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THE SEMIGROUP OF RIGHT IDEALS OF A RIGHT WEAKLY REGULAR RING

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Abstract: The structure of the multiplicative semigroup of all right ideals of a right weakly regular ring is investigated. Relevant background material and some new results for this class of rings are given. Examples are given which illustrate and delimit the theory developed.

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1. Introduction

This paper continues the investigation, began in [11], of connections between the structure of a ring and that of its multiplicative semigroup of right ideals. Here "ring" will mean ring not necessarily with identity, and R will always denote a non-zero ring. We use $\mathbb{R}(R)$ for the multiplicative semigroup of right ideals of R. In [11, Th. 3.1] we showed, for rings with identity, that the semigroup $\mathbb{R}(R)$ is von Neumann regular if and only if it is a band (every element is an idempotent). The same argument holds for rings without identity. Rings for which every right (left) ideal is idempotent are called *right (left) weakly regular rings* in [14]. We abbreviate "right weakly regular" by r.w.r. There are many interesting equivalent conditions to a ring being right weakly regular, some of which are discussed in Sec. 2. In various guises these rings have been studied since at least 1950 [2].

In this paper we investigate the structure of the ring R and the semigroup $\mathbb{R}(R)$ when $\mathbb{R}(R)$ is a band, i.e., when R is right weakly regular. Note that the zero ideal is the zero for the semigroup $\mathbb{R}(R)$. If R has unity or if R is r.w.r., then R is a right identity for $\mathbb{R}(R)$.

In Sec. 2 we present relevant background material on r.w.r. rings, including some methods for constructing such rings. In Sec. 3 we develop conditions for right and two-sided ideals in the context of the r.w.r. condition. Sec. 4 focuses on the structure of the semigroup $\mathbb{R}(R)$, ending with a ring-semigroup decomposition theorem. Examples are given throughout which illustrate and delimit the theory developed.

2. General results and examples

We use $\langle b\rangle_r$ and $\langle b\rangle$ for the right ideal and two-sided ideal of R generated by b, respectively. The first result below gives some useful and illuminating equivalent conditions to being right weakly regular.

Proposition 2.1. *The following are equivalent:*
(a) *R is r.w.r.;*
(b) *if $b \in R$, then $b \in (bR)^2$;*
(c) *if $H, K \in \mathbb{R}(R)$ and $H \subseteq K$, then $HK = H$;*
(d) *every principal right ideal of R is idempotent;*
(e) *if $b \in R$, then $b \in b\langle b\rangle$;*
(f) *every factor ring of R is r.w.r.*

Proof. The equivalence of (a), (b), and (c) is proved in [14, Prop. 1]. Assume (d) and let $B \in \mathbb{R}(R)$. For any $b \in B$ we have $\langle b\rangle_r = (\langle b\rangle_r)^2 = = bRbR + bRb + b^2R + \mathbb{Z}b^2 \subseteq bR$, where $\mathbb{Z}$ is the ring of integers; so $\langle b\rangle_r = bR$. Thus $\langle b\rangle_r = (bR)^2$ and so $b \in (bR)^2$; so (d) implies (b). Since (a) implies (d) is immediate, we have that (a) through (d) are equivalent. Observe that if R is r.w.r., then $\langle b\rangle_r = bR$. So assuming (a) we have $b\langle b\rangle = b^2R + bRb + bRbR$. Since $b \in (bR)^2$ we have that b is in $b\langle b\rangle$. Thus (a) implies (e). Assume (e) and let $b \in R$. From $b\langle b\rangle \subseteq bR$ we have b is in bR and $b\langle b\rangle_r = bR$. A routine calculation then yields $b\langle b\rangle \subseteq (bR)^2$ and hence $bR \subseteq (bR)^2$; so $bR = (bR)^2$ and consequently $\langle b\rangle_r = (\langle b\rangle_r)^2$. Thus (e) implies (d), and hence (a) through (e) are equivalent. The equivalence of (a) and (f) is immediate. $\Diamond$

Corollary 2.2. *If R is a r.w.r. ring, then R is a right identity for $\mathbb{R}(R)$.*

Proof. Use Prop. 2.1. (c)with $K = R$. $\Diamond$

Further equivalent conditions to R being r.w.r. when R has unity can be found in [8, Th. 8], [9, Lemma 1], [14, Rem. 3], and [19, p. 171].

Brown and McCoy [2, Ex. 2] briefly commented on rings that satisfy condition (e) in Prop. 2.1. [2, Ex. 2]. They noted that the class of all such rings, which we call $\mathcal{W}$, is a hereditary Amitsur–Kurosh radical class. This is proved explicitly in [1, Prop. 1] and discussed in [18, p. 197]. (Of course, Brown and McCoy did not use the terminology "hereditary Amitsur–Kurosh radical" in 1950.) We will use $\mathcal{W}^1$ for the class of all rings with unity that are in $\mathcal{W}$. Observe that $\mathcal{W}^1$ as well as $\mathcal{W}$ is closed under homomorphic images. The next result addresses closure under direct products.

Proposition 2.3. *Let Λ be a nonempty index set.*

(a) *If $R_\lambda \in \mathcal{W}^1$ for each $\lambda \in \Lambda$, then the direct product $\prod R_\lambda, \lambda \in \Lambda$, is in $\mathcal{W}^1$.*

(b) *If $R_\lambda \in \mathcal{W}$ for each $\lambda \in \Lambda$, then $\prod R_\lambda, \lambda \in \Lambda$, is in $\mathcal{W}$.*

Proof. (a) Let $x = (\ldots, x_\lambda, \ldots)$, where $x_\lambda \in R_\lambda$, be an arbitrary element in $\prod R_\lambda$. Using (b) in Prop. 2.1., we have that x_λ is in $(x_\lambda R_\lambda)^2$. So x is in $(x \prod R_\lambda)^2$ and hence $\prod R_\lambda$ is in $\mathcal{W}^1$.

The proof for (b) is strictly analogous. $\Diamond$

The classes $\mathcal{W}$ and $\mathcal{W}^1$ are not closed under taking subrings. For example, the rational number field Q, which is in $\mathcal{W}^1$, has the ring of integers, Z, as a subring, and Z is not in $\mathcal{W}^1$. The class $\mathcal{W}$ is not closed under taking right ideals. For example, for any skewfield F the set $B = = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is a right ideal of the full ring of 2×2 matrices over F, which is a regular ring and hence is r.w.r. Since $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ is a nilpotent ideal of B, we see that B is not r.w.r.

For convenience, if I is a (left, right, two-sided) ideal of a ring R and $I \in \mathcal{W}$, then we say I is a *r.w.r.* (left, right, two-sided) ideal. Since $\mathcal{W}$ is an Amitsur–Kurosh radical class, each ring R has a unique largest r.w.r. ideal, the sum of all the r.w.r. ideals, which we denote by $\mathcal{W}(R)$. For more concerning this radical see [1], [2], and [18, p. 197].

The class $\mathcal{W}$ is broad and varied. It contains all rings which are regular, biregular, or simple with unity. The following omnibus result gives several conditions useful in building further examples of r.w.r. rings. (These, of course, can be combined with direct products, direct sums, and homomorphic images to obtain even more examples.)

Proposition 2.4. *Let R be r.w.r.*

(a) *R can be embedded as an ideal in a r.w.r. ring with unity.*

(b) *The full $n \times n$ matrix ring over R, $M_n(R)$, is r.w.r. for each n.*

(c) *If R has unity, G is a locally finite group, and the order of each element of G is a unit in R, then the group ring $R[G]$ is r.w.r.*

(d) *If R has unity and M is a finitely generated unitary right R-module, then $End_R(M)$ is r.w.r.*

Proof. (a) In [14, Prop. 20] this result is stated and it is remarked that it can be proved by adapting ideas of Fuchs and Halperin [7]. Details of a proof of this and more general results will be given in [12].

(b) Embed R as an ideal in a ring R^1 which has unity and is r.w.r. Then $M_n(R^1)$ is r.w.r. by [19, Prop. 20.4(2)]. Since $M_n(R)$ is an ideal of $M_n(R^1)$ and since any ideal of a r.w.r. ring is r.w.r., we have that $M_n(R)$ is r.w.r.

(c) This is given in [8, Th. 9].

(d) This is proved in [19, Prop. 20.4 (3)]. $\Diamond$

Example 2.5. Let $T_1 \subset T_2 \subset \cdots \subset T_n \subset T_{n+1} \subset \cdots$ be a strictly increasing family of r.w.r. rings, and let $T = \cup_{n=1}^{\infty} T_n$. Lift the operations on the T_n to T to obtain a ring. If $x \in T$, then x is in some T_n. Since T_n is r.w.r. we have that $x \in (xT_n)^2$ and hence $x \in (xT)^2$; so T is r.w.r. In particular this can be done where A is any r.w.r. ring and T_n is the full ring of $n \times n$ matrices over A. Then T_n embeds in T_{n+1} as the upper

$n \times n$ block. In this case T can be viewed as a certain ring of infinite matrices.

It is immediate from the definition that a r.w.r. ring is semiprime. Ramamurthi [14] showed that if R is r.w.r., then its Jacobson radical, $J(R)$, is zero. Consequently, every r.w.r. ring is isomorphic to a subdirect product of primitive r.w.r. rings. The converse does not hold since the ring of integers, $\mathbb{Z}$, is isomorphic to a subdirect product of fields. A discussion of $\mathcal{W}$ as an Amitsur–Kurosh radical class is given in [18, p. 197], where it is noted that $\mathcal{W}(R)$ is always contained in the maximal biregular ideal of R, $\mathcal{B}(R)$, and that $\mathcal{B}(\mathcal{R}/\mathcal{W}(R)) = 0$.

3. Ideals and right ideals

Proposition 3.1. *Let R be a simple ring. Then the following are equivalent:*
(a) *R is r.w.r.;*
(b) *$r \in rR$ for each $r \in R$;*
(c) *R is a right identity for $\mathbb{R}(R)$;*
(d) *$\mathbb{R}(R) \setminus \{0\}$ is a left zero semigroup; i.e., $HK = H$ for each nonzero $H, K \in \mathbb{R}(R)$.*

Proof. The equivalence of (a) and (b) is established in [1, Cor. 1]. Cor. 2.2. gives (a) implies (c). Assume (c) and let $H, K \in \mathbb{R}(R)$ with $K \neq 0$. Then $HK = (HR)K = H(RK) = HR = H$. So (c) implies (d). Next, (d) implies (a) is immediate, completing the logical circuit. $\Diamond$

The obvious left-sided version of Prop. 3.1. holds, so a simple ring with unity is both r.w.r. and l.w.r. (left weakly regular). Andruskiewicz and Puczylowski gave an example of a simple ring that is r.w.r. but not l.w.r. [1]. They also give an example of a r.w.r ring with unity that is not l.w.r. The following example illustrates that there are non-trivial simple rings which are neither r.w.r. nor l.w.r.

Example 3.2. Let R be a simple ring, not necessarily having unity, and let $R^2 \neq 0$. If $J(R) = R$, then R cannot be in $\mathcal{W}$. (Recall from [14, Prop. 14] that if $R \in \mathcal{W}$, then $J(R) = 0$, where $J(R)$ is the Jacobson radical.) Examples of such rings were given by Sasiada [15]. (Also see [16], [18, Sec. 32].) Even more extraordinary examples exist: the nil simple algebras constructed by Smoktunowicz [17].

Corollary 3.3. *Let R have a right (left) unity and let M be a maximal ideal of R. Then:*
(a) *R/M is both r.w.r. and l.w.r.;*
(b) *if $M \in \mathcal{W}$, then $R \in \mathcal{W}$.*
Proof. Since R/M is a simple ring with a right (left) unity, it has unity and hence is both r.w.r. and l.w.r. Consequently, when M is also r.w.r. we have R is r.w.r. $\diamondsuit$
Corollary 3.4. *Every ring with a right (left) unity has a nonzero homomorphic image which is r.w.r. and l.w.r.*
Proof. Every ring with a right (left) unity has a maximal ideal. $\diamondsuit$

Since $\mathcal{W}$ is an hereditary Amitsur–Kurosh radical, the sum of r.w.r. ideals is r.w.r. The analogous result does not hold for right ideals, as the next example shows.
Example 3.5. Let $S = \{a, b\}$ be the semigroup with both elements being right identities, i.e., the left zero semigroup of order two, and form the semigroup ring $\mathbb{Z}_2[S]$. The principal right ideals $\langle a\rangle_r$ and $\langle b\rangle_r$ are r.w.r. and their sum is $\mathbb{Z}_2[S]$, which is not r.w.r. since it has a nilpotent ideal, $\langle a + b\rangle_r$.

Next, embed $A = \mathbb{Z}_2[S]$ as an ideal in the ring A^1, using the Dorroh extension and $\mathbb{Z}_2 \times A$ to get a ring with unity. Since right ideals in A are also right ideals in A^1, this gives an example of a ring with unity in which the sum of two r.w.r. right ideals is not r.w.r.

The next results achieve a middle ground between the result for r.w.r. ideals and that for r.w.r. right ideals.
Proposition 3.6. *Let I and B be a r.w.r. ideal and a r.w.r. right ideal of R, respectively. Then $I + B$ is a r.w.r. right ideal of R.*
Proof. Note that $(I+B)/I \cong B/B\cap I$. Since $B/B\cap I$ is a homomorphic image of B, we have that $B/B\cap I$ and hence $I + B/I$ are r.w.r. This and I r.w.r. imply that $I + B$ is r.w.r. (Recall that if I is an ideal of a ring R such that I and R/I are r.w.r., then R is r.w.r. [14, Prop. 5].) $\diamondsuit$
Proposition 3.7. *Let B be a r.w.r. right ideal in R, and let Ω_B be the set of all r.w.r. right ideals of R that contain B. Then $(\Omega_B, \subseteq)$ has a maximal element. In particular, any ring R contains a r.w.r. right ideal which is maximal among all r.w.r. right ideals of R.*
Proof. Let $\mathcal{C}$ be a chain in $(\Omega_B, \subseteq)$ and let $T = \cup\mathcal{C}$. Let $t \in T$. There exists $A \in \mathcal{C}$ such that $t \in A$. Then $t \in (tA)^2$, and hence $t \in (tT)^2$. Thus $T \in \Omega_B$. By Zorn's Lemma $(\Omega_B, \subseteq)$ has a maximal term.

Since the zero ideal is trivially r.w.r., using $B = 0$ gives that there is a maximal element in the set of all r.w.r. right ideals of R. $\diamondsuit$

Corollary 3.8. *Let D be a maximal term in the set of all r.w.r. right ideals. Then either $D = 0$, $\mathcal{W}(R) \subseteq D$, or R is r.w.r.*

Proof. If $D \neq 0$ and $\mathcal{W}(R) \nsubseteq D$, then maximality of D yields $D + \mathcal{W}(R) = R$. $\Diamond$

Since $\mathcal{W}^1$ contains all simple rings and is closed under finite direct sums, we see that all semiprime rings with d.c.c. on right (left) ideals are in $\mathcal{W}^1$. This result can be improved. Recall that R is called an *MHR-ring* if R has d.c.c. on principal right ideals [13, p. 348]. Such rings are also called "semiperfect rings" [6]. A semiprime MHR-ring is regular [13, pp. 350, 364]. Of course, a similar result holds if MHR is replaced by d.c.c. on principal left ideals.

4. The semigroup structure of $\mathbb{R}(R)$

A semigroup S is a *left normal band* if S is a band satisfying $hkl = hlk$ for all $h, k, l \in S$ [10, p. 133].

Proposition 4.1. *If R is r.w.r., then $\mathbb{R}(R)$ is a left normal band.*

Proof. Let $H, K, L \in \mathbb{R}(R)$. Observe that $KL = (KL)^2 \subseteq KLK \subseteq KL$, so $KL = KLK$. Then from $HKL \subseteq HL$ we have $HKLK \subseteq HLK$. But $HKLK = H(KLK) = HKL$, so $HKL \subseteq HLK$. Similarly we get $HLK \subseteq HKL$, and hence $HKL = HLK$. $\Diamond$

Similarly one can show that the semigroup of left ideals of a left weakly regular ring is a *right normal band*, i.e., a band which satisfies the identity $hkl = khl$. Observe that a semigroup which is both a left normal band and a right normal band is a semilattice.

Recall that a *variety of semigroups* is a non-empty class of semigroups which is closed under direct products, subsemigroups, and homomorphic images [3, p. 61]. We use **LN** for the class of all left normal bands, **LZ** for the class of all left zero semigroups (i.e., semigroups satisfying $hk = k$ for all h, k), and **SL** for the class of all semilattices. Observe that **LN** is a variety and **LZ** and **SL** are proper subvarieties of **LN**. (For basic ideas and terminology on varieties, see [3].) The following result, cited in Ćirić and Bogdanović, yields insight into the makeup of the class **LN**.

Proposition 4.2. ([4, p. 53]) *The only proper subvarieties of **LN** are **LZ** and **SL**.*

In light of Prop. 4.2. it is natural to ask if it possible that $\mathbb{R}(R) \in \mathbf{SL}$ or $\mathbb{R}(R) \in \mathbf{LZ}$. The former question has already been answered for rings with identity in [11, Th. 3.5]. A minor change in that argument gives us the same result for rings which do not necessarily contain identity.

Since a non-trivial left zero semigroup does not have a zero element, it is not possible for a r.w.r. ring R to satisfy $\mathbb{R}(R) \in \mathbf{LZ}$. However, we still have a characterization of $\mathbb{R}(R)$ in terms of **LZ**. Recall from Prop. 3.1. that for a simple ring R we have that R is r.w.r. if and only if $\mathbb{R}(R)$ is a left zero semigroup with zero adjoined.

We now consider $\mathbb{I}(R)$, the semigroup of ideals of a ring R, and $\mathbb{L}(R)$, the semigroup of left ideals of R.

Proposition 4.3. *Let $\mathbb{I}(R)$ be von Neumann regular. Then:*
(a) $\mathbb{I}(R)$ *is a semilattice with identity;*
(b) *if $B \in \mathbb{R}(R) \cup \mathbb{L}(R)$, then $B^3 = B^4$;*
(c) *if R has unity, then $B^2 = B^3$ for each $B \in \mathbb{R}(R) \cup \mathbb{L}(R)$.*

Proof. (a) Let $I \in \mathbb{I}(R)$. Then $I = ITI$ for some $T \in \mathbb{I}(R)$ and hence $I \subseteq IT \subseteq I$, or $I = IT$. Note that IT is idempotent.

To see that $\mathbb{I}(R)$ is commutative, again let $I, T \in \mathbb{I}(R)$. Then $IT = = IIT = ITI$ by left normality. But $ITI \subseteq TI$. Thus, $IT \subseteq TI$. The same argument yields $TI \subseteq IT$. It follows that R is the identity for $\mathbb{I}(R)$.

(b) Let $B \in \mathbb{R}(R)$. Then $RB \in \mathbb{I}(R)$ and hence $B^2 \subseteq RB = = (RB)^3 \subseteq RB^3$. So $B^3 \subseteq BRB^3 \subseteq B^4$, and hence $B^3 = B^4$. Proceed similarly for $B \in \mathbb{L}(R)$.

(c) For $B \in \mathbb{R}(R)$, since R has unity we have $B \subseteq RB$. Hence $B \subseteq (RB)^2 \subseteq RB^2$, and so $B^2 \subseteq BRB^2 \subseteq B3$. So $B = B^3$. Proceed similarly for $B \in \mathbb{L}(R)$. $\Diamond$

A ring for which every ideal is idempotent is called a *fully idempotent ring* [5]. (The term "weakly regular" is also used.) We see that R is fully idempotent if and only if $\mathbb{I}(R)$ is von Neumann regular, and in this case each of the semigroups $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are π-regular. (Recall that a semigroup S is *π-regular* if and only if for each $s \in S$ there exists $s' \in S, n \in \mathbb{N}$, such that $s^n = s^n s' s^n$, where n may depend on S.)

A similar argument to that used in part (b) above establishes the following result.

Corollary 4.4. *If $\mathbb{I}(R)$ is π-regular, then $\mathbb{R}(R)$ and $\mathbb{L}(R)$ are π-regular.*

Proof. Let $B \in \mathbb{R}(R)$. Then $RB \in \mathbb{I}(R)$ and hence there exists $T \in \mathbb{I}(R)$ and some $n \in \mathbb{N}$ such that $(RB)^n = (RB)^n T (RB)^n$ and hence $(RB)^n = = (RB)^n T$. Thus $(RB)^n$ is idempotent. So $B^{2n} \subseteq (RB)^n = (RB)^{3n} \subseteq RB^{3n}$, and hence $B^{2n+1} \subseteq BRB^{3n} \subseteq B^{3n+1}$. Since $B^{3n+1} \subseteq B^{2n+1}$ we have $B^{2n+1} = B^{3n+1}$ and so $\mathbb{R}(R)$ is π-regular. Proceed similarly for $\mathbb{L}(R)$. $\Diamond$

Let $\mathbb{L}_2(R)$ denote the set of squares of left ideals of R.

Corollary 4.5. *If R is a r.w.r. ring with identity, then $\mathbb{L}_2(R)$ is a subsemigroup of $\mathbb{L}(R)$ and is a homomorphic image of $\mathbb{L}(R)$.*

Proof. (a) We show that for each $L, T \in \mathbb{L}(R)$ we have $L^2T^2 = (LT)^2$, which establishes that $\mathbb{L}_2(R)$ is a subsemigroup and that the mapping $L \to L^2$ is a homomorphism from $\mathbb{L}(R)$ to $\mathbb{L}_2(R)$. Observe that $L^2T^2 = L(RL)(RT)RT = (LR)(LR)(TR)T = (LR)(TR)(LR)T$, since $\mathbb{R}(R)$ is left normal. But $(LR)(TR)(LR)T = LTLT$, giving the desired result. $\Diamond$

Finally, we present decompositions for R in terms of $\mathbb{R}(R)$ and for $\mathbb{R}(R)$ in terms of R.

Lemma 4.6. *Let H be a right ideal and I an ideal of a r.w.r. ring R. Then $HI = H \cap I$.*

Proof. $HI \subseteq H \cap I = (H \cap I)^2 \subseteq HI$; so $HI = H \cap I$. $\Diamond$

Ramamurthi gives the above result, without proof, for r.w.r. rings with unity [14, Remark 3(b)].

Proposition 4.7. *Let R be r.w.r. If $R = A \oplus B$, as a direct sum of ideals, then $\mathbb{R}(R)$ is isomorphic as a semigroup to $\mathbb{R}(A) \times \mathbb{R}(B)$.*

Proof. Let H be an arbitrary right ideal of R. Then $H = HR = H(A \oplus B) = (HA) \oplus (HB) = (H \cap A) \oplus (H \cap B)$. Define $\phi : \mathbb{R}(R) \to \mathbb{R}(A) \times \mathbb{R}(B)$ via $\phi(H) = (HA, HB)$. Making use of the fact that $\mathbb{R}(R)$ is a left normal band, we have $\phi(HK) = (HKA, HKB) = (HKAA, HKBB) = (HAKA, HBKB) = (HA, HB)(KA, KB) = \phi(H)\phi(K)$ for each $K \in \mathbb{R}(R)$. If $\phi(H) = \phi(K)$, then $H \cap A = K \cap A$ and $H \cap B = K \cap B$; so $H = (H \cap A) + (H \cap B) = (K \cap A) + (K \cap B) = K$. Next, let $T \in \mathbb{R}(A)$, $V \in \mathbb{R}(B)$. Then T, V and $T + V$ are in $\mathbb{R}(R)$. Observe that $\phi(T + V) = ((T + V)A, (T + V)B) = (TA, VB)$. However, since A and B are r.w.r., A is a right identity for $\mathbb{R}(A)$ and B is a right identity for $\mathbb{R}(B)$. So $\phi(T + V) = (T, V)$. So ϕ is the desired isomorphism. $\Diamond$

Lemma 4.8. *Let $A, B \in \mathbb{R}(R)$ such that $AB = 0$. If either* (a) *R is r.w.r. or* (b) *R has unity and is fully idempotent, then $A \cap B = 0$.*

Proof. If R is r.w.r., then $A \cap B = (A \cap B)^2 \subseteq AB = 0$. If R has unity and is fully idempotent, then $RARB \subseteq RAB = 0$. So $A \cap B \subseteq (RA) \cap (RB) = [(RA) \cap (RB)]^2 \subseteq RARB = 0$. $\Diamond$

Proposition 4.9. *Let R be r.w.r. If the semigroup $\mathbb{R}(R)$ is isomorphic to $S_1 \times S_2$, where each S_j is non-trivial, then there exist non-zero ideals A and B of R such that $R = A \oplus B$.*

Proof. Let $\psi : \mathbb{R}(R) \to S_1 \times S_2$ be a semigroup isomorphism. Observe that S_1 and S_2 are semigroups with zero and that $\psi(0) = (0,0)$. Since R is a right identity for $\mathbb{R}(R)$ we have $\psi(R) = (e_1, e_2)$, where e_j is a right identity for $S_j, j = 1,2$. Let $A, B \in \mathbb{R}(R)$ such that $\psi(A) = (e_1, 0)$, $\psi(B) = = (0, e_2)$. Then $\psi(AB) = \psi(A)\psi(B) = (0,0)$ and hence $AB = 0$. So $A \cap B = 0$. Next, $\psi(RA) = \psi(R)\psi(A) = (e_1, e_1)(e_1, 0) = \psi(A)$, and hence $RA = A$ and A is an ideal of R. Similarly, B is an ideal of R.

Let $\psi(A \oplus B) = (a, b) \in S_1 \times S_2$. Then using $(A \oplus B)B = B$, we have $\psi(A \oplus B)\psi(B) = \psi((A \oplus B)B) = \psi(B) = (0, e_2)$. Also, $\psi(A \oplus B)\psi(B) = = (a, b)(0, e_2) = (0, be_2)$. So $(0, e_2) = (0, be_2)$ and hence $e_2 = be_2 = b$. Similarly, $e_1 = a$ and consequently $\psi(A \oplus B) = (e_1, e_2) = \psi(R)$. So $R = A \oplus B$.

Finally, note that if $A = 0$, then $e_1 = 0$, forcing $S_1 = 0$. $\Diamond$

It is worth noting that in neither Prop. 4.7. or 4.9. are the rings assumed to have unity.

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