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DISTRIBUTION OF q-ADDITIVE FUNCTIONS ON SOME SUBSETS OF INTEGERS

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Abstract: Distribution of q-additive functions on the subsets of integers characterized by the values of sum of digits function is investigated.

1. Introduction

Let $q \geq 2$ be an integer, $A_q := \{0, 1, \dots, q-1\}$, $n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j$, $\varepsilon_j(n) \in A_q$ $(j = 0, 1, \dots)$ be the q-ary expansion of n. Let $\mathbb{N}_0 = \mathbb{N} \cup \cup \{0\}$ = set of nonnegative integers. Let $\mathcal{A}_q$, $\mathcal{M}_q$ be the q-additive, q-multiplicative functions, respectively. We say that $f : \mathbb{N}_0 \to \mathbb{R}$ belongs

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to $\mathcal{A}_q$ if $f(0) = 0$, and $f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j)$ $(n \in \mathbb{N})$, furthermore $g :$ $: \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{M}_q$ if $g(0) = 1$, and $g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j)$ $(n \in \mathbb{N})$. We say that $g \in \overline{\mathcal{M}}_q$, if $g \in \mathcal{M}_q$ and $|g(n)| = 1$ $(n = 1, 2, 3, \dots)$.

Let $\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)$, $\beta_l(n) = \#\{j \mid \varepsilon_j(n) = l\}$ $(l = 1, 2, \dots, q-1)$.

Let N be a fixed integer. For some positive integers $r_1, r_2, \dots, r_{q-1}$ let $\underline{r} = (r_1, r_2, \dots, r_{q-1})$,

$$\mathcal{B} = \mathcal{B}(N|\underline{r}) = \{n < q^N \mid \beta_l(n) = r_l,\ l = 1, \dots, q-1\}.$$

Let $r_0 := N - (r_1 + r_2 + \dots + r_{q-1})$. It is clear that $\mathcal{B}$ is empty if $r_0 < 0$, and that

$$B(N|\underline{r}) := \#(\mathcal{B}(N|\underline{r})) = \frac{N!}{r_0! r_1! \dots r_{q-1}!}, \tag{1.1}$$

if $r_0 \geq 0$.

Let $\delta_j (= \delta_{j,N}) = \frac{r_j}{N}$ $(j = 0, 1, \dots, q-1)$. Let $0 < \varepsilon < \frac{1}{2q}$ be a fixed number, and assume that

$$(1.2)_\varepsilon \qquad \delta_j \geq \varepsilon \ (j = 0, \dots, q-1).$$

Let $\underline{\delta}^{(N)} = (\delta_1, \dots, \delta_{q-1})$. Let $f \in \mathcal{A}_q$,

$$F_{\mathcal{B}(N|\underline{r})}(y) := \frac{1}{B(N|\underline{r})} \#\{n \in \mathcal{B}(N|\underline{r}),\ f(n) < y\}. \tag{1.3}$$

Let furthermore

$$Q_{\mathcal{B}(N|\underline{r})}(D) := \sup_{y \in \mathbb{R}} \left(F_{\mathcal{B}(N|\underline{r})}(y + D) - F_{\mathcal{B}(N|\underline{r})}(y) \right). \tag{1.4}$$

A direct consequence of the 3 series theorem of Kolmogorov is that $f \in \mathcal{A}_q$ has a limit distribution, i.e. that

$$\lim_{N \to \infty} \frac{1}{q^N} \#\{n < q^N \mid f(n) < y\} = F(y) \text{ (almost all } y),$$

F is a distribution function, if and only if

$$\sum_{j=0}^{\infty} \sum_{b=1}^{q-1} f(bq^j) \quad \text{is convergent,} \tag{1.5}$$

and

(1.6) $$\sum_{j=0}^{\infty}\sum f^2(bq^j) \quad \text{is convergent.}$$

First we shall give necessary and sufficient conditions for the existence of such distribution function $F_{\underline{\xi}}(y)$ depending on the parameter $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_{q-1})$, where $\xi_i \geq \varepsilon$ $(i = 1, \dots, q-1)$, $\xi_0 := 1 - (\xi_1 + + \dots + \xi_{q-1}) \geq \varepsilon$, for which

(1.7) $$\lim_{\substack{N\to\infty \\ \delta_j \to \xi_j}} F_{\mathcal{B}(N,r)}(y) = F_{\underline{\xi}}(y) \quad \text{(almost all } y\text{)}$$

is satisfied.

Theorem 1. *Let $f \in \mathcal{A}_q$. If there exists some $\xi_1, \dots, \xi_{q-1}$ satisfying $\xi_i \geq \varepsilon$ $(i = 0, 1, \dots, q-1)$, for which* (1.7) *holds, then* (1.5), (1.6) *are satisfied. If* (1.5), (1.6) *hold, then* (1.7) *holds true for all choices of ξ_i satisfying $\xi_i \geq \varepsilon$ $(i = 0, \dots, q-1)$. $F_{\underline{\xi}}(y) := P(\Theta_{\underline{\xi}} < y)$, where $\eta_0, \eta_1 \dots$ are independent random variables, $P(\eta_j = f(aq^j)) = \xi_a$ $(a = = 0, 1, \dots, q-1)$, $\Theta_{\underline{\xi}} = \sum_{j=0}^{\infty} \eta_j$.*

Theorem 2. *Let $g \in \overline{\mathcal{M}}_q$, and for $\xi_i \geq \varepsilon$ $(i = 0, \dots, q-1)$ let*

(1.8) $$M_{N,\underline{\xi}}(g) := \prod_{j=0}^{N-1} \left(\xi_0 + \xi_1 g(1 \cdot q^j) + \dots + \xi_{q-1} g((q-1)q^j)\right).$$

Assume that

(1.9) $$\sum_{j=0}^{\infty}\sum_{a=1}^{q-1}(g(aq^j) - 1) \quad \textit{is convergent.}$$

Then

$(1.2)_\varepsilon$ $$\sup \left| \frac{1}{B(N, \underline{r})} \sum_{n \in \mathcal{B}(N,\underline{r})} g(n) - M_{N,\delta^{(N)}}(g) \right| \to 0 \quad (N \to \infty).$$

Consequently, if $r_j = r_j^{(N)}$ $(j = 1, \dots, q-1)$ are so chosen that $r_j^{(N)}/N \to \xi_j$ $(j = 1, \dots, q-1)$ then

(1.10) $$\frac{1}{B(N, r^{(N)})} \sum_{n \in \mathcal{B}(N, \underline{r}^{(N)})} g(n) = M_{\infty,\underline{\xi}}(g),$$

where $M_{\infty,\underline{\xi}}$ is the limit of $M_{N,\underline{\xi}}$ (defined by (1.8)*) for $N \to \infty$.*

Theorem 3. *Let $f \in \mathcal{A}_q$, $f(bq^j)$ be bounded for $j \in \mathbb{N}_0$, $b \in A_q$. Let*

$$\tau_b = \tau_b^{(N)} := \frac{1}{N} \sum_{j=0}^{N-1} f(bq^j), \tag{1.11}$$

$\tilde{f}(bq^j) = f(bq^j) - \tau_b$, $b \in A_q$, $j = 0, 1, \ldots, N-1$, $\tilde{f}$ be extended to $\mathbb{N}_0$ as a q-additive function.

Let $r_1, \ldots, r_{q-1}$ be satisfying $(1.2)_\varepsilon$,

$$\sigma_N^2(\underline{\delta}^N) := \frac{1}{B(N|\underline{r})} \sum_{n \in \mathcal{B}(N|\underline{r})} \tilde{f}^2(n).$$

We have

$$\sigma_N^2(\underline{\delta}^N) = \frac{N}{N-1} \sum_{l=0}^{N-1} \left(\sum_{b \in A_q} \frac{r_b}{N} \left(\tilde{f}(bq^l) - m_l \right)^2 \right),\ m_l = \sum_{b \in A_q} \frac{r_l}{N} \tilde{f}(bq^l).$$

Assume that

$$\sigma_N^2\left(\frac{1}{q}, \ldots, \frac{1}{q}\right) \to \infty \quad (N \to \infty).$$

Let $h = h_N \in \mathcal{A}_q$, $h(n) := \frac{\tilde{f}(n)}{\sigma_N(\underline{\delta}^N)}$. Then

$$\max_{(1.2)_\varepsilon} \max_{y \in \mathbb{R}} \left| \frac{1}{B(N|\underline{r})} \#\{n \in \mathcal{B}(N|\underline{r}),\ h(n) < y\} - \Phi(y) \right| \to 0$$

as $N \to \infty$.

2. Lemmata

Lemma 1. *Let $f \in \mathcal{A}_q$, $D > 0$ be fixed. If $f \in \mathcal{A}_q$, $\limsup_{bq^j \to \infty} |f(bq^j)| = \infty$, then*

$$\max_{(1.2)_\varepsilon} \frac{Q_{B(N|\underline{r})}(D)}{B(N|\underline{r})} \to 0 \quad (N \to \infty).$$

Proof. Let $b^* \in A_q \backslash \{0\}$ be such coefficient for which $\limsup_{j \to \infty} |f(b^* q^j)| = \infty$. By changing the sign of f, if needed, we may assume that $\limsup_{j \to \infty} f(b^* q^j) = \infty$.

Let $l_1 < l_2 < \ldots$ be such a sequence of integers for which $2D \leq f(b^* q^{l_1})$, $f\left(b^* q^{l_{h+1}}\right) \geq 2f(b^* q^{l_h})$.

Let N be a large integer, T be defined such that $l_T \le N-1 < < l_{T+1}$. Then $T = T_N \to \infty$. We may assume that $T_N | \log N \to 0$ (say). Let

$$U = \{l_1, l_2, \dots, l_T\}, \quad V = \{0, 1, \dots, N-1\} \backslash U.$$

Consider all those $n \in \mathcal{B}(N, \underline{r})$ for which $f(n) \in [y, y+D]$. Let $s_0, s_1, \dots, s_{q-1}$ be nonnegative integers such that $s_0 + s_1 + \ldots + s_{q-1} = = T$. Let

$$\mathcal{E}^{(U)}_{s_0,s_1,\dots,s_{q-1}} = \left\{ m \mid m = \sum_{j=1}^{T} \varepsilon_{l_j}(m) q^{l_j},\ \beta_b(m) = s_b,\ b \in \mathbb{A}_q \right\}$$

$$\mathcal{F}^{(U)}_{s_0,\dots,s_{q-1}} = \left\{ \nu \mid \nu = \sum_{r \in V} \varepsilon_r(\nu) q^r,\ \beta_b(\nu) = r_b - s_b,\ b \in A_q \right\}.$$

We have

$$\#\mathcal{F}^{(U)}_{s_0,\dots,s_{q-1}} = B(N-T \mid \underline{r} - \underline{s}) = \frac{(N-T)!}{(r_0 - s_0)!(r_1 - s_1)! \dots (r_{q-1} - s_{q-1})!}$$

$$\#\mathcal{E}^{(U)}_{s_0,\dots,s_{q-1}} = B(T | \underline{s}) = \frac{T!}{s_0! s_1! \dots s_{q-1}!}.$$

It is clear that every $n \in \mathcal{B}(N|\underline{r})$ can be written uniquely as $n = m + \nu$, where $m \in \mathcal{E}^{(U)}_{s_0,\dots,s_{q-1}}$ and $\nu \in \mathcal{F}^{(U)}_{s_0,\dots,s_{q-1}}$. Let us fix a ν with

$$\nu = \sum_{r \in V} \varepsilon_r(\nu) q^r,\ \beta_b(\nu) = r_b - s_b, \quad b \in A_q.$$

Let $U_{s_0, s_{b^*}}$ be an arbitrary subset of U having exactly $s_0 + s_{b^*}$ elements,

$$U_{s_0,s_{b^*}} = \{ j_1 < j_2 < \ldots < j_{s_0 + s_{b^*}} \},$$

$$H_{s_0,s_{b^*}} = U \backslash U_{s_0,s_{b^*}}, \quad H_{s_0,s_{b^*}} = \{ k_1 < k_2 < \ldots < k_{T-(s_0+s_{b^*})} \}.$$

We shall write every $m \in \mathcal{E}^{(U)}_{s_0,\dots,s_{q-1}}$ as $\kappa + \rho$, where $\kappa = \sum_{h=1}^{s_{b^*}} b^* q^{r_h}$, $r_1 < r_2 < \ldots < r^*_{s_b}$ is an arbitrary sequence of the elements of U_{s_0,s^*_b}, and $\rho = \sum \varepsilon_p(\rho) q^p$, where p runs over all elements of $H_{s_0,s_{b^*}}$, $\varepsilon_p(\rho) \in \in A_q \backslash \{0, b^*\}$, and $\beta_l(\rho) = s_l$ if $l \in A_q \backslash \{0, b^*\}$.

Let $H_{s_0,s_{b^*}}$ be fixed, and $r^{(i)}_1 < r^{(i)}_2 < \ldots < r^{(i)}_{s_{b^*}}$ $(i = 1, 2)$ be two subsequences and $\kappa^{(1)}$, $\kappa^{(2)}$ be the corresponding integers: $\kappa^{(j)} = = \sum_{h=1}^{s_{b^*}} b^* q^{r^{(j)}_h}$ $(j = 1, 2)$.

From the definition of the sequence U we obtain that $|f(\kappa^{(1)})-$ $-f(\kappa^{(2)})|>D$.

Assume that $f(n)\in[y,y+D]$, $in\in\mathcal{B}(N|\underline{r})$. Then n can be written in the form

$$n=\kappa+\rho+\nu.$$

Let ν be fixed, $\beta_l(\nu)=r_\nu-s_\nu$, $s_0,\dots,s_{q-1}$ are determined by ν. We can form exactly $\binom{T}{s_0+s_{b^*}}$ different sets U_{s_0},s_{b^*}.

Assume that $U_{s_0,s_{b^*}}$ is fixed. Then the number of ρ is

$$\frac{(T-(s_0+s_{b^*}))!}{\prod\limits_{j\neq 0,b^*}s_j!}.$$

Let us assume now that ν,ρ,s_0,s_b^* and U_{s_0,s_b^*} are fixed. Then no more than one κ is appropriate. Thus we have

$$Q_{\mathcal{B}(N,\underline{r})}(d)\le$$
$$\le\sum_{s_0,\dots,s_{q-1}}\frac{(N-T)!}{(r_0-s_0)!\dots(r_{q-1}-s_{q-1})!}\binom{T}{s_0+s_{b^*}}\frac{(T-(s_0+s_{b^*}))!}{\prod\limits_{j\neq 0,b^*}s_j!},$$

$$\frac{Q_{\mathcal{B}(N,r)}(D)}{B(N|r)}\le$$
$$\le 2\sum_{s_0+\dots+s_{q-1}=T}\frac{T!}{s_0!\dots s_{q-1}!}\left(\frac{r_0}{N}\right)^{s_0}\dots\left(\frac{r_{q-1}}{N}\right)^{s_{q-1}}\cdot\frac{s_0!s_{b^*}!}{(s_0+s_b^*)!}. \tag{2.1}$$

We subdivide the sum on the right hand side of (2.1) as $\sum_1+\sum_2+$ $+\sum_3+\sum_4$, where in $\sum_1$ $s_0=0$, in $\sum_2$ $s_b^*=0$; in $\sum_3$ $s_0+s_b^*\le H$ and $s_0,s_{b^*}\ge 1$; and in $\sum_4:$ $s_0+s_b^*>H$, $s_0s_{b^*}\neq 0$.

One can see easily that $\sum_1,\sum_2,\sum_3=o_N(1)$.

Since $\frac{s_0!s_{b^*}!}{(s_0+s_b)!}=\frac{1}{\binom{s_0+s_b}{s_0}}\le\frac{1}{s_0+s_b}\le\frac{1}{H}$, we obtain that $\sum_4\le 2/H$.

Since H is an arbitrary large fixed number, therefore Lemma 1 is true.

Lemma 2. *Let* $f\in\mathcal{A}_q$, $\tilde f$ *be defined as in* Th. 3.

Let $m_l:=\sum\limits_{b\in A_q}\frac{r_b}{N}\tilde f(bq^l)$. Then

$$\frac{1}{B(N|\underline{r})}\sum_{n\in\mathcal{B}(N|r)}\tilde{f}^2(n)=\frac{N}{N-1}\sum_{l=0}^{N-1}\sum_{b}\frac{r_b}{N}\left(\tilde{f}(bq^l)-m_l\right)^2. \tag{2.2}$$

Proof. Since

$$\frac{1}{B(N|\underline{r})}\sum_{n\in\mathcal{B}(N|\underline{r})}\tilde{f}(n)=\sum_{j=0}^{N-1}\sum_{b\in A_q}\tilde{f}(bq^j)\frac{r_b}{N}=0,$$

we have

$$\frac{1}{B(N|\underline{r})}\sum_{n\in\mathcal{B}(N|\underline{r})}\tilde{f}^2(n)=\sum_{b_1\neq b_2}\frac{r_{b_1}}{N}\frac{r_{b_2}}{(N-1)}\sum_{l_1\neq l_2}\tilde{f}(b_1q^{l_1})\tilde{f}(b_2q^{l_2})+$$

$$+\sum_{b\in A_q}\frac{(r_b-1)r_b}{(N-1)N}\sum_{l_1\neq l_2}\tilde{f}(bq^{l_1})\tilde{f}(bq^{l_2})+\sum_{b}\frac{r_b}{N}\sum_{l=0}^{N-1}\tilde{f}^2(bq^l).$$

Since $\sum_{j=0}^{N-1}\tilde{f}(bq^j)=0$ $(b\in A_q)$, therefore

$$\sum=-\sum_{b_1\neq b_2}\frac{r_{b_1}}{N}\cdot\frac{r_{b_2}}{(N-1)}\sum_{l=0}^{N-1}\tilde{f}(b_1q^l)\tilde{f}(b_2q^l)+$$

$$+\sum_{b}\frac{r_b}{N}\left(1-\frac{r_{b-1}}{N-1}\right)\sum_{l}\tilde{f}^2(bq^l)$$

whence we obtain that

$$\sum=\frac{N}{N-1}\sum_{l=0}^{N-1}\sum_{b}\frac{r_b}{N}\left(\tilde{f}(bq^l)-m_l\right)^2,$$

thus Lemma 2 is true.

3. Proof of Theorem 3

We shall use the Frechet–Shohat theorem. (See [1].)

Let

$$m_l:=\sum_{b\in A_q}\frac{r_b}{N}\tilde{f}(bq^l), \tag{3.1}$$

and

$$g_l(b)=\tilde{f}(bq^l)-m_l\quad(b\in A_q). \tag{3.2}$$

For $n<q^N$ let

$$g(n) := \sum_{j=0}^{N-1} g_j(\varepsilon_j(n)). \tag{3.3}$$

We have

$$\sum_{l=0}^{N-1} m_l = \sum_{b\in A_q} \frac{r_b}{N} \sum_l (f(bq^l) - \tau_b) = 0.$$

Let

$$K(n) := \frac{g(n)}{\sigma_N} \quad (n = 0, 1, \dots, q^N - 1), \tag{3.4}$$

and

$$S_h(N) := \frac{1}{B(N|\underline{r})} \sum_{n<q^N} K^h(n). \tag{3.5}$$

We shall prove that

$$\max_{(1.2)_\varepsilon} |S_h(N) - \mu_h| \to 0 \quad (N \to \infty), \tag{3.6}$$

where

$$\mu_h = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^h e^{-u^2/2} du$$

We have

$$\sum_{n\in\mathcal{B}(N|r)} K^h(n) = \sum_{s=1}^{h} \sum_{a_1+\ldots+a_s=h} d(a_1, \dots, a_s) \sum_{b_1,\ldots,b_s\in A_q} \sum_{l_1,\ldots,l_s} K^{a_1}(b_1q^{l_1}) \ldots K^{a_s}(b_sq^{l_s}) E_N(s, \underline{b}, \underline{a}, \underline{l}),$$

where $a_1, \dots, a_s$ are positive integers, $d(a_1, \dots, a_s)$ the coefficient coming from the polynomial theorem, $b_1, \ldots, b_s$ run over the possible values of A_q, independently, $l_1, \ldots, l_s$ run over $\{0, 1, \ldots, N-1\}$ such that $l_i \neq l_j$ if $i \neq j$, and

$$E_N(s, \underline{b}, \underline{a}, \underline{l}) = \frac{(N-s)!}{\prod_{b=0}^{q-1} (r_b - e_b)!},$$

where $e_b := \#\{b \text{ among } b_1, \dots, b_s\}$.

We have

$$
(3.7)\quad
\begin{aligned}
\psi_a(s,\underline{b}) = \frac{E(s,\underline{b},\underline{a},\underline{l})}{B(N|\underline{r})} &= \prod_{b=0}^{q-1}\prod_{j=0}^{e_b-1}(r_b-j)\cdot\prod_{j=0}^{s-1}\frac{1}{(N-j)} = \\
&= \prod_{b=0}^{q-1}\left(\frac{r_b}{N}\right)^{e_b}\cdot\prod_{j=0}^{s-1}\frac{1}{(1-j/N)}\cdot\prod_{b=0}^{q-1}\prod_{j=0}^{e_b-1}(1-j/r_b)\,.
\end{aligned}
$$

Thus

$$
\frac{1}{B(N|\underline{r})}\sum_{n\in\mathcal{B}(N|\underline{r})}K^h(n) = \sum_{s=1}^{h}\sum_{a_1,\ldots,a_s=h} d(a_1,\ldots,a_s)H(a_1,\ldots,a_s),
$$

$$
H(a_1,\ldots,a_s) = \sum_{b_1,\ldots,b_s} T(a_1,\ldots,a_s \mid b_1,\ldots,b_s),
$$

(3.8)

$$
T(a_1,\ldots,a_s \mid b_1,\ldots,b_s) = \sum_{l_1,\ldots,l_s} K^{a_1}(b_1q^{l_1})\ldots K^{a_s}(b_sq^{l_s})\psi_a(s,b).
$$

Let $a_j = 1$ for some $j\in\{1,\ldots,s\}$. We have

$$
\sum_{l\neq\{l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_s\}} K(b_jq^l) =
$$

$$
= -K(b_jq^{l_1}) - \ldots - K(b_jq^{l_{j-1}}) - K(b_jq^{l_{j+1}}) - \ldots - K(b_jq^{l_s}).
$$

Iterating this procedure we can rewrite (3.8) as no more than $O(h)$ sums of type

$$
(3.9)\qquad \psi_a(s,\underline{b})\cdot\sum_{t_1,t_2,\ldots,t_p}\prod_{j=1}^{p}\left(\prod_{u=1}^{m_j}K^{q_{u,j}}(c_{u,j}q^{t_j})\right) =: Q,
$$

where $c_{u,j}\in A_q$, $\sum_{u=1}^{m_j} q_{u,j}\geq 2$, and $\#\{c_{u,j}=b \mid u,j\} = e_b$ and the summation is over those $t_j\in\{0,\ldots,N-1\}$ $(j=1,\ldots,p)$ for which $t_i\neq t_j$ $(i\neq j)$.

We shall prove that $Q = o_N(1)$ if $\max_j \sum_{u=1}^{m_j} q_{u,j}\geq 3$.

Indeed

$$
|K(b_1q^t)K(b_2q^t)| \leq K^2(b_1q^t) + K^2(b_2q^t),
$$

and in general

$$
|K(b_1q^t)\ldots K(b_vq^t)| \leq \left(\left|K^v(b_1q^t)\right| + \ldots + \left|K^v(b_vq^t)\right|\right).
$$

Furthermore $\max_{b,l}|K(bq^l)| \leq \frac{c}{\sigma_N}\to 0$ $(N\to\infty)$, and hence our assertion directly follows.

It remains to consider the case when $\sum_{u=1}^{m_j} q_{u,j} = 2$ holds for every $j = 1, 2, \dots, p$.

Let Q be such a sum in which there is an l for which $q_{1,l} = q_{2,l} = 1$. Observe that

$$\psi_a(s,b) = \prod_{b=0}^{q-1} \left(\frac{r_b}{N}\right)^{e_b} \left(1 + O\left(\frac{1}{N}\right)\right). \tag{3.10}$$

Then

$$\begin{aligned} & Q\left(\begin{matrix} c_l \\ d_l \end{matrix} \middle| . \right) = \\ & = \psi_a(s,b) \sum_{t_l} K(c_l q^{t_l}) K(d_l q^{t_l}) \sum_{\substack{t_1,\dots,t_{l-1},\\ t_{l+1},\dots,t_p}}^{*} \prod_{j \neq l} \prod_{u=1}^{m_j} K^{q_{u,j}}(c_{u,j} q^{t_j}) \end{aligned} \tag{3.11}$$

and $*$ means that $t_\nu \neq t_l$ if $\nu \neq l$.

Let

$$\begin{aligned} & Q_0\left(\begin{matrix} c_l \\ d_l \end{matrix} \middle| . \right) = \\ & = \sum \left(\frac{r_{c_l}}{N} K(c_l q^{t_l})\right) \frac{r_{d_l}}{N} K(d_1 q^{t_l}) \sum^{*} \prod_{j \neq l} \prod_{u=1}^{m_j} \left(\frac{r_{c_{u,j}}}{N} K^{q_{u,j}}(c_{u,j} q^{t_j})\right). \end{aligned} \tag{3.12}$$

Then

$$\begin{aligned} Q\left(\begin{matrix} c_l \\ d_l \end{matrix} \middle| . \right) = Q_0 & \left(\begin{matrix} c_l \\ d_l \end{matrix} \middle| . \right) + \\ & + O\Bigg(\frac{1}{N} \sum_{t_l} \left|\frac{r_{c_l}}{N} K(c_l q^{t_l})\right| \left|\frac{r_{d_l}}{N} K(d_l q^{t_l})\right| \cdot \\ & \cdot \sum^{*} \prod_{j \neq l} \prod_{u=1}^{m_j} \left|\frac{r_{c_{u,j}}}{N} K(c_{u,j} q^{t_j})^{q_{u,j}}\right| \Bigg). \end{aligned}$$

The error term is clearly $o_N(1)$. Furthermore Q_0 does not depend on the numbers e_j. In the definition of $H(a_1, \dots, a_s)$ we have to sum $T(a_1, \dots, a_s \mid b_1, \dots, b_s)$ over all possible values of $b_1, \dots, b_s \in A_q$. Since

$$\left(\sum_{c_l=0}^{q-1} \frac{r_{c_l}}{N} K(c_l q^{t_l})\right)\left(\sum_{d_l=0}^{q-1} \frac{r_{d_l}}{N} K(d_l q^{t_l})\right) = 0,$$

the effect of these summands can be ignored.

Hence we obtain that (3.6) holds with $\mu_h = 0$ if $h =$ odd, and for $h = 2s$

$$\begin{aligned}(3.13)\qquad &\frac{1}{B(N|\underline{r})} \sum_{n\in\mathcal{B}(N,r)} K^{2s}(n) = d(2,2,\dots,\overset{s}{2})\\ &\sum_{b_1,\dots,b_s\in A_q} \sum_{l_1,\dots,l_s} K^2(b_1 q^{l_1})\dots K^2(b_s q^{l_s})\psi_{\underline{2}}(s,\underline{b}) + o_N(1).\end{aligned}$$

Substituting $\psi_2(s,b)$ by $\prod_{b=0}^{q-1}\left(\frac{r_b}{N}\right)^{e_b}$, and omitting the condition on the right hand side of (3.13), the difference is $o_N(1)$. Thus the left hand side of (3.13) equals to

$$d(2,\dots,2)\left(\sum_{l}\sum_{b\in A_q} \frac{r_b K^2(bq^l)}{N}\right)^s + o_N(1) = \frac{(2s)!}{2^s} + o_N(1).$$

This proves the theorem.

4. Proof of Theorem 1, necessity

In Lemma 1 we proved that if f has a limit distribution according to Th. 1, then $f(bq^j)$ is bounded. If (1.6) would be divergent, then we would be able to show that f satisfied the conditions of Th. 3, which would imply that $Q_{\mathcal{B}(N,\underline{r})}(D) \to 0$ $(N \to \infty)$.

Therefore (1.6) is convergent.

The proof of the convergence of (1.5) easily follows from Lemma 2. We omit the details.

Proof of Theorem 2 and the sufficiency part of Theorem 1

Let $g_R(n) := \prod_{j=0}^{R-1} g(\varepsilon_j(n)q^j)$. Let $g(aq^j) = e^{i\psi(aq^j)}$, $\psi(aq^j) \in$ $\in [-\pi,\pi]$. From (1.9) we obtain that $\sum_j \sum_a \psi(aq^j)$ is convergent, and that $\sum\limits_j \sum\limits_a \psi^2(aq^j)$ is convergent as well. Hence, by using Lemma 2 we can deduce that

$$\sup_{(1.2)_\varepsilon} \frac{1}{B(N,\underline{r})} \sum_{n<q^N} |g(n)-g_R(n)| \le \kappa_R(N),$$

where $\kappa_R(N) \to 0$ if $R, N \to \infty$, and this implies Th. 2.

The sufficiency part of Th. 1 can be proved by defining $g(n) := := g_\tau(n) = e^{i\tau f(n)}$, and considering

$$\frac{1}{B(N|\underline{r})} \sum_{n<q^N} g_\tau(n)$$

as the characteristic function of the distribution function of f. Since

$$\sum_{j=0}^{\infty} \sum_{a=0}^{q-1} (g_\tau(aq^j)-1)$$

converges, we can apply Th. 1. This completes the proof. $\Diamond$

Reference

[1] GALAMBOS, J.: Advanced Probability Theory, Marcel Dekker, Inc., New York, 1988.