gamma_q,q}, Q)$
$\Pi_Q \leftarrow$ all cycles of $G(A_{\gamma_q,q}, Q)$
remove all cycles of the shape $(a, -a) \rightarrow (-a, a) \rightarrow (a, -a)$ from Π_Q

In [3, Sec. 4.1] the set R was analysed for $x \leq 5/6$ without having found any cutout. Additionally the sets R_{γ_i} for $i = 3, \ldots, 6$ have also been recognised not to have cutouts ([3, Th. 4.8]). It is possible to continue this series. For $i = 3, \ldots, 11$ fix rational numbers γ_i' with $\gamma_i' < \gamma_i$ and $|\gamma_i' - \gamma_i| < 10^{-6}$ and for $j = 4, \ldots, 11$ fix rational numbers γ_j'' with $\gamma_j'' > \gamma_j$ and $|\gamma_j'' - \gamma_j| < 10^{-6}$.

Theorem 2.11. $\mathrm{Ak}(\overline{R_{\gamma_i''} \setminus R_{\gamma_{i-1}'}}, i)$ *terminates for* $i = 7, \ldots, 11$ *yielding no periods.*

The approximation of γ_i by rational numbers is necessary for the calculation by a computer to avoid rounding errors. Th. 2.11 as well as [3, Th. 4.8] left some areas between the sets R_{γ_i} and the parable $x = \frac{y^2}{4}$ uninvestigated. These "holes" are very close to the boundary of $\mathcal{D}_2$ causing troubles for an analysis by Algorithm 2. This explains the choice of Q_D in Th. 2.8. We investigate these areas by another usage of Algorithm 3.

Theorem 2.12. *For* $i = 4, \ldots, 11$ *let*

$$E_i = \left\{(x, y) \in R \,\middle|\, x \geq \gamma_{i-1}' y - (\gamma_{i-1}')^2, x \geq \gamma_i'' y - (\gamma_i'')^2 \right\}.$$

$\mathrm{Ak}(E_i, i)$ *terminates yielding no periods.*

Summing up we gain

Theorem 2.13. $\bigcup_{0<\kappa\leq\gamma_{11}} R_\kappa \supset \left\{(x, y) \in R \,\middle|\, x \leq \frac{19}{20}\right\}$ *is fully contained in* $\mathcal{D}_2^0$.

3. An area near the point (1, 1)

3.1. Statement of the theorem. For areas near the boundary of $\mathcal{D}_2$ an application of the presented algorithms fails. Often the only possible way to prove a set Q to be a subset of $\mathcal{D}_2^0$ is to do this directly by observing

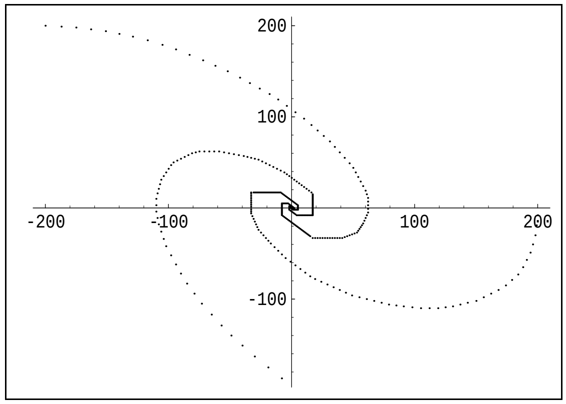

Figure 5. The orbit of the point $(-200, 200)$

the behaviour of the orbits of the mapping $\tau_{\mathbf{r}}$ for $\mathbf{r} \in Q$. An example for the usage of this strategy is [3, Subsec. 4.2], where it is shown that a small area near the point $(1, -1)$ is a subset of $\mathcal{D}_2^0$. In that style, we will prove the following

Theorem 3.1.

$$P := \left\{ (1 - T, 1 + \delta T) \,\middle|\, 0 < T < \frac{1}{30}, 0 \leq \delta \leq 1 \right\} \subset \mathcal{D}_2^0.$$

Proof. Fix $T \in (0, \frac{1}{30})$ and $\delta \in [0, 1]$. Let $\mathbf{r} = (1 - T, 1 + \delta T) \in P$. Furthermore define

$$\begin{aligned} A &:= \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x \leq 0, y < 0 \right\}, \\ B &:= \left\{ (x, y) \in \mathbb{Z}^2 \,|\, x \geq 0, y > 0 \right\}. \end{aligned}$$

These sets represent the third quadrant with the negative y-axis and the first quadrant with the positive y-axis, respectively. We will prove the statement by showing that $\tau_{\mathbf{r}}^p$ sends each point of $\mathbb{Z}^2$ to $\mathbf{0}$ for some $p \in \in \mathbb{N}$. The idea is not very complicated, but several technical lemmas are needed. These lemmas are proven afterwards. Everything is based on the fact that the application of $\tau_{\mathbf{r}}^3$ changes a point only little. Fig. 5 shows the orbit of the point $(-200, 200)$ for $T = \frac{1}{50}$ and $\delta = 1$. It is divided into three branches. After three applications of $\tau_{\mathbf{r}}$ we return to one branch.

We now look at the sequence $\{\tau_{\mathbf{r}}^n(\mathbf{z})\}_{n \in \mathbb{N}}$ of a point $\mathbf{z} \in \mathbb{Z}^2$. We will show the existence of a finite subsequence $\{\mathbf{z}_0, \ldots, \mathbf{z}_{q_0}\}$ that ends up in $\mathbf{0}$. This proves the theorem. We first assert that each point in $\mathbb{Z}^2$ has an orbit that intersects with $A \cup B \cup \{\mathbf{0}\}$. This is shown in Lemma 3.12. Hence, without loss of generality, we can start our subsequence with $\mathbf{z}_0 \in A$ (for B the proof runs analogously). For a $\mathbf{z}_q \in A$, $q > 0$, construct $\mathbf{z}_{q+1}$ in the following way: Let $(u_0, v_0) := \mathbf{z}_q$. For an $i \geq 0$ set $(u_{i+1}, v_{i+1}) := \tau_{\mathbf{r}}^3(u_i, v_i)$. Then for $(u_i, v_i) \in A$ the following points (which are shown in the mentioned lemmas) are true:

$$u_{i+1} \leq 0 \quad \text{(Lemma 3.2)}, \tag{3.1}$$

$$u_{i+1} + v_{i+1} \geq -\|(u_i, v_i)\|_1 \quad \text{(Lemma 3.4)}, \tag{3.2}$$

$$v_{i+1} - v_i \geq 1 \quad \text{(Lemma 3.6)}. \tag{3.3}$$

Formula (3.3) ensures that there is no repetition possible and hence there cannot exist a period within the set A. By (3.1) and (3.3) can further be seen that either $(u_{i+1}, v_{i+1}) \in A$ or $v_{i+1} \geq 0$. Thus there exists a $j \in \mathbb{N}$ with $(u_i, v_i) \in A$ for $i \leq j$ and $(u_{j+1}, v_{j+1}) \notin A$ where (u_{j+1}, v_{j+1}) lies on or above the x-axis. Additionally the length of (u_i, v_i) is not growing with respect to the 1-norm. Now apply $\tau_{\mathbf{r}}$ once. Then Lemma 3.8 says that either $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) = \mathbf{0}$ or $\tau_{\mathbf{r}}(u_{j+1}, v_{j+1}) \in B$. Moreover we always have $\|\tau_{\mathbf{r}}(u_{j+1}, v_{j+1})\|_1 \leq \|(u_j, v_j)\|_1$ which is shown in Lemma 3.10. Now, if $(x_0, y_0) = \mathbf{0}$, set $\mathbf{z}_{q+1} := \mathbf{0}$. Otherwise we proceed in an analogous manner as before for the set B. Start with $(x_0, y_0) := \tau_{\mathbf{r}}(u_{j+1}, v_{j+1})$ and define $(x_{k+1}, y_{k+1}) := \tau_{\mathbf{r}}^3(x_k, y_k), k \geq 0$. Then for each $(x_k, y_k) \in B$ we have

$$x_{k+1} \geq 0 \quad \text{(Lemma 3.3)}, \tag{3.4}$$

$$x_{k+1} + y_{k+1} \leq \|(x_k, y_k)\|_1 \quad \text{(Lemma 3.5)}, \tag{3.5}$$

$$y_{k+1} - y_k \leq -1 \quad \text{(Lemma 3.7)}. \tag{3.6}$$

Thus, again, there exists an $l \in \mathbb{N}$ with $(x_k, y_k) \in B$ for $k \leq l$ and $(x_{l+1}, y_{l+1}) \notin B$. (3.5) ensures that $\|(x_{k+1}, y_{k+1})\|_1 \leq \|(x_k, y_k)\|_1$ for $k < l$. We set $\mathbf{z}_{q+1} := \tau_{\mathbf{r}}(x_{l+1}, y_{l+1})$ and see that $\mathbf{z}_{q+1} \in A$ or $\mathbf{z}_{q+1} = \mathbf{0}$ (Lemma 3.9) and this time $\|\mathbf{z}_{q+1}\|_1 < \|(x_l, y_l)\|_1$ (Lemma 3.11). Hence we have

$$\begin{aligned} \|\mathbf{z}_q\|_1 &= \|(u_0, v_0)\|_1 \leq \|(u_1, v_1)\|_1 \leq \ldots \leq \|(u_j, v_j)\|_1 \\ &\leq \|(x_0, y_0)\|_1 \leq \|(x_1, y_1)\|_1 \leq \ldots \leq \|(x_l, y_l)\|_1 < \|\mathbf{z_{q+1}}\|_1 . \end{aligned}$$

It is easy to see that any $\mathbf{z}_q$ is a member of our sequence $\{\tau_{\mathbf{r}}^n \mathbf{z}\}_{n \in \mathbb{N}}$ and there exists an $q_0 > 0$ with $\|\mathbf{z}_0\|_1 < \|\mathbf{z}_1\|_1 < \ldots < \|\mathbf{z}_{q_0}\|_1 = 0$. Hence $\{\mathbf{z}_0, \ldots, \mathbf{z}_{q_0}\}$ really ends up in $\mathbf{0}$. ♢

3.2. Supporting lemmas. We need some preparing definitions. Let

$(u, v) \in \mathbb{Z}^2$. Using the abbreviations

$$\begin{aligned} \iota(u,v) &:= v\delta T - uT, \\ \kappa(u,v) &:= -u\delta T - v(T+\delta T) - \lfloor \iota(u,v) \rfloor \delta T, \\ \lambda(u,v) &:= u(T+\delta T) + vT + \lfloor \iota(u,v) \rfloor (T+\delta T) - \lfloor \kappa(u,v) \rfloor \delta T \end{aligned}$$

yields

$$\begin{aligned} \tau_{\mathbf{r}}(u,v) &= (v, -u-v-\lfloor \iota(u,v) \rfloor), \\ \tau_{\mathbf{r}}^2(u,v) &= (-u-v-\lfloor \iota(u,v) \rfloor, u + \lfloor \iota(u,v) \rfloor - \lfloor \kappa(u,v) \rfloor), \\ \tau_{\mathbf{r}}^3(u,v) &= (u + \lfloor \iota(u,v) \rfloor - \lfloor \kappa(u,v) \rfloor, v + \lfloor \kappa(u,v) \rfloor - \lfloor \lambda(u,v) \rfloor). \end{aligned}$$

For some proofs it is better to choose another representation. By direct calculation we gain

$$\tau_{\mathbf{r}}^3((u,v)) = (u + \alpha_1 u + \alpha_2 v + \alpha_3, v + \beta_1 u + \beta_2 v + \beta_3) \tag{3.7}$$

with

$$\begin{aligned} \alpha_1 &:= T(-1+\delta) - T^2\delta, \\ \alpha_2 &:= T(1+2\delta) + T^2\delta^2, \\ \beta_1 &:= T(-1-2\delta) + T^2(1+2\delta-\delta^2) + T^3\delta^2, \\ \beta_2 &:= T(-2-\delta) + T^2(-2\delta-3\delta^2) - T^3\delta^3, \\ \alpha_3 &:= (-1-\delta T)\{\iota(u,v)\} + \{\kappa(u,v)\}, \\ \beta_3 &:= (T+2\delta T+\delta^2T^2)\{\iota(u,v)\} + (-1-\delta T)\{\kappa(u,v)\} + \{\lambda(u,v)\}. \end{aligned}$$

where $\{a\}$ denotes the fractional part of a. These expressions satisfy the following inequalities:

$$-T \le \alpha_1 < 0, \tag{3.8}$$

$$T \le \alpha_2 < 4T, \tag{3.9}$$

$$-3T < \beta_1 < 0, \tag{3.10}$$

$$-4T < \beta_2 \le -2T, \tag{3.11}$$

$$-T \le \alpha_2 + \beta_2 < 0. \tag{3.12}$$

The estimations are partly very crude, but easy to verify and sufficient for our aims. Because of monotonicity the extreme values of α_3 and β_3 can only occur, if $\{\iota(u,v)\}$, $\{\kappa(u,v)\}$ and $\{\lambda(u,v)\}$ take extreme values. From this consideration we gain the following table:

$\{\lambda(u,v)\}$	$\{\kappa(u,v)\}$	$\{\iota(u,v)\}$	α_3	β_3
0	0	0	0	0
0	0	1	$-1-\delta T$	$T+2\delta T+\delta^2T^2$
0	1	0	1	$-1-\delta T$
0	1	1	$-\delta T$	$-1+T+\delta T+\delta^2T^2$
1	0	0	0	1
1	0	1	$-1-\delta T$	$1+T+2\delta T+\delta^2T^2$
1	1	0	1	$-\delta T$
1	1	1	$-\delta T$	$+T+\delta T+\delta^2T^2$

This table shows that

$$-1-\delta T < \alpha_3 < 1, \tag{3.13}$$

$$-1-\delta T < \beta_3 < 1+T+2\delta T+\delta^2T^2, \tag{3.14}$$

$$-1+T+\delta^2T^2 < \alpha_3+\beta_3 < 1. \tag{3.15}$$

Note that $\{\iota(u,v)\},\{\kappa(u,v)\}$ and $\{\iota(u,v)\}$ cannot be equal to 1 hence all inequalities are strict. While proving the lemmas, we always have to keep track of the signs of the α_i and β_i as well as the possible values δ and T can obtain.

Lemma 3.2. *Let* $(u_i,v_i)\in A$ *and* $(u_{i+1},v_{i+1})=\tau_{\mathbf{r}}^3(u_i,v_i)$. *Then* $u_{i+1}\leq 0$.

Proof. $u_{i+1}=u_i+u_i\alpha_1+v_i\alpha_2+\alpha_3=u_i(1+\alpha_1)+v_i\alpha_2+\alpha_3$. By the definition of A we have $u_i\leq 0$ and $v_i<0$. Because v_i is an integer this implies $v_i\leq -1$. $(1+\alpha_1)>0$ and $\alpha_2>0$ by (3.8) and (3.9). Since by (3.13) we have $\alpha_3<1$ we obtain

$$u_{i+1}<-\alpha_2+1=-T(1+2\delta)-T^2\delta^2+1<1.$$

The fact that u_{i+1} is an integer allows the final conclusion $u_{i+1}\leq 0$. ♢

Lemma 3.3. *Let* $(x_k,y_k)\in B$ *and* $(x_{k+1},y_{k+1})=\tau_{\mathbf{r}}^3(x_k,y_k)$. *Then* $x_{k+1}\geq 0$.

Proof. Analogously to Lemma 3.2, by using (3.8), (3.9) and (3.13), we get

$$\begin{aligned} x_{k+1} &= x_k+x_k\alpha_1+y_k\alpha_2+\alpha_3= \\ &= x_k(1+\alpha_1)+y_k\alpha_2+\alpha_3> \\ &> \alpha_2-1-\delta T= \\ &= -1+T(1+\delta)+T^2\delta^2>-1 \end{aligned}$$

and therefore $x_{k+1}\geq 0$. ♢

Lemma 3.4. *Let* $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i)$*. Then*

$$(u_i, v_i) \in A \quad and \quad \|(u_i, v_i)\|_1 = m$$

implies that $u_{i+1} + v_{i+1} \geq -m$*.*

Proof. Since $(u_i, v_i) \in A$ we have $\|(u_i, v_i)\|_1 = -u_i - v_i$. Thus

$$\begin{aligned} u_{i+1} + v_{i+1} &= u_i + u_i\alpha_1 + v_i\alpha_2 + \alpha_3 + v_i + u_i\beta_1 + v_i\beta_2 + \beta_3 > \\ &> u_i(\alpha_1 + \beta_1) + v_i(\alpha_2 + \beta_2) - m - 1 + T + \delta^2 T^2 \end{aligned}$$

where (3.15) gives the lower bound for $\alpha_3 + \beta_3$. Considering (3.8), (3.9) and (3.12) yields $u_{i+1} + v_{i+1} > -m - 1$ and for integer values u_{i+1}, v_{i+1}, m we get $u_{i+1} + v_{i+1} \geq -m$. ♢

Lemma 3.5. *Let* $(x_{k+1}, y_{k+1}) = \tau_{\mathbf{r}}^3(x_k, y_k)$*. If* $(x_k, y_k) \in B$ *and* $\|(x_k, y_k)\|_1 = m$*, then* $x_{k+1} + y_{k+1} \leq m$*.*

Proof. Since $(x_k, y_k) \in B$ we have $\|(x_k, y_k)\|_1 = x_k + y_k$. Again (3.8), (3.9), (3.12) and (3.15) are used for the following estimation.

$$\begin{aligned} x_{k+1} + y_{k+1} &= x_k + x_k\alpha_1 + y_k\alpha_2 + \alpha_3 + y_k + x_k\beta_1 + y_k\beta_2 + \beta_3 < \\ &< x_k(\alpha_1 + \beta_1) + y_k(\alpha_2 + \beta_2) + m + 1 < \\ &< m + 1 \end{aligned}$$

and thus $x_{k+1} + y_{k+1} \leq m$. ♢

Lemma 3.6. *Let* $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i)$*. Then* $(u_i, v_i) \in A$ *implies that* $v_{i+1} - v_i \geq 1$*.*

Proof. Since $(u_i, v_i) \in A$ we have $u_i \leq 0$ and $v_i < 0$. Thus

$$\begin{aligned} \kappa(u_i, v_i) &= -u_i\delta T - v_i(T + \delta T) - \lfloor \iota(u_i, v_i) \rfloor \delta T \geq \\ &\geq -u_i\delta T - v_i(T + \delta T) - \iota(u_i, v_i)\delta T = \\ &= u_i(-\delta T + \delta T^2) + v_i(-T - \delta T - \delta^2 T^2) > 0 \Rightarrow \lfloor \kappa(u_i, v_i) \rfloor \geq 0 \\ \lambda(u_i, v_i) &= u_i(T + \delta T) + v_i T + \lfloor \iota(u_i, v_i) \rfloor (T + \delta T) - \lfloor \kappa(u_i, v_i) \rfloor \delta T \leq \\ &\leq u_i(T + \delta T) + v_i T + \iota(u_i, v_i)(T + \delta T) = \\ &= u_i(T + \delta T - T^2 - \delta T^2) + v_i(T + \delta T^2 + \delta^2 T^2) \leq \\ &\leq -T - \delta T^2 - \delta^2 T^2 < 0 \Rightarrow \lfloor \lambda(u_i, v_i) \rfloor \leq -1. \end{aligned}$$

Finally the simple computation

$$v_{i+1} - v_i = \lfloor \kappa(u_i, v_i) \rfloor - \lfloor \lambda(u_i, v_i) \rfloor \geq 1$$

shows the statement. ♢

Lemma 3.7. *Let* $(x_k, y_k) \in B$ *and* $(x_{k+1}, y_{k+1}) = \tau_{\mathbf{r}}^3(x_k, y_k)$*. Then* $y_{k+1} - y_k \leq -1$*.*

Proof.

$$\begin{aligned}\kappa(x_k,y_k) =& -x_k\delta T - y_k(T+\delta T) - \lfloor \iota(x_k,y_k)\rfloor \delta T \leq\\ \leq& -x_k\delta T - y_k(T+\delta T) - (\iota(x_k,y_k)-1)\delta T =\\ =& x_k(-\delta T+\delta T^2) + y_k(-T-\delta T-\delta^2T^2) + \delta T \leq\\ \leq& -T-\delta^2T^2 < 0 \Rightarrow \lfloor\kappa(x_k,y_k)\rfloor \leq -1\\ \lambda(x_k,y_k) =& x_k(T+\delta T) + y_kT + \lfloor\iota(x_k,y_k)\rfloor(T+\delta T) - \lfloor\kappa(x_k,y_k)\rfloor\delta T \geq\\ \geq& x_k(T+\delta T) + y_kT + (\iota(x_k,y_k)-1)(T+\delta T) + \delta T =\\ =& x_k(T+\delta T-T^2-\delta T^2) + y_k(T+\delta T^2+\delta^2T^2) - T \geq\\ \geq& \delta T^2+\delta^2T^2 \geq 0 \Rightarrow \lfloor\lambda(x_k,y_k)\rfloor \geq 0.\end{aligned}$$

Hence

$$y_{k+1}-y_k = \lfloor\kappa(x_k,y_k)\rfloor - \lfloor\lambda(x_k,y_k)\rfloor \leq -1. \quad \Diamond$$

Lemma 3.8. *If* $(u_j,v_j)\in A$ *and* $(u_{j+1},v_{j+1}) = \tau_{\mathbf{r}}^3(u_j,v_j)\notin A$ *then* $\tau_{\mathbf{r}}(u_{j+1},v_{j+1})\in B$ *or* $\tau_{\mathbf{r}}(u_{j+1},v_{j+1}) = \mathbf{0}$.

Proof. Let $(u',v') := \tau_{\mathbf{r}}(u_{j+1},v_{j+1})$. We will show that $u'\geq 0$ and $v'>0$. We have $u_{j+1}\leq 0$ (according to Lemma 3.2) and $v_{j+1}>v_j$ (according to Lemma 3.6). Since $(u_{j+1},v_{j+1})\notin A$ we can conclude that $v_{j+1}\geq 0$ and therefore $u' = v_{j+1}\geq 0$. The proof of the other statement requires some more estimations. Set $m := -u_j - v_j$. Suppose first that $m<2$. This is only for $u_j+v_j=-1$ and therefore $u_i=0$ and $v_i=-1$. Then $\lfloor\iota(u_j,v_j)\rfloor\in\{-1,0\}$, $\lfloor\kappa(u_j,v_j)\rfloor = 0$ and $\lfloor\lambda(u_j,v_j)\rfloor = -1$. Hence either $(u_{j+1},v_{j+1}) = \mathbf{0}$, which means that $(u',v') = \mathbf{0}$, or $(u_{j+1},v_{j+1}) = (1,0)$, which implies that $v' = -\lfloor -1+T\rfloor = 1>0$. If $m\geq 2$ then

$$\begin{aligned}u_{j+1} =& u_j(1+\alpha_1)+v_j\alpha_2+\alpha_3 >\\ >& (-v_j-m)(1+\alpha_1)+v_j\alpha_2-1-\delta T =\\ =& v_j(-1-\alpha_1+\alpha_2)-m(1+\alpha_1)-1-\delta T.\end{aligned}$$

Note that $(u_j,v_j)\in A$ and so $v_j\leq -1<0$. Thus

$$\begin{aligned}u_{j+1} \geq& 1+\alpha_1-\alpha_2-m-m\alpha_1-1-\delta T =\\ =& -m+\alpha_1(1-m)-\alpha_2-\delta T \geq\\ \geq& -m-\alpha_2-\delta T.\end{aligned}$$

Since u_{j+1} and m are integers, the conclusion

$$u_{j+1}\geq -m \tag{3.16}$$

holds. Furthermore by (3.7) and (3.15)

$$\begin{aligned} u_{j+1}+v_{j+1} &= u_j(1+\alpha_1+\beta_1)+v_j(1+\alpha_2+\beta_2)+\alpha_3+\beta_3 < \\ &< (-m-v_j)(1+\alpha_1+\beta_1)+v_j(1+\alpha_2+\beta_2)+1 = \\ &= -m+1-m\alpha_1-m\beta_1+v_j(\alpha_2+\beta_2-\alpha_1-\beta_1). \end{aligned}$$

Inserting $v_j \le -1$ and using (3.8)–(3.11) yields

$$\begin{aligned} u_{j+1}+v_{j+1} &\le -m+1-m\alpha_1-m\beta_1-(\alpha_2+\beta_2-\alpha_1-\beta_1) = \\ &= (-m+1)(1+\alpha_1+\beta_1)-\alpha_2-\beta_2 \le \\ &\le (-m+1)(1-4T)+3T = \\ &= m(-1+4T)+1-T. \end{aligned}$$

Together with (3.16) this implies

$$\begin{aligned} u_{j+1}(1-T) &+ v_{j+1}(1+\delta T) \le \\ &\le u_{j+1}(1-T)+(m(-1+4T)+1-T-u_{j+1})(1+\delta T) = \\ &= m(-1+4T)(1+\delta T)+(1-T)(1+\delta T)-u_{j+1}(T+\delta T) \le \\ &\le m(-1+4T)(1+\delta T)+(1-T)(1+\delta T)+m(T+\delta T) = \\ &= m(-1+5T+4\delta T^2)+1-T+\delta T-\delta T^2 \le \\ &\le -2+10T+8\delta T^2+1-T+\delta T-\delta T^2 = \\ &= -1+9T+\delta T+7\delta T^2 < 0 \end{aligned}$$

and therefore $v' = -\lfloor u_{j+1}(1-T)+v_{j+1}(1+\delta T)\rfloor \ge 1 > 0$. Hence $\tau_{\mathbf{r}}(u_{j+1},v_{j+1}) = (u',v')$ is really inside B, when it is not $\mathbf{0}$. ◊

Lemma 3.9. *$(x_l,y_l) \in B$ and $(x_{l+1},y_{l+1}) = \tau_{\mathbf{r}}^3(x_l,y_l) \notin B$ implies that $\tau_{\mathbf{r}}(x_{l+1},y_{l+1}) \in A$ or $\tau_{\mathbf{r}}(x_{l+1},y_{l+1}) = \mathbf{0}$.*

Proof. Let $(x',y') := \tau_{\mathbf{r}}(x_{l+1},y_{l+1})$. Analogously to Lemma 3.8 we have to show that $x' \le 0$ and $y' < 0$. The claim $(x_{l+1},y_{l+1}) \notin B$ together with Lemma 3.3 and Lemma 3.7 implies that $y_{l+1} \le 0$ and therefore $x' = y_{l+1} \le 0$. The second estimation comes from the following computations: Let $m := x_l + y_l$. Suppose $m < 3$. There are three possibilities:

$\boldsymbol{(x_l, y_l) = (0,1)}$**:**

$$\begin{aligned} \lfloor \iota(0,1)\rfloor &= \lfloor \delta T\rfloor = 0 \\ \lfloor \kappa(0,1)\rfloor &= \lfloor -T-\delta T\rfloor = -1 \\ \lfloor \lambda(0,1)\rfloor &= \lfloor T+\delta T\rfloor = 0 \end{aligned}$$

and therefore $(x_{l+1},y_{l+1}) = (0+0+1, 1-1+0) = (1,0)$ and further $(x',y') = \tau_r(1,0) = (0, -\lfloor(1-T)\rfloor) = \mathbf{0}$.

$(x_l, y_l) = (1,1)$:

$$\begin{aligned}\lfloor \iota(1,1)\rfloor &= \lfloor \delta T - T\rfloor \in \{0,-1\}\\ \lfloor \kappa(1,1)\rfloor &= \lfloor -T - 2\delta T - \lfloor \iota(1,1)\rfloor \delta T\rfloor = -1\\ \lfloor \lambda(1,1)\rfloor &= \lfloor 2T + 2\delta T + \lfloor \iota(1,1)\rfloor (T+\delta T)\rfloor = 0.\end{aligned}$$

So either $(x_{l+1}, y_{l+1}) = (1,0)$, which goes to $\mathbf{0}$ by the calculation above, or $(x_{l+1}, y_{l+1}) = (2,0)$ and $y' = -\lfloor 2-2T\rfloor = -1 < 0$.

$(x_l, y_l) = (0,2)$:

$$\begin{aligned}\lfloor \iota(0,2)\rfloor &= \lfloor 2\delta T\rfloor = 0\\ \lfloor \kappa(0,2)\rfloor &= \lfloor -2T - 2\delta T\rfloor = -1\\ \lfloor \lambda(0,2)\rfloor &= \lfloor 2T + \delta T\rfloor = 0.\end{aligned}$$

This yields $(x_{l+1}, y_{l+1}) = (1,1)$. This case does not fulfill the condition $(x_{l+1}, y_{l+1}) \notin B$. Thus it is irrelevant for the present lemma.

Now let $m \geq 3$. Note that always $y_l \geq 1$.

$$\begin{aligned}x_{l+1} &= x_l(1+\alpha_1) + y_l\alpha_2 + \alpha_3 <\\ &< (-y_l + m)(1+\alpha_1) + y_l\alpha_2 + 1 =\\ &= y_l(-1-\alpha_1+\alpha_2) + m(1+\alpha_1) + 1 \leq\\ &\leq -1 - \alpha_1 + \alpha_2 + m + m\alpha_1 + 1 =\\ &= m + \alpha_1(m-1) + \alpha_2 \leq\\ &\leq m + \alpha_2.\end{aligned}$$

Again x_{l+1} and m are integers and from there follows

$$x_{l+1} \leq m. \tag{3.17}$$

Analogously to Lemma 3.8, we need a lower bound for $x_{l+1} + y_{l+1}$:

$$\begin{aligned}x_{l+1} + y_{l+1} &= x_l(1+\alpha_1+\beta_1) + y_l(1+\alpha_2+\beta_2) + \alpha_3 + \beta_3 >\\ &> (m - y_l)(1+\alpha_1+\beta_1) + y_l(1+\alpha_2+\beta_2) - 1 + T + \delta^2T^2 \geq\\ &\geq m - 1 + m\alpha_1 + m\beta_1 + y_l(\alpha_2+\beta_2-\alpha_1-\beta_1) + T \geq\\ &\geq m - 1 + m\alpha_1 + m\beta_1 + (\alpha_2+\beta_2-\alpha_1-\beta_1) + T =\\ &= (m-1)(1+\alpha_1+\beta_1) + \alpha_2 + \beta_2 + T \geq\\ &\geq (m-1)(1-4T) - 2T =\\ &= m(1-4T) - 1 + 2T\end{aligned}$$

where (3.8)–(3.11) are used to gain the last lines. With the help of these two results we show the estimation

$$\begin{aligned} x_{l+1}(1-T) + y_{l+1}(1+\delta T) &\geq \\ &\geq x_{l+1}(1-T) + (m(1-4T) - 1 + 2T - x_{l+1})(1+\delta T) = \\ &= m(1-4T)(1+\delta T) + (-1+2T)(1+\delta T) - x_{l+1}(T+\delta T) \geq \\ &\geq m(1-4T)(1+\delta T) + (-1+2T)(1+\delta T) - m(T+\delta T) = \\ &= m(1-5T-4\delta T^2) - 1 + 2T - \delta T + 2\delta T^2 \geq \\ &\geq 3 - 15T - 12\delta T^2 - 1 + 2T - \delta T + 2\delta T^2 = \\ &= 2 - 13T - \delta T - 10\delta T^2 > 1. \end{aligned}$$

Therefore $y' = -\lfloor x_{l+1}(1-T) + y_{l+1}(1+\delta T)\rfloor \leq -1 < 0$. ♢

Lemma 3.10. *If $(u_i, v_i) \in A$ and $(u_{i+1}, v_{i+1}) = \tau_{\mathbf{r}}^3(u_i, v_i) \notin A$ then we have $\|\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})\|_1 \leq \|(u_i, v_i)\|_1 = m$.*

Proof. Let $(u', v') := \tau_{\mathbf{r}}(u_{i+1}, v_{i+1})$. Lemma 3.8 says that $\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})$ is element of B and therefore $\|\tau_{\mathbf{r}}(u_{i+1}, v_{i+1})\|_1 = u' + v'$, while the inside A lying point (u_i, v_i) induces the condition $u_i + v_i = -m$. According to Lemma 3.4, $u_{i+1} + v_{i+1} \geq -m$ is valid, although the point is not an element of A.

$$\begin{aligned} u' + v' &= v_{i+1} - u_{i+1} - v_{i+1} - \lfloor -u_{i+1}T + v_{i+1}\delta T\rfloor = \\ &= -\lfloor u_{i+1}(1-T) + v_{i+1}\delta T\rfloor \leq \\ &\leq -\lfloor u_{i+1}(1-T) + (-u_{i+1} - m)\delta T\rfloor = \\ &= -\lfloor u_{i+1}(1-T-\delta T) - m\delta T\rfloor. \end{aligned}$$

Using (3.16) yields

$$\begin{aligned} u' + v' &\leq -\lfloor -m(1-T-\delta T) - m\delta T\rfloor \leq \\ &\leq m - \lfloor mT\rfloor \leq m \end{aligned}$$

and shows that $u' + v' \leq m$ holds. ♢

Lemma 3.11. *If $(x_l, y_l) \in B$ and $(x_{l+1}, y_{l+1}) = \tau_{\mathbf{r}}^3(x_l, y_l) \notin B$ then we have $\|\tau_{\mathbf{r}}(x_{l+1}, y_{l+1})\|_1 < \|(x_l, y_l)\|_1 = m$.*

Proof. Let $(x', y') = \tau_{\mathbf{r}}(x_{l+1}, y_{l+1})$. Referring to Lemma 3.9 we have $(x', y') \in A$ and therefore $\|(x', y')\|_1 = -x' - y'$. On the other hand $\|(x_l, y_l)\|_1 = x_l + y_l =: m$. According to Lemma 3.5, $x_{l+1} + y_{l+1} \leq m$ holds, although the point is not an element of B.

$$\begin{aligned} x' + y' &= y_{l+1} - x_{l+1} - y_{l+1} - \lfloor -x_{l+1}T + y_{l+1}\delta T\rfloor = \\ &= -\lfloor x_{l+1}(1-T) + y_{l+1}\delta T\rfloor \geq \\ &\geq -\lfloor x_{l+1}(1-T) + (-x_{l+1} + m)\delta T\rfloor = \\ &= -\lfloor x_{l+1}(1-T-\delta T) + m\delta T\rfloor. \end{aligned}$$

Now we use (3.17) to get

$$\begin{aligned} x' + y' \geq & -\lfloor m(1 - T - \delta T) + m\delta T \rfloor \geq \\ & \geq -m - \lfloor -mT \rfloor \geq -m + 1. \end{aligned}$$

This shows the validity of $\|(x', y')\|_1 \leq m - 1 < m$. $\Diamond$

Lemma 3.12. *Let $(u, v) \in \mathbb{Z}^2$. Then there is an $i \in \mathbb{N}$ with either $\tau_{\mathbf{r}}^i(u, v) \in A \cup B$ or $\tau_{\mathbf{r}}^i(u, v) = \mathbf{0}$.*

Proof. Consider the line $x + y + \iota(x, y) = 0$. It runs through the origin and splits the second quadrant into two pieces for each possible T and δ. It allows the partition of $\mathbb{Z}^2$ into $\mathbf{0}$ and the sets

$$\begin{aligned} B &:= \left\{(x, y) \in \mathbb{Z}^2 \,|x \geq 0, y > 0\right\}, \\ A &:= \left\{(x, y) \in \mathbb{Z}^2 \,|x \leq 0, y < 0\right\}, \\ U_1 &:= \left\{(x, y) \in \mathbb{Z}^2 \,|x < 0, y \geq 0, x + y + \iota(x, y) < 0\right\}, \\ U_2 &:= \left\{(x, y) \in \mathbb{Z}^2 \,|x < 0, y \geq 0, x + y + \iota(x, y) \geq 0\right\}, \\ U_3 &:= \left\{(x, y) \in \mathbb{Z}^2 \,|x > 0, y \leq 0\right\}. \end{aligned}$$

There are the following cases:

$\boldsymbol{(u, v) \in U_1}$**:** We have $v \geq 0$ and $-\lfloor u + v + \iota(u, v)\rfloor \geq 1 > 0$. Thus $\tau_{\mathbf{r}}(u, v) = (v, -\lfloor u + v + \iota(u, v)\rfloor) \in B$.

$\boldsymbol{(u, v) \in U_2}$**:** (u, v) cannot be an element of the x-axis. Suppose $v = 1$ and $u \leq -2$. Then $u + v + \iota(u, v) \leq -2 + 2T + 1 + \delta T = -1 - 2T - \delta T < 0$ shows that (u, v) does not lie in U_2. If $u = -1$ then $(u, v) = (-1, 1)$. This point is an element of U_2. $\tau_{\mathbf{r}}(-1, 1) = (1, -1 + 1 - \lfloor T + \delta T\rfloor) = (1, 0)$ and $\tau_{\mathbf{r}}(1, 0) = (0, -1 - \lfloor -T\rfloor) = \mathbf{0}$ shows that this point goes to $\mathbf{0}$ after 2 applications of $\tau_{\mathbf{r}}$. For the rest of U_2 we can assume $u \leq -1$ and $v \geq 2$.

$$\begin{aligned} v + u\beta_1 + v\beta_2 + \beta_3 \geq\, & 2 + 2\beta_2 - \beta_1 - 1 - \delta T = \\ =\, & 1 - T + T^2(-1 - 4\delta - 2\delta^2) + T^3(-\delta^2 - \delta^3) > 0 \end{aligned}$$

and therefore $v + u\beta_1 + v\beta_2 + \beta_3 \geq 1$. Further we have

$$\begin{aligned} \iota(u, v) =\, & v\delta T - uT \geq 2\delta T + T > 0 \Rightarrow \lfloor \iota(u, v)\rfloor \geq 0, \\ \kappa(u, v) =\, & -u\delta T - v(T + \delta T) - \lfloor \iota(u, v)\rfloor \delta T = \\ =\, & -vT - \lfloor u + v + \iota(u, v)\rfloor \delta T \leq \\ \leq\, & -vT < 0 \end{aligned}$$

which shows that $\lfloor \kappa(u, v)\rfloor \leq -1$. Furthermore $\lfloor \iota(u, v)\rfloor - \lfloor \kappa(u, v)\rfloor \geq 1$. Thus we can conclude that the point

$$\tau_{\mathbf{r}}^3(u, v) = (u + \lfloor \iota(u, v)\rfloor - \lfloor \kappa(u, v)\rfloor, v + \beta_1 u + \beta_2 v + \beta_3)$$

is above the x-axis and right of (u,v). This induces that $\tau_{\mathbf{r}}^m(u,v)$ is either $\mathbf{0}$, an element of U_1 or an element of B for an $m \in \mathbb{N}$.

$\boldsymbol{(u,v) \in U_3}$**:** This implies that $v \le 0$. Thus $\tau_{\mathbf{r}}(u,v) = (v, -u-v- - \lfloor \iota(u,v) \rfloor) \notin U_3$.

Hence $\exists i \in \mathbb{N}$ with $\tau_{\mathbf{r}}^i(u,v) \in A \cup B \cup \{\mathbf{0}\}$ for each $(u,v) \in \mathbb{Z}^2$. $\Diamond$

4. A family of cutouts

In this section we construct a family of pairwise disjoint periods which all yield nonempty cutouts. Afterwards we use this to show that the point $K'_d := (0,\dots,0,1,1)$ is a critical point. At the end we turn to another family of periods. It was presented in [1] and we will provide the shape of the corresponding polygons.

4.1. Quadrangles near the point (1, 1). For each $n \in \mathbb{N}$, $n \ge 1$, consider the period

$$\omega_n := (2n+1,-2n); \bigsqcup_{i=1}^{n} (2i-1, 2n-2i+1, -2n), \bigsqcup_{i=1}^{n-1} (2n+1, -2i, -2n+2i)$$

where $\bigsqcup$ denotes the sequence gained by concatenation, e.g. $\bigsqcup_{i=1}^n a_i = = a_1,\dots,a_n$. For an empty set of indices the corresponding sequence is empty. Hence for each $n \ge 1$ we have a period of length $6n-1$.

$$\begin{aligned}
\omega_1 &= (3,-2); 1, 1, -2,\\
\omega_2 &= (5,-4); 1, 3, -4, 3, 1, -4, 5, -2, -2,\\
\omega_3 &= (7,-6); 1, 5, -6, 3, 3, -6, 5, 1, -6, 7, -2, -4, 7, -4, -2,\\
&\ \ \vdots
\end{aligned}$$

Thus the set $P(\omega_n)$ consists of the points satisfying the following system of inequalities, deduced from (1.2):

$$\begin{aligned}
&(4.1)\quad 0 \le && x - 2ny + 2n + 1 && < 1, &&\\
&(4.2)\quad 0 \le && -2nx + (2n+1)y - 2 && < 1, &&\\
&(4.3)\quad 0 \le && -2x + (2n+1)y - 2n && < 1, &&\\
&(4.4)\quad 0 \le && (2n+1)x - 2ny + 1 && < 1, &&\\
&(4.5)\quad 0 \le && -2nx + (2j+1)y + 2n - 2j - 1 && < 1 && (0 \le j < n),\\
&(4.6)\quad 0 \le && (2j+1)x + (2n-2j-1)y - 2n && < 1 && (0 \le j < n),\\
&(4.7)\quad 0 \le && (2n+1)x - 2jy - 2n + 2j && < 1 && (0 < j < n),\\
&(4.8)\quad 0 \le && -2jx + (-2n+2j)y + 2n + 1 && < 1 && (0 < j < n).
\end{aligned}$$

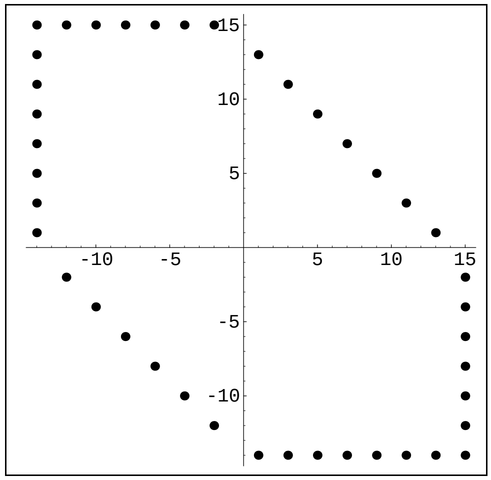

Figure 6. The points of the period ω_7

For $n = 1$, (4.2) and (4.3) are equal and (4.7) as well as (4.8) do not exist. For a point $\mathbf{r} \in P(\omega_n)$, the function $\tau_{\mathbf{r}}$ maps as follows:

$$\begin{aligned}
(1, -2n) &\mapsto (-2n, 2n+1), \\
(-2n, 2n+1) &\mapsto (2n+1, -2), \\
(-2, 2n+1) &\mapsto (2n+1, -2n), \\
(2n+1, -2n) &\mapsto (-2n, 1), \\
(-2n, 2j+1) &\mapsto (2j+1, 2n-2j-1) \quad (0 \le j < n), \\
(2j+1, 2n-2j-1) &\mapsto (2n-2j-1, -2n) \quad (0 \le j < n), \\
(2n+1, -2j) &\mapsto (-2j, -2n+2j) \quad (0 < j < n), \\
(-2j, -2n+2j) &\mapsto (-2n+2j, 2n+1) \quad (0 < j < n).
\end{aligned}$$

Fig. 6 shows these points for $n = 7$.

We will see that the set $P(\omega_n)$ of points satisfying the inequalities (4.1)–(4.8) equals a nonempty polygon for all $n \ge 1$. Let

$$\begin{aligned}
\mathbf{x}_n^{(1)} &:= \left(1, \frac{2n+1}{2n}\right), \\
\mathbf{x}_n^{(2)} &:= \left(\frac{2n(2n+1)}{4n^2+2n-1}, \frac{(2n+1)^2}{4n^2+2n-1}\right), \\
\mathbf{x}_n^{(3)} &:= \left(\frac{2n(2n-1)}{4n^2-2n+1}, \frac{4n^2}{4n^2-2n+1}\right),
\end{aligned}$$

$$\mathbf{x}_n^{(4)} := \begin{cases} \left(\frac{3}{4}, \frac{3}{2}\right) & (n=1) \\ \left(1, \frac{2n}{2n-1}\right) & (\text{otherwise}) \end{cases}.$$

Denote by $\square(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ the convex hull of the points $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Theorem 4.1. *For any $n \geq 1$, $P(\omega_n)$ equals the open set*

$$S := \operatorname{int} \square(\mathbf{x}_n^{(1)}, \ldots, \mathbf{x}_n^{(4)}).$$

Proof. We chose from our list the four right hand (strict) inequalities (4.1), (4.4), (4.5) with $j = n-1$ and (4.6) with $j = 0$. For $n = 1$ take (4.2) instead of the last one. They form a subsystem of the system (4.1)–(4.8). Each of these inequalities describes an open half plane:

$$\begin{aligned} U_n^{(1)} &: \{(x,y)\,|x - 2ny + 2n + 1 < 1\}, \\ U_n^{(2)} &: \{(x,y)\,|(2n+1)x - 2ny + 1 < 1\}, \\ U_n^{(3)} &: \{(x,y)\,|-2nx + (2n-1)y + 1 < 1\}, \\ U_n^{(4)} &: \begin{cases} \{(x,y)\,|-2x + 3y - 2 < 1\} & (n=1) \\ \{(x,y)\,|x + (2n-1)y - 2n < 1\} & (\text{otherwise}) \end{cases}. \end{aligned}$$

Obviously we have $P(\omega_n) \subset \bigcap_{i=1}^4 U_n^{(i)}$. The lines

$$\begin{aligned} g_n^{(1)} &: & x - 2ny + 2n &= 0, \\ g_n^{(2)} &: & (2n+1)x - 2ny &= 0, \\ g_n^{(3)} &: & -2nx + (2n-1)y &= 0, \\ g_n^{(4)} &: & &\begin{cases} -2x + 3y - 3 = 0 & (n=1) \\ x + (2n-1)y - 2n - 1 = 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

are the boundary lines of these half planes: $g_n^{(i)}$ bound $U_n^{(i)}$ for $i = 1, \ldots, 4$. Now it is quickly verified that

$$\begin{aligned} x_n^{(1)} &= g_n^{(1)} \wedge g_n^{(2)},\ x_n^{(1)} \in U_n^{(3)} \cap U_n^{(4)}, \\ x_n^{(2)} &= g_n^{(2)} \wedge g_n^{(4)},\ x_n^{(2)} \in U_n^{(1)} \cap U_n^{(3)}, \\ x_n^{(3)} &= g_n^{(1)} \wedge g_n^{(3)},\ x_n^{(3)} \in U_n^{(2)} \cap U_n^{(4)}, \\ x_n^{(4)} &= g_n^{(3)} \wedge g_n^{(4)},\ x_n^{(4)} \in U_n^{(1)} \cap U_n^{(2)}. \end{aligned}$$

This shows that $S = \bigcap_{i=1}^4 U_n^{(i)}$ and thus $S \supset P(\omega_n)$. On the other hand simple calculations show that for each $i = 1, \ldots, 4$, $x_n^{(i)}$ satisfies all the other inequalities of the system (4.1)–(4.8). Hence we also have $S \subset P(\omega_n)$. $\Diamond$

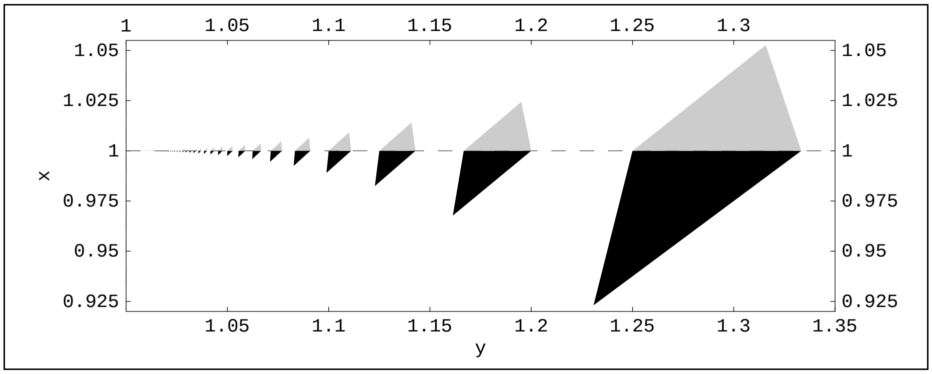

Figure 7. The cutouts $P(\omega_n)$

From the x-values of $\mathbf{x}_n^{(2)}$ and $\mathbf{x}_n^{(3)}$

$$\frac{2n(2n+1)}{4n^2+2n-1} = 1 + \frac{1}{4n^2+2n-1}$$
$$\frac{2n(2n-1)}{4n^2-2n+1} = 1 - \frac{1}{4n^2-2n+1}$$

we see that only a part of $\Pi(\omega_n)$ lies within $\mathcal{D}_2$. Fig. 7 shows the cutout polygons for $n > 1$. Again the axes are reversed to save space. The very big Polygon $P(\omega_1)$ can be seen in Fig. 1. Together with a second one, it forms the rightmost cutout. As it can be seen, the polygons cut out triangles out of $\mathcal{D}_2$ (black).

4.2. A critical point. In [1, Th. 7.5] it was shown that $K_d :=$ $:= (0, \ldots, 0, 1, 0) \in \overline{D_d}$ is a critical point, i.e. for any neighbourhood $U \in \mathcal{D}_d$ infinitely many cutout polyhedra are necessary to gain $U \cap D_d^0$. We will use the results of Subsec. 4.1 to show $K_d' := (0, \ldots, 0, 1, 1) \in \overline{D_d}$ to be a critical point, too. This was already stated in [1, Rem. 7.6] but without being explicitly proven. We will accomplish that proof now. The following lemma is a basic fact:

Lemma 4.2. *Let π be any period. Then* $\operatorname{int} P(\pi) \cap \mathbb{Z}^d = \emptyset$.

Proof. $P(\pi)$ is described be several inequalities of the form

$$0 \le a_1 x_1 + \ldots + a_d x_d + a_{d+1} < 1$$

with integers $a_1, \ldots, a_d$. To ensure that a point is an inner point of $P(\pi)$, both inequalities have to be strict. For a point of $\mathbb{Z}^d$ this is impossible to fulfill. $\Diamond$

Theorem 4.3. *The point K_d' is critical.*

Proof. Because of the lifting Th. [1, Th. 6.2] it suffices to show the assertion only for $K_2' = (1,1)$. From Th. 4.1 we know that we can construct a sequence of points $(x_k, y_k)_{k\in\mathbb{N}}$ converging to K_2' with $x_k < 1$, $y_k > 1$ and $(x_k, y_k) \notin \mathcal{D}_2^0$ for all $k \in \mathbb{N}$. Suppose K_2' were not a critical point. Then there must exist a period $\pi = (a_0, a_1); a_2, \dots, a_{n-1}$ such that $P(\pi)$ includes all but finitely many elements of the sequence. For this period we can deduce the following properties (for the rest of the proof, take all indices modulo n):

(1) $P(\pi) \supset Q := (1-\delta, 1) \times (1, 1+\epsilon)$ for some $\delta, \epsilon > 0$. The set $P(\pi)$ is described by inequalities

$$0 \le a_{i-1}x + a_i y + a_{i+1} < 1$$

with $i \in \{0, \dots, n-1\}$. The points of Q have to suffice all of these inequalities. Together with Lemma 4.2 we gain

$$\begin{array}{lcl} a_i < 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 1, \\ a_i > 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 0, \\ a_i = 0, a_{i-1} < 0 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 0, \\ a_i = 0, a_{i-1} > 1 & \Rightarrow & a_{i-1} + a_i + a_{i+1} = 1. \end{array}$$

Especially we have

(4.9)

$$\begin{array}{lllll} a_i \le 0, a_{i+1} < 0 & \Rightarrow & a_{i+2} = 1 - a_i - a_{i+1} > 0 & \Rightarrow & a_{i+3} = a_i - 1 < 0 \\ & \Rightarrow & a_{i+4} = a_{i+1} + 1 & & \end{array},$$

(4.10)

$$\begin{array}{lllll} a_i \ge 0, a_{i+1} > 0 & \Rightarrow & a_{i+2} = -a_i - a_{i+1} < 0 & \Rightarrow & a_{i+3} = a_i + 1 > 0 \\ & \Rightarrow & a_{i+4} = a_{i+1} - 1 & & \end{array}.$$

(2) π has to include zeros. To see this, we first note that π consists of positive and negative numbers. Summing up over all triples $a_{i-1} + a_i + a_{i+1}$ and observing the rules from (1) yields

$$3 \sum_{i=1}^{n} a_i = |\{ i \mid a_i < 0 \vee a_i = 0, a_{i-1} > 1 \}| > 0.$$

For any $i \in \{0, \dots, n-1\}$ we have

$$\begin{array}{ccccl} a_i & + a_{i+1} & + a_{i+2} & & \in \{0,1\} \\ & - a_{i+1} & - a_{i+2} & - a_{i+3} & \in \{0,-1\} \\ \hline a_i & & & - a_{i+3} & \in \{0,1,-1\}. \end{array}$$

Therefore, the element a_{i+3} differs from the element a_i by one at most. If $n \not\equiv 0(3)$, the elements of π can be rearranged as

$$a_0, a_3, \dots, a_{3j+3(\mathrm{mod}\, n)}, \dots, \ (0 \le j \le n-1).$$

Observing that this list includes positive numbers as well as negative ones and that neighboured elements differ by one at most shows that necessarily it has to include at least one zero. In the case $n \equiv 0(\mathrm{mod}\, 3)$ we can make such a rearrangement for each of the sets $A_k := \{a_i | i \equiv \equiv k(\mathrm{mod}\, 3)\}$, $k = 0, 1, 2$ separately. Suppose all of these sets consist of equal signed integers only. Then there must be at least one such set including only positive numbers, say A_1. The calculation

$$0 < \sum_{i=0}^{n-1} a_i = \sum_{j=0}^{n/3-1} (a_{3j} + a_{3j+1} + a_{3j+2}) = 0,$$

where $a_{3j+1} \in A_1$ for all j, shows the impossibility of this. Hence A_1 has to include both positive and negative numbers and therefore zeros too.

We are now at the point to prove that there cannot exist a period π performing the desired properties. We do this by constructing a sequence $(b_i)_{i \ge 0}$ of integers sufficing points (1) and (2) and showing that this sequence ends up in a series of zeros. Without loss of generality we may start with $b_0 = 0$. Suppose $b_1 = k > 0$. Successive application of (4.10) shows $b_{3k} = k > 0$ and $b_{3k+1} = 0$. Hence $b_{3k+2} = -k + 1$. Now we apply (4.9) sufficiently often to see that $b_{6k-2} = -k + 1, b_{6k-1} = 0$. This implies $b_{6k} = k - 1$. Repeating this procedure we gain $b_{3k^2+2k-1} = 0$ and $b_{3k^2+2k} = 0$. Similar calculations for the case $k < 0$ yield the same result finishing the proof. $\Diamond$

Conjecture 4.4. K and K' are the only critical points in the 2-dimensional case.

4.3. The shape of another family of cutouts. In [1] the family of periods

$$\zeta_n = (n+1, 1); -n, -1, n, \bigsqcup_{i=2}^{n} (i, -n-1+i, -i, n+1-i)$$

of nonempty cutout polyhedra was investigated. Contrary to the cutouts $P(\omega_n)$, which are located in the upper half plane, we have $y < 0$ for $P(\zeta_n)$. But the shape of these cutouts has not been investigated yet. Therefore we will add an analogue analysis for the periods ζ_n here.

We have the pairs of inequalities

$$(4.11)\quad 0 \le \quad x + (n+1)y + 1 \quad < 1,$$

$$(4.12)\quad 0 \le \quad jx + (j-n-1)y - j \quad < 1 \quad (0 < j \le n),$$

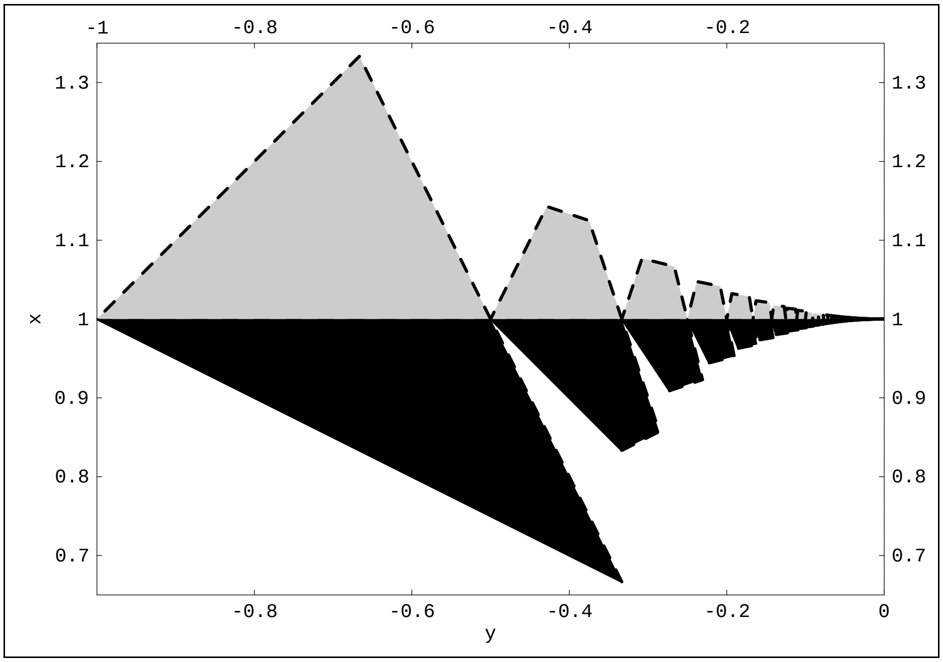

Figure 8. The cutouts $P(\zeta_n)$

$$
\begin{array}{llcll}
(4.13) & 0 \leq & (j-n-1)x - jy + n + 1 - j & < 1 & (0 < j \leq n), \\
(4.14) & 0 \leq & -jx + (n+1-j)y + (j+1) & < 1 & (0 < j \leq n), \\
(4.15) & 0 \leq & (n+1-j)x + (j+1)y + j - n & < 1 & (0 \leq j < n).
\end{array}
$$

Note that the length of ζ_n is $4n+1$ and that we have exactly one inequality of each type for the case $n = 1$, which we will see to behave a little different again. We direct our attention to five lines, for $n = 1$ to three lines, respectively. They are deduced from the strict side of inequality (4.11) and (4.12) with $j = 1$, and the not strict side of inequality (4.15) with $j = n - 1$ and additionally, for $n > 1$, the strict sides of (4.14) with $j = n$ and (4.15) with $j = n - 1$. In particular we have:

$$
\begin{aligned}
h_1^{(1)} &: x + 2y = 0, \\
h_1^{(2)} &: x - y = 2, \\
h_1^{(3)} &: 2x + y = 1
\end{aligned}
$$

and for $n > 1$

$$
h_n^{(1)} : (n+1)x + y = n + 1,
$$

$$h_n^{(2)} : x - ny = 2,$$
$$h_n^{(3)} : 2x + ny = 1,$$
$$h_n^{(4)} : nx - y = n,$$
$$h_n^{(5)} : x + (n+1)y = 0.$$

Further define the points:

$$\mathbf{y}_n^{(1)} := \left(\frac{n^2+n+2}{n^2+n+1}, -\frac{n+1}{n^2+n+1}\right),$$
$$\mathbf{y}_n^{(2)} := \left(1, -\frac{1}{n}\right),$$
$$\mathbf{y}_n^{(3)} := \left(\frac{n^2+1}{n^2+2}, -\frac{n}{n^2+2}\right),$$
$$\mathbf{y}_n^{(4)} := \left(\frac{n(n+1)}{n^2+n+1}, -\frac{n}{n^2+n+1}\right),$$
$$\mathbf{y}_n^{(5)} := \left(\frac{(n+1)^2}{n(n+2)}, -\frac{n+1}{n(n+2)}\right).$$

The points $\mathbf{y}_1^{(1)}$ and $\mathbf{y}_1^{(5)}$ as well as $\mathbf{y}_1^{(3)}$ and $\mathbf{y}_1^{(4)}$ are identical, such that there are only the three different points $\mathbf{y}_1^{(i)}, i = 1, 2, 3$ for the case $n = 1$.

Theorem 4.5.

$$P(\zeta_1) = \square(\mathbf{y}_1^{(1)}, \mathbf{y}_1^{(2)}, \mathbf{y}_1^{(3)}) \setminus (h_1^{(1)} \cup h_1^{(2)}),$$
$$P(\zeta_n) = \square(\mathbf{y}_n^{(1)}, \dots, \mathbf{y}_n^{(5)}) \setminus (h_n^{(1)} \cup h_n^{(2)} \cup h_n^{(4)} \cup h_n^{(5)}), \quad n > 1.$$

Proof (sketch). For $n > 1$ we have $g_n^{(i)} \wedge g_n^{(i+1)} = \mathbf{y}_n^{(i)}$ for $i = 1, \dots, 5$ (upper indices are taken modulo 5) and $g_1^{(i)} \wedge g_1^{(i+1)} = \mathbf{y}_1^{(i)}$ for $i = 1, 2, 3$ (upper indices are taken modulo 3), respectively. Therefore the five (three, resp.) lines bound the stated area. Only $h_n^{(3)}$ is deduced from a not strict inequality, all other lines come from strict ones. Hence these lines have to be removed. Additionally all points satisfy the rest of the inequalities. $\Diamond$

ζ_1 describes a triangle, the others form pentagons. In Fig. 8 this is shown graphically, starting with $P(\zeta_1)$ on the left (reversed axes). As easy can be verified, ζ_n cuts out a quadrangle from $\mathcal{D}_2$ for $n > 1$, for $n = 1$ it is an triangle (black parts).

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