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A NOTE ON NORMAL NUMBERS IN MATRIX NUMBER SYSTEMS

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Abstract: We consider a generalization of normal numbers to matrix number systems. In particular we show that the analogue of the Theorem of Copeland and Erdős also holds in this setting. As a consequence, this generalization holds true also for canonical number systems.

1. Introduction

Among the properties of number systems mainly investigated are finiteness, periodicity and randomness. In this paper we concentrate on the third property. When dealing with an infinite expansion, one is interested whether a certain block of digits will occur asymptotically equally often. If this is true for every possible block, then this number is called normal to that number system.

The first two questions coming up are how many normal numbers are there and how to construct such a normal number. For the first it is known that almost every real number is normal to a given base $q \geq 2$.

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The second one was first answered by Champernowne [1] who was able to show that the concatenation of the integers, i.e.,

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\ldots$$

is normal in the decimal system ($q = 10$). He conjectures that the same holds true for the concatenation of the primes. Copeland and Erdős [2] even proved more that this. They were able to show that one can construct a normal number with any increasing sequence which satisfies a certain restraint in connection with its growth rate (see Lemma 5.1). Ways to construct numbers as concatenation of the integer part of polynomials were considered by several authors in [2, 3, 14, 15]. Finally a construction by the integer part of entire functions of bounded logarithmic growth is given by Madritsch et al. in [12].

In this paper we want to generalize the result of Copeland and Erdős to matrix number systems (MNS). These number systems are strongly connected with canonical number systems (CNS). Knuth [9] was one of the first who considered CNS for the Gaussian integers when he was investigating the properties of the "twin-dragon" fractal. These considerations were extended to quadratic number fields by Kátai, Kovács, and Szabó [6, 7, 8]. The extension to the integral domains of algebraic number fields was shown by Kovács and Pethő in [11]. The connection of MNS and CNS is based on the following observation by Kovács [10]: if β is a base of a CNS in a number field then $\{1, \beta, \ldots, \beta^{n-1}\}$ forms an integral basis for this number field. Furthermore the connection of MNS to lattice tilings was worked out for instance by Gröchenig and Haas in [4].

2. Definitions of number systems and normality

As these definitions are standard in this area, we mainly follow [13].

Let $B \in \mathbb{Z}^{n\times n}$ be an expanding matrix (i.e., its eigenvalues have all modulus greater than 1). Let $\mathcal{D} \subset \mathbb{Z}^n$ be a complete set of residues mod B with $0 \in \mathcal{D}$. We call the pair $(B, \mathcal{D})$ a *(matrix) number system* if every $m \in \mathbb{Z}^n$ admits a representation of the form

$$m = \sum_{j=0}^{k} B^j a_j, \quad (a_j \in \mathcal{D}).$$

We set $\ell(m) := k + 1$ for the *length* of m. As $\mathcal{D}$ is a complete set of residues modulo B, this representation is unique and we furthermore get that $|\mathcal{D}| = [\mathbb{Z}^n : B\mathbb{Z}^n] = |\det B| > 1$.

By $\mathcal{F} := \left\{ \sum_{j \geq 1} B^{-j} a_j : a_j \in \mathcal{D} \right\}$ we denote the *fundamental domain* of $(B, \mathcal{D})$. Furthermore for every $a \in \mathbb{Z}^n$, we denote by $\mathcal{F}_a := B^{-\ell(a)}(\mathcal{F} + a)$ the elements of $\mathcal{F}$ whose $(B, \mathcal{D})$ expansion starts with the same digits as a. For $\alpha \in \mathbb{R}^n$ with $\alpha = \sum_{j=-\infty}^{k} B^j a_j$, we denote by

$$\lfloor \alpha \rfloor := \sum_{j=0}^{k} B^j a_j, \quad \{\alpha\} := \sum_{j \geq 1} B^{-j} a_j,$$

the integral and the fractional part of α, respectively.

For our generalization it is not necessary that $(B, \mathcal{D})$ is a number system. We are interested in a wider class of pairs $(B, \mathcal{D})$, which Indlekofer et al. [5] call *just touching covering systems* (JTCS). A pair $(B, \mathcal{D})$ is a JTCS if

$$\lambda((m_1 + \mathcal{F}) \cap (m_2 + \mathcal{F})) = 0, \quad (m_1 \neq m_2, \quad m_1, m_2 \in \mathbb{Z}^n)$$

where λ denotes the n-dimensional Lebesgue measure.

Now we are ready to define normal numbers in $(B, \mathcal{D})$. Let $\theta \in \mathcal{F}$, then we denote by $\mathcal{N}(\theta; a, N)$ the number of blocks in the first N digits of θ which are equal to the expansion of a. Thus

$$\mathcal{N}(\theta; a, N) := \left| \{0 \leq n < N : \{B^n \theta\} \in \mathcal{F}_a\} \right|.$$

We call $\theta \in \mathcal{F}$ *normal in* $(B, \mathcal{D})$ if for every $k \geq 1$

$$\sup_{\ell(a)=k} \left| \mathcal{N}(\theta; a, N) - \frac{N}{|\mathcal{D}|^k} \right| = o(N), \tag{2.1}$$

where the supremum is taken over all $a \in \mathbb{Z}^n$ whose $(B, \mathcal{D})$ expansion has length k.

As the representation of an element is not necessarily unique in a JTCS, we have to define and to consider ambiguous expansions. Later we will show that an element with an ambiguous expansion cannot be normal.

3. Numbering the elements of a JTCS

To show the structure of elements of $(B, \mathcal{D})$ we mainly follow [13]. First we define the map

$$\begin{aligned} \Phi : \mathbb{Z}^n &\to \mathbb{Z}^n \\ x &\mapsto B^{-1}(x - a) \end{aligned}$$

where $a \in \mathcal{D}$ is the representative of the congruence class of x (i.e., $x - a \in B\mathbb{Z}^n$).

We define $\mathcal{P} := \left\{m \in \mathbb{Z}^n : \exists k \in \mathbb{N} : \Phi^k(m) = m\right\}$ to be the set of *periodic* elements, which is finite (cf. [13]). Now we construct a unique representation of every $m \in \mathbb{Z}^n$. Therefore let $r = r(m) \geq 0$ be the least integer such that $\Phi^r(m) = p \in \mathcal{P}$. Then every $m \in \mathbb{Z}^n$ has a unique representation as follows:

$$m = \sum_{j=0}^{r-1} B^j a_j + B^r p \quad (a_j \in \mathcal{D}, p \in \mathcal{P})$$

with $\Phi^{r-1}(m) = a_{r-1} + Bp \notin \mathcal{P}$ if $r \geq 1$.

We denote by

$$\mathcal{R} := \left\{\sum_{j=0}^{k} B^j a_j : k \geq 0, a_j \in \mathcal{D}\right\}$$

the set of all *properly representable* elements of $\mathbb{Z}^n$.

We want to define an ordering on this set. Therefore let $q := |\det B|$ and let τ be a bijection from $\mathcal{D}$ to $\{0, \ldots, q-1\}$ such that $\tau(0) = 0$. Then we extend τ on $\mathcal{R}$ by setting $\tau(a_k B^k + \cdots + a_1 B + a_0) := \tau(a_k)q^k + \cdots + \tau(a_1)q + \tau(a_0)$. We also pull back the relation $\leq$ from $\mathbb{N}$ to $\mathcal{R}$ by setting

$$a \preceq b :\Leftrightarrow \tau(a) \leq \tau(b) \quad (a, b \in \mathcal{R}). \tag{3.1}$$

Then we define a sequence $\{z_i\}_{i\geq 0}$ of elements in $\mathcal{R}$ with $z_i := \tau^{-1}(i)$. This sequence is increasing, i.e., $i \leq j \Rightarrow \ell(z_i) \leq \ell(z_j)$ and $z_i \preceq z_j$ for $i, j \in \mathbb{N}$.

Now we can state our main result.

Theorem 3.1. *Let $(B, \mathcal{D})$ be a JTCS and let $\{a_i\}_{i\geq 0}$ be an increasing subsequence of $\{z_i\}_{i\geq 0}$. If for every $\varepsilon > 0$ the number of a_i with $a_i \preceq z_N$ exceeds N^ε for N sufficiently large, then*

$$\theta = 0.[a_0][a_1][a_2][a_3][a_4][a_5][a_6][a_7]\cdots$$

is normal in $(B, \mathcal{D})$ where $[\cdot]$ denotes the expansion in $(B, \mathcal{D})$.

Before we state the proof of the theorem we have to exclude the case that θ is ambiguous (i.e., has two different representations). In the next section we will show that any $\theta \in \mathcal{F}$ with two different representations cannot be normal.

4. Ambiguous expansions in JTCS

We call a $\theta \in \mathcal{F}$ *ambiguous* (with ambiguous expansion) if there exists a $l \geq 0$ such that

$$\{B^l\theta\} \in \partial\mathcal{F}. \tag{4.1}$$

In the following lines we will justify our definition. If a $\theta \in \mathcal{F}$ has two different expansions this means that there exist $l \geq 1$ and $a_i, b_i \in \mathcal{D}$ for $i = 1, 2, \ldots$ with

$$\theta = \sum_{i=1}^{\infty} B^{-i} a_i = \sum_{i=1}^{\infty} B^{-i} b_i \quad \text{and} \quad a_l \neq b_l.$$

This equals saying that there exist an $m \in \mathbb{Z}^n$ and a $l \geq 0$ such that

$$\{B^l\theta\} \in \mathcal{F} \cap (m + \mathcal{F}).$$

We set $S := \{m \in \mathbb{Z}^n \setminus \{0\} : \mathcal{F} \cap (m + \mathcal{F}) \neq \emptyset\}$, $S_0 := S \cup \{0\}$, $B_m := \mathcal{F} \cap (m + \mathcal{F})$. By Lemma 3.1 of [13] we see that

$$\partial\mathcal{F} = \bigcup_{m \in S} B_m.$$

Thus all $\theta \in \mathcal{F}$, which satisfy (4.1), have at least two different expansions.

Since l is finite and we are interested in the asymptotical distribution of blocks in the digital expansion and since $B^l\mathcal{F} \cap \mathcal{F} = \mathcal{F}$ we may assume without loss of generality that $l = 0$.

The goal of this section is to show the following

Theorem 4.1. *If $\theta \in \mathcal{F}$ is ambiguous, then θ is not normal.*

We follow [13] to construct the graph $G(\mathbb{Z}^n)$, which provides a tool for constructing the representation of an element of S_0. For this graph $\mathbb{Z}^n$ is its set of vertices and $\mathcal{B} := \mathcal{D} - \mathcal{D}$ its set of labels. The rule for drawing an edge is the following

$$m_1 \xrightarrow{b} m_2 :\Longleftrightarrow Bm_1 - m_2 = b \in \mathcal{B} \quad (m_1, m_2 \in \mathbb{Z}^n).$$

By $G(S)$ and $G(S_0)$ we define the restrictions of $G(\mathbb{Z}^n)$ to the sets S and S_0, respectively.

By Rem. 3.4 of [13] we get that any infinite walk $m \xrightarrow{b_1} m_2 \xrightarrow{b_2} \xrightarrow{b_2} m_3 \xrightarrow{b_3} \cdots$ in $G(S_0)$ yields a representation

$$m = \sum_{j \geq 1} B^{-j} b_j.$$

Vice versa, by looking at such a representation of m we get an infinite walk in $G(S_0)$, starting at m.

Now we construct the graph $\overline{G}(S_0)$ to determine all points of B_m. Therefore we define for every pair (m_1, m_2) the set $\mathcal{C}(m_1, m_2) := \{a \in \mathcal{D} : (Bm_1 + \mathcal{D}) \cap (m_2 + a) \neq \emptyset\}$ and its cardinality $c_{m_1,m_2} := |\mathcal{C}(m_1, m_2)|$. Now the graph $\overline{G}(S_0)$ results from $G(S_0)$ by replacing every edge $m_1 \xrightarrow{b} m_2$ by c_{m_1,m_2} edges $m_1 \xrightarrow{a} m_2$ with $a \in \mathcal{C}(m_1, m_2)$. By the considerations in Rem. 3.4 of [13] we furthermore get that every infinite walk $m \xrightarrow{a_1} \xrightarrow{a_1} m_2 \xrightarrow{a_2} m_3 \xrightarrow{a_3} \cdots$ in $\overline{G}(S_0)$ yields a point

$$\theta = \sum_{j \geq 1} B^{-j} a_j \in B_m \subset \partial \mathcal{F}.$$

We denote by $C := (c_{k,l})_{k,l \in S}$ the accompanying matrix of $\overline{G}(S)$ and call it the *contact matrix* (cf., (6) of [4]). Similarly we call $\overline{G}(S)$ the *contact graph* of $(B, \mathcal{D})$.

Thus every ambiguous point $\theta \in \mathcal{F}$ can be constructed by an infinite walk in $\overline{G}(S_0)$. If we can show that there exists a sufficiently long walk which could not be constructed by $\overline{G}(S_0)$, then we get that the corresponding block does not appear in any ambiguous point and hence the ambiguous points cannot be normal.

Therefore we denote by $W_k(m)$ the set of all different walks of length k starting at m in $\overline{G}(S_0)$. Further let W_k be the total set of walks of length k in $\overline{G}(S_0)$. Then we simply get

$$|W_k| = \sum_{m \in S} |W_k(m)|.$$

By the definition of the contact matrix C and noting that

$$(|W_0(m)|)_{m \in S} = (1, \ldots, 1)^t$$

we get the recurrence

$$(|W_{k+1}(m)|)_{m \in S} = C \cdot (|W_k(m)|)_{m \in S}.$$

Let $\mu_{\max}$ be the eigenvalue of largest modulus of C. Then there exists a constant $c > 0$ such that

$$|W_k| = \sum_{m \in S} |W_k(m)| = c\mu_{\max}^k (1 + o(1)). \tag{4.2}$$

Thus we are left with an estimation of $\mu_{\max}$. Therefore we justify our naming of C and use the following result.

Lemma 4.2 ([4, Th. 2.1]). *If* $(B, \mathcal{D})$ *is a JTCS, then* $|\mu_{\max}| < |\det B|$.

Now the proof of Theorem 4.1 follows easily.

Proof of Theorem 4.1. In order to show that an ambiguous number θ is not normal we need to show that there exists a block of length k that cannot occur in the $(B, \mathcal{D})$-expansion of θ.

By the considerations above this is equivalent to showing that there exists a block of length k that cannot occur as a labeling of a walk of length k in $\overline{G}(S)$.

Since the number of possible blocks of length k is $|\mathcal{D}|^k$ and the number of walks of length k is $|W_k|$ it suffices to show

$$|W_k| \leq |\mathcal{D}|^k - 1.$$

Thus putting (4.2) and Lemma 4.2 together we get that there exists a $k_0 > 0$ such that

$$|W_k| = c\mu_{\max}^k(1 + o(1)) \leq |\det B|^k - 1 = |\mathcal{D}|^k - 1 \quad (k \geq k_0). \quad \Diamond$$

5. Proof of Theorem 3.1

The proof works in three steps.

1. We start by using the ordering function τ to transfer the number to a number in q-ary expansion for $q := |\det B|$.

2. Then we apply the Theorem of Copeland and Erdős to show the normality of this transferred number.

3. Finally transferring the number back to a JTCS we show that this does not affect normality.

First we transpose the problem in the setting of q-ary expansions where $q := |\det B| > 1$. Therefore we use our numbering function τ to transfer θ into a q-ary expansion. Thus

$$\tau(\theta) := 0.[\tau(a_0)][\tau(a_1)][\tau(a_2)][\tau(a_3)][\tau(a_4)][\tau(a_5)][\tau(a_6)][\tau(a_7)]\cdots,$$

where $[\cdot]$ denotes the q-ary expansion. As it will always be clear we use $[\cdot]$ for the $(B, \mathcal{D})$- and the q-ary expansion simultaneously.

By the assumptions of the theorem we get that $\{\tau(a_i)\}_{i \geq 0}$ is an increasing sequence and we can apply the Theorem of Copeland and Erdős.

Lemma 5.1 ([2, Th.]). *If $a_1, a_2, \ldots$ is an increasing sequence of integers such that for every $\varepsilon < 1$ the number of a's up to N exceeds N^ε provided N is sufficiently large, then the infinite decimal*

$$0.a_1a_2a_3a_4a_5a_6\ldots$$

is normal with respect to the base q in which these integers are expressed.

Applying Lemma 5.1 gives that $\tau(\theta)$ is normal. Thus for $k \geq 1$, $M \geq k$ and $(d_1, \ldots, d_k) \in \{0, 1, \ldots, q-1\}^k$

$$\left|\left\{k \leq n \leq M+k \middle| \exists\, x \in \mathbb{Z} : \lfloor q^n \tau(\theta) \rfloor = xq^k + \sum_{i=0}^{k-1} d_i q^i \right\}\right| = \frac{M}{q^k} + o(M). \tag{5.1}$$

For an $x \in \mathbb{Z}$ with $x = \sum_{i=0}^{k} a_i q^i$, where $0 \leq a_i < q$ for every i, we define $\ell(x) := k+1$ to be the q-ary length of x. Then it is clear that $\ell(a) = \ell(\tau(a))$ for all $a \in \mathcal{R}$.

For $k \geq 1$ and $a \in \mathcal{R}$ with $\ell(a) = k$ we get together with (5.1) that

$$\begin{aligned}\mathcal{N}(\theta; a, N) &= |\{0 \leq n < N | \{B^n \theta\} \in \mathcal{F}_a\}| = \\ &= \left|\left\{k \leq n \leq N+k \middle| \exists\, x \in \mathbb{Z} : \lfloor q^n \tau(\theta) \rfloor = xq^k + \tau(a) \right\}\right| = \\ &= \frac{N}{q^k} + o(N) = \frac{N}{|\mathcal{D}|^k} + o(N).\end{aligned}$$

By noting the definition of normality in (2.1) the theorem is proven.

References

[1] CHAMPERNOWNE, D. G.: The construction of decimals normal in the scale of ten, *J. Lond. Math. Soc.* **8** (1933), 254–260 (English).

[2] COPELAND, A. H. and ERDŐS, P.: Note on normal numbers, *Bull. Amer. Math.* Soc. **52** (1946), 857–860.

[3] DAVENPORT, H. and ERDŐS, P.: Note on normal decimals, *Canadian J. Math.* **4** (1952), 58–63.

[4] GRÖCHENIG, K. and HAAS, A.: Self-similar lattice tilings, *J. Fourier Anal. Appl.* **1** (1994), no. 2, 131–170.

[5] INDLEKOFER, K.-H., KÁTAI, I. and RACSKÓ, P.: Number systems and fractal geometry, Probability theory and applications, Math. Appl., vol. 80, Kluwer Acad. Publ., Dordrecht, 1992, pp. 319–334.

[6] KÁTAI, I. and KOVÁCS, B.: Kanonische Zahlensysteme in der Theorie der quadratischen algebraischen Zahlen, *Acta Sci. Math. (Szeged)* **42** (1980), no. 1-2, 99–107.

[7] KÁTAI, I. and KOVÁCS, B.: Canonical number systems in imaginary quadratic fields, *Acta Math. Acad. Sci. Hungar.* **37** (1981), no. 1-3, 159–164.

[8] KÁTAI, I. and SZABÓ, J.: Canonical number systems for complex integers, *Acta Sci. Math. (Szeged)* **37** (1975), no. 3-4, 255–260.

[9] KNUTH, D. E.: The art of computer programming, Vol. 2, second ed., Seminumerical algorithms, Addison-Wesley Series in Computer Science and Information Processing, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[10] KOVÁCS, B.: Canonical number systems in algebraic number fields, *Acta Math. Acad. Sci. Hungar.* **37** (1981), no. 4, 405–407.

[11] KOVÁCS, B. and PETHŐ, A.: Number systems in integral domains, especially in orders of algebraic number fields, *Acta Sci. Math. (Szeged)* **55** (1991), no. 3-4, 287–299.

[12] MADRITSCH, M. G., THUSWALDNER, J. M. and TICHY, R. F.: Normality of numbers generated by the values of entire functions, *J. Number Theory*, to appear.

[13] MÜLLER, W., THUSWALDNER, J. M. and TICHY, R. F.: Fractal properties of number systems, *Period. Math. Hungar.* **42** (2001), no. 1-2, 51–68.

[14] NAKAI, Y.-N. and SHIOKAWA, I.: Discrepancy estimates for a class of normal numbers, *Acta Arith.* **62** (1992), no. 3, 271–284.

[15] NAKAI, Y.-N. and SHIOKAWA, I.: Normality of numbers generated by the values of polynomials at primes, *Acta Arith.* **81** (1997), no. 4, 345–356.