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FINITE NEIGHBORHOOD GAMES WITH BINARY CHOICES

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Abstract: Finite neighborhood repeated games with large number of players are examined. Each agent has the choice between two actions and its payoff depends on the number of other players with the same choice. We first showed that the number of different types of games is finite and then we derived tight upper bounds for the number of game types in the general case and also with monotonic and linear payoff functions. The different game types are identified by equivalence classes generated by canonical forms, and in the linear case a system of linear inequalities is derived that characterizes all games being equivalent to any given game. Numerical examples illustrate the theoretical results. Both bounded and unbounded examples are shown.

1. Introduction

It is well known ([5]) that there are 576 different types of two-person games when each player has exactly two available choices (strategies) and only preference orderings of the payoffs are taken into account. This number can be reduced to 78 if we interchange the players and their payoffs.

A game is symmetric if both players get the same playoff for the same behavior. Such games can easily be extended to an arbitrary number of players (N-person games). Each of the N players has a choice between two actions. As a result of its choice, each of them receives a reward or punishment (payoff) that is dependent on its choice as well as everybody else's. As the players can be individuals, collectives of persons, organizations, or anything else, they are simply called agents.

To represent the fact that for any agent the payoff to a choice depends on how many others make the same choice, it is convenient to display the playoffs in the form of two functions (one for each choice) depending on the ratio of agents choosing one of the actions ([6]) (see Fig. 1).

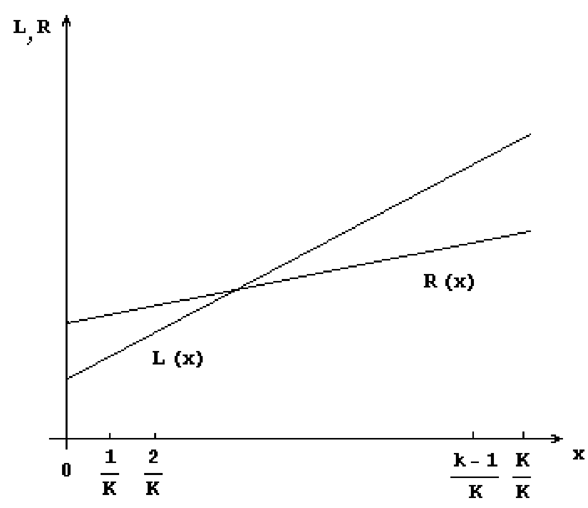

Figure 1. The payoff functions

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At a first glance, such a game looks a well-defined problem. However, at least the following questions arise immediately ([8]):

(1) Are the choices and actions of the agents simultaneous or distributed in time?

(2) Can individual agents see and adapt to the actions of others?

(3) Can they form coalitions?

(4) What are the agents' goals in the game: to maximize their payoffs, to win a competition, to do better than their neighbors, to behave like the majority, or any other goal?

(5) Is it a one-shot game or a repeated one? If it is a repeated game, how will the next action be determined?

(6) Can an agent refuse participation in the game?

(7) Are the payoff curves the same for all agents?

(8) What are the payoff curves?

(9) How is the total payoff to all agents related to the number of agents choosing a certain strategy?

(10) How are the agents distributed in space? Can they move?

(11) Do the agents interact with everyone else or just with their neighbors?

(12) How is neighborhood defined?

We will assume that the agents will face the same repeated "stage game"(see [2]), the agents are distributed in and fully occupy a finite two-dimensional space, the updates are simultaneous, all agents greedily seek maximum rewards for their actions, and they cannot refuse participation in any iteration. These restrictions leave only the problems of the information structure, the payoff curves and the neighborhood open for investigation.

It is assumed that each agent is able to observe the past actions and payoffs of its neighbors. The number of neighborhood layers around each agent and the agent's location determine the number of its neighbors. The depth of agent A's neighborhood is defined as the maximum distance, in two orthogonal directions, that agent B can be from agent A and still be in its neighborhood. An agent at the edge or in the corner of the available space has fewer neighbors than one in the middle. The neighborhood may even extend to the entire array of agents.

When everything else is fixed, the payoff curves determine the game. There is an infinite variety of payoff curves. In addition, stochastic factors can be introduced to represent stochastic responses from the environment. Zero stochastic factors mean a deterministic environment and will be the object of analysis in this paper. Even in the almost trivial case when both payoff curves are straight lines and the stochastic factors are both zero, four parameters specify the environment. If we choose the end points of both curves as the four parameters and exclude equal values of these parameters, their simple ordering results in $4! = 24$ different games. Twelve of these are shown in Schelling's seminal paper [6]. One of these games is the famous Prisoners' Dilemma, another is the Chicken Dilemma; both can be considered stage games in repeated games. For an introduction to repeated games the reader may refer to [2].

There have been some simplistic attempts to describe the two payoff functions with a single parameter [3], [4] however the choice of this parameter is always arbitrary. For the case of greedy agents we may assume that one of the payoff lines starts at zero and it has a positive or negative unit slope. The two remaining parameters that determine the other payoff line are still free. Therefore, the number of possible games still seems to be infinitely large.

Computer simulation of N-person games for greedy agents has been attempted before [4], [9]. In this paper we will examine the structure of the set of such N-person games.

2. The mathematical model

We are looking at a society consisting of agents. Agents are located in a square grid with coordinates (k, l) where $k = 0, 1, \dots, n-1$ and $l = 0, 1, \dots, m-1$. Agents will be identified by their coordinates.

Definition 2.1. *Let $\mathcal{N}$ be a given subset of $\mathbb{Z}^2$. The neighborhood of any agent (k, l) contains agent (i, j) if and only if $(i-k, j-l) \in \mathcal{N}$.*

As a special case we say that the agents have Moore neighborhoods of range p if and only if

$$\mathcal{N} = \left\{(s,t) : (s,t) \in \mathbb{Z}^2, |s| \le p, |t| \le p\right\}$$

We assume therefore that all agents have the same finite neighborhood $\mathcal{N}$.

As in [6] we have a population of a finite number of individuals (agents) each with a choice between L and R ("Left" and "Right"). For any individual the payoff to a choice of Left or Right depends on the choices of the other agents in its neighborhood[1].

[1] In [6] the payoff depends on how many others have a specific choice in the entire population.

The corresponding payoff functions are
$$l\,(x) = \alpha x + \beta \qquad \text{and} \qquad r\,(x) = \gamma x + \delta$$
where x is the fraction of agents playing L in the neighborhood.

We assume that the agents are profit maximizers[2]. Each agent looks around in its neighborhood, finds the agent with maximum payoff and copies the choice that led to that payoff. For example assume payoff functions
$$l\,(x) = 9x \qquad \text{and} \qquad r\,(x) = 18x$$
and consider the central agent in a Moore neighborhood of range 1 depicted in Fig. 2, then the highest payoff is 8 and resulted from a R behavior. Therefore the central agent will change its behavior from L to R at the next iteration.

$-_L$	$-_R$	$-_R$	$-_R$	$-_R$
$-_R$	3_L	4_R	6_R	$-_L$
$-_R$	6_R	$\mathbf{2}_L$	8_R	$-_L$
$-_L$	6_R	4_R	6_R	$-_L$
$-_R$	$-_L$	$-_R$	$-_R$	$-_R$

Figure 2. Choices and payoff in a range 1 Moore neighborhood

Since agents have the same finite neighborhood $\mathcal{N}$ and the same payoff function, the game depends on only the neighborhood definition and the payoff functions. We may use the symbol
$$G = \{m, n, l\,(\cdot)\,, r\,(\cdot)\,, \mathcal{N}\}$$
for describing the game.

Assume there are K agents in each neighborhood. Since the payoff of each agent depends on ratio of the agents with choice L in its neighborhood, the actual behavior of each agent and the long term outcome of the game depends only on the relative order of the values $l\,(k/K)$ and $r\,(k/K)$ for $k = 0, 1, \ldots, K$.

[2] As in [9] they are called *greedy agents*.

We assume that the $2(k+1)$ values are different. We will relax this assumption later in Sec. 4.

First notice that the function values $l(0/K)$ and $r(K/K)$ are defined but actually do not occur when agents compare payoffs, so the number of relevant columns and rows is actually K. Therefore there are at most $(2K)!$ different games, since some of the comparisons never occur.

If we assume that both functions l and r are strictly monotonic, then there are $\binom{2K}{K}$ different orderings of the $2K$ numbers when keeping the monotonicity of the functions, so we have at most $\binom{2K}{K}$ different games.

Finally, when the payoff functions are linear the number of games is sharply bounded by $2a(n)$ where $a(n)$ is the number of ways to divide the points of an $(n \times n)$ square grid in the plane into two sets using a straight line (see sequence A114043 in [7]). This is also the number of the two-dimensional threshold functions[3] [1].

This fact, that the number of possible game types is finite is very important, therefore we report it below as

Proposition 2.2. *The number of different types of games is finite.*

Notice that in the case of finitely many agents (which is the case of finite m and n) there are only finitely many initial configurations, therefore we may have only finitely many possible trajectories.

Each agent updates its choice based on comparing its own payoff to the maximum payoff in its neighborhood. When the choice leading to this payoff is the same, then the agent does not change choice. More interesting is the case when those payoffs come from different choices. As a consequence, the agents' decisions depend on comparing two different payoffs with different choices. The relative order of magnitudes of all possible values can be conveniently represented by a matrix where the rows corresponds to the l values, columns to the r values and each matrix element is either L or R depending on which of the corresponding values is the larger:

[3]We are grateful to Neil Sloane for his help in finding this reference.

(2.1)

	$r(0)$	$r(1/K)$	$\dots$	$r((K-1)/K)$
$l(1/K)$	m_{10}	m_{11}		m_{1K-1}
$\vdots$				$\vdots$
$l(K/K)$	m_{K0}	m_{K1}	$\dots$	m_{KK-1}

The entry (i, j) is defined as follows:

$$m_{ij} = \begin{cases} L & \text{if } l(i/K) > r(j/K) \\ R & \text{if } l(i/k) < r(j/K) \end{cases} .$$

We note that the elements of the matrix are not independent of each other, since certain transitivity relations must be satisfied. Furthermore, depending on the value of K and the number of agents who are playing L in the neighborhood, not all comparisons may actually take place.

Definition 2.3. Given a game $G = \{m, n, l(\cdot), r(\cdot), \mathcal{N}\}$, we call matrix (2.1) the *Update Table of game* G and refer to it as UT(G).

We have already shown that the number of different games is finite. Since the parameters for the payoff functions are continuous variables with infinitely many possible values, it is an important question to find and characterize all games being equivalent to a given game.

Definition 2.4. Two games are equivalent if and only if they have the same dimension and their Update Tables are the same. This is clearly an equivalence relation. Given a game G we call $\mathbb{G}(G)$ the equivalence class identified by G.

Proposition 2.5. *Given a game G, there exists an equivalent game $G^* \in \mathbb{G}(G)$ with payoff functions*

$$l(x) = \alpha^* x + \beta^* \qquad \text{and} \qquad r(x) = \gamma^* x + \delta^*$$

such that $\beta^* = 0$ and either $\alpha^* = -1$ or $\alpha^* = 1$.

Proof. We assumed that the values $l(i/K)$ and $r(j/K)$, are different, $i = 1, 2, \dots, K$, $j = 0, 1, 2, \dots, K-1$, so $\alpha \neq 0$. Therefore two cases are considered:

(1) $\alpha > 0$: then subtracting β from both payoff functions and dividing them by α, UT(G) does not change;

(2) $\alpha < 0$: then subtracting β from both payoff functions and dividing them by $-\alpha$, UT(G) remains the same. $\Diamond$

Definition 2.6. We call game $G^* \in \mathbb{G}(G)$ a canonical form game.

Remark 2.7. Games with canonical form $(\alpha^* = -1) \wedge (\beta^* = 0)$ are not equivalent to games with canonical form $(\alpha^* = 1) \wedge (\beta^* = 0)$.

As we have said earlier, it is an important question to find all equivalent games to any given game. Without losing generality we can assume that the given game is transformed into canonical form. We will next show that the parameters of the equivalent games can be obtained as the solutions of a certain system of linear inequalities. The following algorithm can be suggested to construct this system.

Algorithm 1. *Any game G is identified by its payoff curves $l(x) = \alpha x + \beta$ and $r(x) = \gamma x + \delta$, that is, by four parameters $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$. Given G we will provide conditions for $\bar{G} \in \mathbb{G}(G)$ where $\bar{G}$ is identified by $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$. We assume next that G is in canonical form.*

Assume $\alpha > 0$ $(\alpha < 0)$ and consider $UT(G)$. From each of its columns we will obtain one constraint for the game parameters.

Consider entries in column $j \in \{0, 1, 2, \ldots, K-1\}$. There are three possible cases:

(1) Every element in the column is L. Than for all $i \in \{1, 2, \ldots, K\}$, $l(i/K) > r(j/K)$. Since $\alpha > 0$ $(\alpha < 0)$, any $\bar{G} \in \mathbb{G}(G)$ must be such that $\bar{l}(0) > \bar{r}(j/K)$ $\left(\bar{l}(1) > \bar{r}(j/K)\right)$, that is,

$$0 > \bar{\gamma} j/K + \bar{\delta} \qquad \left(-1 > \bar{\gamma} j/K + \bar{\delta}\right). \tag{2.2}$$

(2) All elements in the column are R. Then for all $i \in \{1, 2, \ldots, K\}$, $l(i/K) < r(j/K)$. Since $\alpha > 0$ $(\alpha < 0)$, any $\bar{G} \in \mathbb{G}(G)$ must be such that $\bar{l}(1) < \bar{r}(j/K)$ $\left(\bar{l}(0) < \bar{r}(j/K)\right)$, that is,

$$1 < \bar{\gamma} j/K + \bar{\delta} \qquad \left(0 < \bar{\gamma} j/K + \bar{\delta}\right). \tag{2.3}$$

(3) Otherwise there is at least one L and R in the column. If $\alpha > 0$ $(\alpha < 0)$ then, in the column, the R (L) elements are followed by L (R) elements; in this case there is an $i^* \in \{1, 2, \ldots, K-1\}$ such that $l(i^*/K) < r(j/K)$ and $l((i^*+1)/K) > r(j/K)$ $\left(l(i^*/K) > r(j/K)\right.$ and $\left.l((i^*+1)/K) < r(j/K)\right)$. Therefore any $\bar{G} \in \mathbb{G}(G)$ must be such that

$$\begin{cases} \bar{l}(i^*/K) < \bar{r}(j/K) \\ \bar{l}((i^*+1)/K) > \bar{r}(j/K) \end{cases} \qquad \left(\begin{cases} \bar{l}(i^*/K) > \bar{r}(j/K) \\ \bar{l}((i^*+1)/K) < \bar{r}(j/K) \end{cases}\right)$$

which can be written as

$$\begin{cases} i^*/K < \bar{\gamma} j/K + \bar{\delta} \\ (i^*+1)/K > \bar{\gamma} j/K + \bar{\delta} \end{cases} \qquad \left(\begin{cases} -i^*/K > \bar{\gamma} j/K + \bar{\delta} \\ -(i^*+1)/K < \bar{\gamma} j/K + \bar{\delta} \end{cases}\right)$$

that is,

$$i^*/K < \bar{\gamma} j/K + \bar{\delta} < (i^*+1)/K \tag{2.4}$$
$$\left(-(i^*+1)/K < \bar{\gamma} j/K + \bar{\delta} < -i^*/K\right).$$

Thus the equivalent game is characterized by a set of constraints, each of them has the form (2.2), (2.3) *or* (2.4).

Finally notice that class $\mathbb{G}(G)$ *is not empty since* $G \in \mathbb{G}(G)$.

As an illustration of the above algorithm we provide two simple examples.

Example 2.8. Consider the following game with an $n \times m$ grid, Moore neighborhood of range 1 and payoff functions

$$\begin{cases} l(x) = -x + 2 \\ r(x) = 3.1x \end{cases}.$$

The UT of the game is the following:

	$r(0)$	$r\left(\frac{1}{9}\right)$	$r\left(\frac{2}{9}\right)$	$r\left(\frac{3}{9}\right)$	$r\left(\frac{4}{9}\right)$	$r\left(\frac{5}{9}\right)$	$r\left(\frac{6}{9}\right)$	$r\left(\frac{7}{9}\right)$	$r\left(\frac{8}{9}\right)$
$l\left(\frac{1}{9}\right)$	L	L	L	L	L	L	R	R	R
$l\left(\frac{2}{9}\right)$	L	L	L	L	L	L	R	R	R
$l\left(\frac{3}{9}\right)$	L	L	L	L	L	R	R	R	R
$l\left(\frac{4}{9}\right)$	L	L	L	L	L	R	R	R	R
$l\left(\frac{5}{9}\right)$	L	L	L	L	L	R	R	R	R
$l\left(\frac{6}{9}\right)$	L	L	L	L	R	R	R	R	R
$l\left(\frac{7}{9}\right)$	L	L	L	L	R	R	R	R	R
$l\left(\frac{8}{9}\right)$	L	L	L	L	R	R	R	R	R
$l(1)$	L	L	L	R	R	R	R	R	R

By using the above procedure we obtain 12 inequalities:

$$\begin{cases} -1 > \bar{\gamma} j/9 + \bar{\delta} \qquad j = 0,1,2 \\ -9/9 < 3/9\bar{\gamma} + \bar{\delta} < -8/9 \\ -6/9 < 4/9\bar{\gamma} + \bar{\delta} < -5/9 \\ -3/9 < 5/9\bar{\gamma} + \bar{\delta} < -2/9 \\ 0 < \bar{\gamma} j/9 + \bar{\delta} \qquad j = 6,7,8 \end{cases}.$$

By simple algebra we can show that some of them can be eliminated as the consequences of the others. The feasible region can be depicted as shown in Fig. 3, where the dot indicates the original game given in canonical form.

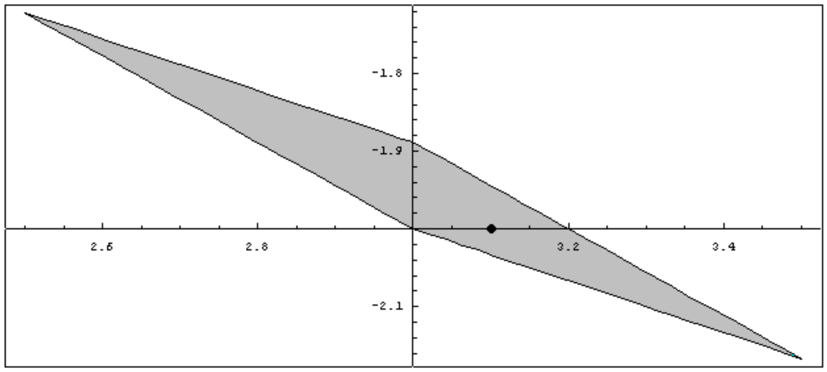

Figure 3. Feasible parameter region for equivalent games

We notice that this region is bounded, in other cases however it might be unbounded as the following example illustrates.

Example 2.9. Consider now the game with an $n \times m$ grid, Moore neighborhood of range 1 and payoff functions

$$\begin{cases} l(x) = -2x + 2 \\ r(x) = 3.1x + 3 \end{cases}.$$

The UT of the game is the following:

	$r(0)$	$r\left(\frac{1}{9}\right)$	$r\left(\frac{2}{9}\right)$	$r\left(\frac{3}{9}\right)$	$r\left(\frac{4}{9}\right)$	$r\left(\frac{5}{9}\right)$	$r\left(\frac{6}{9}\right)$	$r\left(\frac{7}{9}\right)$	$r\left(\frac{8}{9}\right)$
$l\left(\frac{1}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{2}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{3}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{4}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{5}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{6}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{7}{9}\right)$	R	R	R	R	R	R	R	R	R
$l\left(\frac{8}{9}\right)$	R	R	R	R	R	R	R	R	R
$l(1)$	R	R	R	R	R	R	R	R	R

By using the above procedure we obtain 9 inequalities:

$$0 < \gamma' i/9 + \delta' \qquad i = 0, 1, 2, \ldots, 8.$$

By simple algebra it is possible again to reduce the number of constraints and the region can be depicted as given in Fig. 4; as in the previous example the dot indicates the original game given in the canonical form.

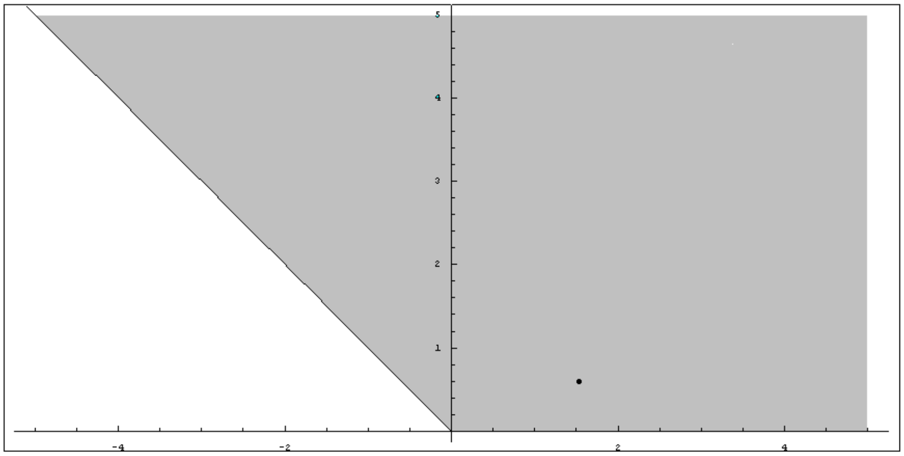

Figure 4. Unbounded feasible parameter region for equivalent games

3. Borders

While considering finite toroidal grids the analysis remain mainly identical, we have to make minor changes when considering borders. The neighborhood for an agent in the border proximity might be different than that of agents in the middle.

Fig. 5 shows the number of neighbors when considering range 1 Moore neighborhood. In this case in addition to $0/9, 1/9, \ldots, 9/9$ we also need the values of functions l and r at $1/4, 2/4, 3/4, 1/6$ and $5/6$. In general, as a consequence, the UT will have more possible values but always from a finite set.

4	6	6	6	4
6	9	9	9	6
6	9	9	9	6
6	9	9	9	6
4	6	6	6	4

Figure 5. Number of neighbors when considering borders in a range

Even if we have infinitely many agents with identical neighborhoods, then there is still only a finite number of UT matrices. However the number of initial configurations is infinite, therefore the number of possible trajectories also becomes infinite.

4. Games with ties

If some of the l and r values are equal then the agents might need to select between the two possible choices with equal payoffs. In such cases a certain rule has to be assigned, for example by selecting the current strategy or following the majority of its neighborhood. Such rules may be embedded in the UT, which remains however finite given the finiteness of the neighborhood.

A special case occurs when at least one of the functions l and r is constant. In this case similar canonical form can be obtained with zero slope.

The number of the game types in the general case when allowing ties also remains finite.

5. Conclusion

In this paper finite neighborhood games were considered with finitely many agents and with binary choices. We proved first that there is only a finite number of game types. We could also show tight upper bounds for the number of games. These bounds were different without or with monotonicity of the payoff functions and also in the linear case.

For games with linear payoffs we have also developed an algorithm to characterize all games which are equivalent to any given game. This characterization is based on a system of linear inequalities. We have also illustrated this algorithm with a bounded and an unbounded example. Some remarks were made finally about neighborhood changes around the boundaries and also with games with equal payoff values.

The algorithm introduced in the paper for the linear case can be extended to monotonic case, the details of which will be the subject of a future paper.

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