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A FAMILY OF LIMIT DISTRIBUTIONS IN THE METRICAL THEORY OF CONTINUED FRACTIONS

Gabriela Ileana **Sebe**

Department of Mathematics I, University Politehnica of Bucharest, Splaiul Independenţei 313, 060042 Bucharest, Romania

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Abstract: We give a generalization of Th. 2.5.8 from [6], namely, we derive the asymptotic behaviour of $\gamma_a(\tau^{n+m} \le x, s_n^a \le y)$ as $n \to \infty$ for any $a, x, y \in I$ and $m \in \mathbb{N}_+$. We also derive corresponding upper and lower bounds which are of order $O(g^{2n})$ as $n \to \infty$, too.

1. Introduction

Let Ω denote the collection of irrational numbers in the unit interval $I = [0, 1]$. Given $\omega \in \Omega$, let $a_1(\omega), a_2(\omega), \dots$ be the sequence of the incomplete quotients of the continued fraction expansion of ω. That is, defining the continued fraction transformation $\tau : \Omega \to \Omega$ by $\tau(\omega) = \dfrac{1}{\omega}$ (mod 1) = fractionary part of $\dfrac{1}{\omega}$, $\omega \in \Omega$, we have $a_{n+1}(\omega) = a_1(\tau^n(\omega))$, $n \in \mathbb{N}_+ = \{1, 2, \dots\}$, with $a_1(\omega) =$ integer part of $\dfrac{1}{\omega}$. Here τ^n denotes the nth iterate of τ. Then, by the very definition,

E-mail address: gisebe@mathem.pub.ro

$$\omega = \frac{1}{a_1(\omega) + \tau(\omega)} = \cfrac{1}{a_1(\omega) + \cfrac{1}{a_2(\omega) + \ddots + \cfrac{1}{a_n(\omega) + \tau^n(\omega)}}}, \quad n \geq 2,$$

and we have

$$\omega = \lim_{n \to \infty} \frac{p_n(\omega)}{q_n(\omega)} := [a_1(\omega), a_2(\omega), \ldots], \ \omega \in \Omega,$$

where

$$\frac{p_n(\omega)}{q_n(\omega)} = \cfrac{1}{a_1(\omega) + \cfrac{1}{a_2(\omega) + \ddots + \cfrac{1}{a_n(\omega)}}},$$

with g.c.d. $(p_n(\omega), q_n(\omega)) = 1$, $\omega \in \Omega$, $n \in \mathbb{N}_+$.

Clearly, the a_n, $n \in \mathbb{N}_+$, can be viewed as random variables on $(I, \mathcal{B}_I)$, where $\mathcal{B}_I$ is the collection of Borel subsets of I, that are defined almost surely with respect to any probability measure on $\mathcal{B}_I$ assigning measure 0 to the set of rationals in I. Such a probability measure is Lebesgue measure λ, but a more important one in the present context is the Gauss measure γ defined by

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}, \quad A \in \mathcal{B}_I.$$

We have $\gamma = \gamma\tau^{-1}$, that is, $\gamma(A) = \gamma(\tau^{-1}(A))$, $A \in \mathcal{B}_I$. Therefore, by its very definition, $(a_n)_{n \in \mathbb{N}_+}$ is a strictly stationary sequence on $(I, \mathcal{B}_I, \gamma)$. Note that

$$G(x) := \gamma([0, x]) = \int_0^1 \gamma_a([0, x])\gamma(da), \ x \in I,$$

where $(\gamma_a)_{a \in I}$ is the family of probability measures on $\mathcal{B}_I$ defined by their distribution functions

$$\gamma_a([0, x]) = \frac{(a+1)x}{ax+1}, \quad x \in I, \ a \in I.$$

In particular, we have $\gamma_0 = \lambda$, the Lebesgue measure on $\mathcal{B}_I$. For any $a \in I$ and $n \in \mathbb{N}_+$ we have

$$\gamma_a(\tau^n < x | a_1, \ldots, a_n) = \frac{(s_n^a + 1)x}{s_n^a x + 1}, \quad x \in I,$$

(see Prop. 1.3.8 in [6]) where the s_n^a are defined recursively by $s_0^a = a$ and

$$s^a_{n+1} = \frac{1}{a_{n+1} + s^a_n}, \quad a \in I,\ n \in \mathbb{N}.$$

Since $\tau^n(\omega) = [a_{n+1}(\omega), a_{n+2}(\omega), \dots]$, $n \in \mathbb{N}$, $\omega \in \Omega$, it follows that

$$\gamma_a(a_{n+1} = i|a_1, \dots, a_n) = \gamma_a\left(\frac{1}{i+1} < \tau^n < \frac{1}{i}|a_1, \dots, a_n\right)$$
$$= \frac{s^a_n + 1}{(s^a_n + i)(s^a_n + i + 1)} := P_i(s^a_n)$$

for any $a \in I$ and $i, n \in \mathbb{N}_+$. Hence for any $a \in I$ the sequence $(s^a_n)_{n\in\mathbb{N}}$ on $(I, \mathcal{B}_I, \gamma_a)$, with $\mathbb{N} = \{0\} \cup \mathbb{N}_+$, is an I-valued Markov chain which starts at $s^a_0 = a$ and has the following transition mechanism: from state $s \in I$ the possible transitions are to any state $1/(s+i)$ with corresponding transition probability $(s+1)/(s+i)(s+i+1)$, $i \in \mathbb{N}_+$.

In a series of papers (see [2], [3], [4], [5]) explicit lower and upper bounds are derived for the convergence rate of the distribution function of s^a_n to its limit, the Gauss distribution function $G(x) = \frac{1}{\log 2}\log(x+1)$, $0 \le x \le 1$, as $n \to \infty$. A survey of this subject is presented in Sec. 2.5.3 of the monograph [6].

We recall Th. 2.5.5 from [6] according to which

(1)
$$\frac{a+1}{2(F_n + aF_{n-1})(F_{n+1} + aF_n)} \le \sup_{x\in I} |\gamma_a(s^a_n \le x) - G(x)| \le \frac{k_0}{F_n F_{n+1}}$$

for any $a \in I$ and $n \in \mathbb{N}$, where k_0 is a constant not exceeding 14.8 and F_n, $n \in \mathbb{N}$, are the Fibonacci numbers defined by $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \ge 2$. Both lower and upper bounds in (1) are $O(g^{2n})$ as $n \to \infty$ with $g = (\sqrt{5}-1)/2$, $g^2 = (3-\sqrt{5})/2 = 0.38196\dots$, thus yielding the optimal convergence rate.

Inequalities (1) allow a quick derivation of the asymptotic behaviour of

$$\gamma_a(\tau^n \le x, s^a_n \le y)$$

as $n \to \infty$ for any $a, x, y \in I$, and of the optimal convergence rate, the same as above. Generalizing the main result in [1], Th. 2.5.8 from [6] establishes that

$$\frac{a+1}{2(F_n+aF_{n-1})(F_{n+1}+aF_n)} \le \sup_{x,y\in I}\left|\gamma_a(\tau^n \le x, s^a_n \le y) - \frac{\log(xy+1)}{\log 2}\right| \le$$
$$\le \frac{k_0}{F_n F_{n+1}}$$

for any $a \in I$ and $n \in \mathbb{N}$.

In this paper Th. 2.5.8 from [6] is generalized. We derive the asymptotic behaviour of

$$\gamma_a(\tau^{n+m} \leq x,\ s_n^a \leq y)$$

as $n \to \infty$ for any $a, x, y \in I$ and $m \in \mathbb{N}_+$. We also derive upper and lower bounds which are of order $O(g^{2n})$ as $n \to \infty$, too. In the last section we derive the asymptotic behaviour as both n and $m \to \infty$.

2. A few prerequisites

The transition operator U of the Markov chain $(s_n^a)_{n\in\mathbb{N}_+}$ is

$$Uf(x) = \sum_{i\in\mathbb{N}_+} P_i(x) f(u_i(x)),\ x \in I,\ f \in B(I),$$

where $B(I)$ denotes the collection of all bounded measurable functions $f : I \to \mathbf{C}$, and where the functions u_i and P_i, $i \in \mathbb{N}_+$, are defined by

$$u_i(x) = \frac{1}{x+i},\ P_i(x) = \frac{x+1}{(x+i)(x+i+1)},\ x \in I.$$

Let us consider for any $x \in I$ and $m \geq 2$ the functions

$$\begin{gathered} u_{i_m \ldots i_1} = u_{i_m} \circ \ldots \circ u_{i_1}, \\ P_{i_1 \ldots i_m}(x) = P_{i_1}(x) P_{i_2}(u_{i_1}(x)) \ldots P_{i_m}(u_{i_{m-1}} \ldots i_1(x)). \end{gathered} \tag{2}$$

Let us put

$$s_{n+m}^a(i^{(m)}) = \cfrac{1}{i_m + \cfrac{}{\ddots + \cfrac{1}{i_1 + s_n^a}}},\ n, m \in \mathbb{N}_+,\ a \in I, \tag{3}$$

where $i^{(m)} = (i_1, \ldots, i_m) \in \mathbb{N}_+^m$.

Proposition 1. *For any $a \in I$ and $n, m \in \mathbb{N}_+$ we have*

$$\gamma_a(\tau^{n+m} < x | a_1, \ldots, a_n) = \sum_{i_1,\ldots,i_m \in \mathbb{N}_+} \frac{x(s_{n+m}^a(i^{(m)}) + 1)}{s_{n+m}^a(i^{(m)})x + 1} P_{i_1 \ldots i_m}(s_n^a). \tag{4}$$

Proof. For any $a \in I$ and $n, m \in \mathbb{N}_+$ we have

(5)

$$\gamma_a(\tau^{n+m} < x|a_1, \dots , a_n) =$$
$$= \sum_{i_1,\dots,i_m \in \mathbb{N}_+} \gamma_a(\tau^{n+m} < x, a_{n+1} = i_1, \dots , a_{n+m} = i_m|a_1, \dots , a_n) =$$
$$= \sum_{i_1,\dots,i_m \in \mathbb{N}_+} \gamma_a(\tau^{n+m} < x|a_1, \dots , a_n, a_{n+1} = i_1, \dots , a_{n+m} = i_m)\times$$
$$\times \gamma_a(a_{n+m} = i_m, \dots , a_{n+1} = i_1|a_1, \dots , a_n).$$

Using (3), it follows from the generalized Brodén–Borel–Lévy formula (Prop. 1.3.8 in [6]) that

$$\gamma_a(\tau^{n+m} < x|a_1, \dots , a_n, a_{n+1} = i_1, \dots , a_{n+m} = i_m) = \tag{6}$$
$$= \frac{x(s^a_{n+m}(i^{(m)}) + 1)}{s^a_{n+m}(i^{(m)})x + 1}.$$

By Cor. 1.3.9 in [6], we have

$$\gamma_a(A|a_1, \dots , a_n) = \gamma_{s^a_n}(\tau^n(A)),\ a \in I,\ n \in \mathbb{N}_+, \tag{7}$$

for any set A belonging to the σ-algebra generated by the random variables $a_{n+1}, a_{n+2}, \dots$.

Now, using (7) and equation (2.5.4) in [6], i.e.,

$$P_{i_1\dots i_m}(a) = \gamma_a(I(i^{(m)})),$$

where $I(i^{(m)}) = (\omega \in \Omega : a_1(\omega) = i_1, \dots , a_m(\omega) = i_m)$ is the fundamental interval of rank m, $m \in \mathbb{N}_+$, we obtain

$$\gamma_a(a_{n+m} = i_m, \dots , a_{n+1} = i_1|a_1, \dots , a_n) = P_{i_1\dots i_m}(s^a_n). \tag{8}$$

From (5), (6) and (8), equation (4) follows. $\Diamond$

Now, by (1.2.4) in [6], $I(i^{(m)})$ is the collection of irrationals in the interval with end-points p_m/q_m and $(p_m + p_{m-1})/(q_m + q_{m-1})$. Since

$$\frac{p_m}{q_m} = [i_1, \dots , i_m] = \begin{cases} \dfrac{1}{i_1}, & m = 1 \\ \dfrac{1}{i_1 + p_{m-1}(i_2, ..., i_m)/q_{m-1}(i_2, ..., i_m)}, & m > 1 \end{cases}$$

and

$$\frac{p_m + p_{m-1}}{q_m + q_{m-1}} = \begin{cases} \dfrac{1}{i_1 + 1}, & m = 1 \\ [i_1, \dots , i_{m-1}, i_m + 1], & m > 1 \end{cases} =$$

$$= \begin{cases} \dfrac{1}{i_1+1}, & m=1 \\ \dfrac{1}{i_1+p_m(i_2,\ldots,i_m,1)/q_m(i_2,\ldots,i_m,1)}, & m>1 \end{cases}$$

we have

(9)
$$P_{i_1\ldots i_m}(x) = (x+1) \times \frac{1}{q_{m-1}(i_2,\ldots,i_m)(x+i_1)+p_{m-1}(i_2,\ldots,i_m)} \times \\ \times \frac{1}{q_m(i_2,\ldots,i_m,1)(x+i_1)+p_m(i_2,\ldots,i_m,1)}$$

for any $m \geq 2$, $i^{(m)} \in \mathbb{N}_+^m$ and $x \in I$.

Finally, note that $u_{i_1\ldots i_m}(x)$ can be written as

$$u_{i_1\ldots i_m}(x) = \frac{p_{m-1}x+p_m}{q_{m-1}x+q_m},\ x \in I,\ m \in \mathbb{N}_+, \tag{10}$$

with $p_0 = 0$, $q_0 = 1$.

In the sequel we also need the well-known equation

$$p_m q_{m-1} - p_{m-1} q_m = (-1)^{m+1},\ m \in \mathbb{N}. \tag{11}$$

3. The main result

We are now in a position to prove our main result which reads as follows.

Theorem 1. *For any $a \in I$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_+$ we have*

$$\begin{aligned} &\frac{a+1}{2(F_n+aF_{n-1})(F_{n+1}+aF_n)} \leq \\ &\leq \sup_{x,y\in I} |\gamma_a(\tau^{n+m} \leq x, s_n^a \leq y) - H_m(x,y)| \leq \frac{6k_0}{F_nF_{n+1}}, \end{aligned} \tag{12}$$

where

$$H_m(x,y) = \frac{1}{\log 2} \log \prod_{i_1,\ldots,i_m} \left(1 + \frac{(-1)^m xy}{(q_{m-1}x+q_m)(p_my+q_m)}\right)^{(-1)^m}, \\ x,y \in I.$$

The probability density of the limiting distribution H_m is

$$h_m(x,y)=\frac{1}{\log 2}\sum_{i_1,\dots,i_m}\frac{1}{(p_{m-1}xy+q_{m-1}x+p_my+q_m)^2},\ x,y\in I.$$

Proof. Set $G_n^a(y)=\gamma_a(s_n^a\le y), H_n^a(y)=G_n^a(y)-G(y),\ a,y\in I$, $n\in\mathbb{N}$. By (1) we have

$$|H_n^a(y)|\le\frac{k_0}{F_nF_{n+1}},\ a,y\in I,\ n\in\mathbb{N}, \tag{13}$$

where k_0 is a constant not exceeding 14.8.

By Prop. 1 and equations (2) and (3), for any $a,x,y\in I$ and $n\in\mathbb{N}$, $m\in\mathbb{N}_+$, we have

$$\begin{aligned}\gamma_a(\tau^{n+m}\le x, s_n^a\le y)&=\int_0^y\gamma_a(\tau^{n+m}\le x|s_n^a=z)dG_n^a(z)=\\&=\int_0^y\sum_{i_1,\dots,i_m\in\mathbb{N}_+}\frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1}P_{i_1\dots i_m}(z)dG_a^n(z).\end{aligned}$$

When applying Prop. 1 we used the fact that the σ-algebras generated by $(a_1,\dots,a_n)$ and by s_n^a are identical for any $a\in I$ and $n\in\mathbb{N}_+$.

Using equation (9), the right-hand member above can be written as

$$\begin{aligned}&\frac{1}{\log 2}\int_0^y\sum_{i_1,\dots,i_m\in\mathbb{N}_+}\frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1}\times\frac{1}{q'_{m-1}(z+i_1)+p'_{m-1}}\times\\&\qquad\times\frac{dz}{q''_m(z+i_1)+p''_m}+\\&+\int_0^y\sum_{i_1,\dots,i_m\in\mathbb{N}_+}\frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1}\times\frac{z+1}{q'_{m-1}(z+i_1)+p'_{m-1}}\times\\&\qquad\times\frac{dH_n^a(z)}{q''_m(z+i_1)+p''_m},\end{aligned}$$

where, to simplify notation, we put

$$q_{m-1}(i_2,\dots,i_m)=q'_{m-1}, p_{m-1}(i_2,\dots,i_m)=p'_{m-1},$$
$$q_m(i_2,\dots,i_m,1)=q''_m,\quad p_m(i_2,\dots,i_m,1)=p''_m.$$

Also, using elementary properties of s_n^a and q_n, we have

$$\begin{aligned}u_{i_m\dots i_1}(z)&=\frac{q_{m-1}(z+i_1,i_2,...,i_{m-1})}{q_m(z+i_1,i_2,...,i_m)}=\frac{q_{m-1}(i_{m-1},...,i_2,z+i_1)}{q_m(i_m,...,i_2,...,i_m)}=\\&=\frac{(z+i_1)q'_{m-2}+q_{m-3}(i_3,...,i_{m-1})}{(z+i_1)q'_{m-1}+q_{m-2}(i_3,...,i_m)}=\frac{(z+i_1)q'_{m-2}+p'_{m-2}}{(z+i_1)q'_{m-1}+p'_{m-1}}\end{aligned}$$

hence

(14)
$$\frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1} = \frac{x((z+i_1)q''_m+p''_m)}{x[q'_{m-2}(z+i_1)+p'_{m-2}]+q'_{m-1}(z+i_1)+p'_{m-1}}.$$

(i) *The upper bound.* We start with computing

$$S_1 := \frac{1}{\log 2}\int_0^y \sum_{i_1,\dots,i_m} \frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1} \times \frac{1}{q'_{m-1}(z+i_1)+p'_{m-1}} \times \frac{dz}{q''_m(z+i_1)+p''_m}.$$

Using (10), (11) and (14), it is easy to check that

$$S_1 = \frac{1}{\log 2}\sum_{i_1,\dots,i_m}(-1)^m\int_0^y\left(\frac{1}{z+i_1+u_{i_2\dots i_m}(x)} - \frac{1}{z+i_1+u_{i_2\dots i_m}(0)}\right)dz =$$
$$= \frac{1}{\log 2}\sum_{i_1,\dots,i_m}(-1)^m\Big[\log(z+i_1+u_{i_2\dots i_m}(x)) - \log(z+i_1+u_{i_2\dots i_m}(0))\Big]\Big|_{z=0}^{z=y} =$$
$$= \frac{1}{\log 2}\log\prod_{i_1,\dots,i_m}\left(\frac{y+i_1+u_{i_2\dots i_m}(x)}{y+i_1+u_{i_2\dots i_m}(0)}\cdot\frac{i_1+u_{i_2\dots i_m}(0)}{i_1+u_{i_2\dots i_m}(x)}\right)^{(-1)^m} =$$
$$= \frac{1}{\log 2}\log\prod_{i_1,\dots,i_m}\left(\frac{y+(q_{m-1}x+q_m)/(p_{m-1}x+p_m)}{y+q_m/p_m}\cdot\frac{q_m/p_m}{(q_{m-1}x+q_m)/(p_{m-1}x+p_m)}\right)^{(-1)^m} =$$
$$= \frac{1}{\log 2}\log\prod_{i_1,\dots,i_m}\left(1+\frac{(-1)^m xy}{(q_{m-1}x+q_m)(p_my+q_m)}\right)^{(-1)^m} =$$
$$= H_m(x,y).$$

Now, put

$$S_2 = \int_0^y \sum_{i_1,\dots,i_m} \frac{x(u_{i_m\dots i_1}(z)+1)}{xu_{i_m\dots i_1}(z)+1} \times \frac{z+1}{q'_{m-1}(z+i_1)+p'_{m-1}} \times \\ \times \frac{dH_n^a(z)}{q''_m(z+i_1)+p''_m}.$$

Using again (10) and (14), we obtain

$$S_2 = \int_0^y \sum_{i_1,\dots,i_m} \frac{(z+1)/q'_{m-2}}{z+i_1+p'_{m-2}/q'_{m-2}} \Big[\frac{1/q'_{m-1}}{z+i_1+u_{i_2\dots i_m}(0)} - \\ - \frac{1/(q'_{m-2}x+q'_{m-1})}{z+i_1+u_{i_2\dots i_m}(x)} \Big] \times dH_n^a(z).$$

Integrating by part now yields

$$S_2 = \sum_{i_1,\dots,i_m} \Bigg\{ \frac{(y+1)/q'_{m-2}}{y+i_1+p'_{m-2}/q'_{m-2}} \Big[\frac{1/q'_{m-1}}{y+i_1+u_{i_2\dots i_m}(0)} - \\ - \frac{1/(q'_{m-2}x+q'_{m-1})}{y+i_1+u_{i_2\dots i_m}(x)} \Big] H_n^a(y) - \\ - \int_0^y (-1)^m \frac{d}{dz} \Big[(z+1) \Big(\frac{1}{z+i_1+u_{i_2\dots i_m}(x)} - \\ - \frac{1}{z+i_1+u_{i_2\dots i_m}(0)} \Big) \Big] H_n^a(z)dz \Bigg\}.$$

Next, put

$$S_3 = \sum_{i_1,\dots,i_m} (-1)^m \int_0^y \frac{d}{dz} \Big[(z+1) \Big(\frac{1}{z+i_1+u_{i_2\dots i_m}(x)} - \\ - \frac{1}{z+i_1+u_{i_2\dots i_m}(0)} \Big) \Big] H_n^a(z)dz = \\ = \sum_{i_1,\dots,i_m} (-1)^m \int_0^y \Big[\frac{i_1+u_{i_2\dots i_m}(x)-1}{(z+i_1+u_{i_2\dots i_m}(x))^2} - \\ - \frac{i_1+u_{i_2\dots i_m}(0)-1}{(z+i_1+u_{i_2\dots i_m}(0))^2} \Big] H_n^a(z)dz = \\ = \sum_{i_1,\dots,i_m} (-1)^m \int_0^y \big[(A(z)-B(z))(1-(z+1)(A(z)+B(z))) \big] H_n^a(z)dz,$$

where

$$A(z) = \frac{1}{z + i_1 + u_{i_2 \dots i_m}(x)} \quad \text{and} \quad B(z) = \frac{1}{z + i_1 + u_{i_2 \dots i_m}(0)}.$$

Using (13), since $|1-(z+1)(A(z)+B(z))| \le 1$ and $A(z)-B(z)$ preserves a constant sign, for any fixed $m \in \mathbb{N}_+$ and any $i_1, \dots, i_m \in \mathbb{N}_+$ we have

$$\begin{aligned}
&|\gamma_a(\tau^{n+m} \le x, s_n^a \le y) - H_m(x,y)| \le \\
&\le \frac{k_0}{F_n F_{n+1}} \left(\left| \sum_{i_1,\dots,i_m} \frac{(y+1)/q'_{m-2}}{y + i_1 + p'_{m-2}/q'_{m-2}} \left[\frac{1/q'_{m-1}}{y + i_1 + u_{i_2\dots i_m}(0)} - \right.\right.\right. \\
&\qquad\qquad \left.\left. - \frac{1/(q'_{m-2}x + q'_{m-1})}{y + i_1 + u_{i_2\dots i_m}(x)} \right] \right| + \\
&\quad \left. + \sum_{i_1,\dots,i_m} \left| \log \frac{y + i_1 + u_{i_2\dots i_m}(x)}{i_1 + u_{i_2\dots i_m}(x)} - \log \frac{y + i_1 + u_{i_2\dots i_m}(0)}{i_1 + u_{i_2\dots i_m}(0)} \right| \right) = \\
&= \frac{k_0}{F_n F_{n+1}} \left(\left| \sum_{i_1,\dots,i_m} (-1)^m (y+1) \left[\frac{1}{y + i_1 + u_{i_2\dots i_m}(x)} - \right.\right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{y + i_1 + u_{i_2\dots i_m}(0)} \right] \right| + \\
&\quad + \sum_{i_1,\dots,i_m} \left| \log\left(1 + \frac{u_{i_2\dots i_m}(x) - u_{i_2\dots i_m}(0)}{y + i_1 + u_{i_2\dots i_m}(0)} \right) - \right. \\
&\qquad\qquad \left.\left. - \log\left(1 + \frac{u_{i_2\dots i_m}(x) - u_{i_2\dots i_m}(0)}{i_1 + u_{i_2\dots i_m}(0)} \right) \right| \right) = \\
&= \frac{k_0}{F_n F_{n+1}} (|S_4| + S_5),
\end{aligned}$$

where S_4 and S_5 are the two series occurring on the right-hand side above. Since $\displaystyle\sum_{i_1,\dots,i_m} \frac{1}{q_m(q_{m-1}+q_m)} = 1$, it follows that

$$\begin{aligned}
|S_4| &\le 2 \sum_{i_1,\dots,i_m} \left| \frac{1}{y + (q_{m-1}x + q_m)/(p_{m-1}x + p_m)} - \frac{1}{y + q_m/p_m} \right| \le \\
&\le 2 \sum_{i_1,\dots,i_m} \frac{1}{[(p_{m-1}x + p_m)y + q_{m-1}x + q_m](p_m y + q_m)} \le
\end{aligned}$$

$$\leq 2 \sum_{i_1,\dots,i_m} \frac{1}{q_m^2} \leq 4 \sum_{i_1,\dots,i_m} \frac{1}{q_m(q_{m-1}+q_m)} = 4.$$

Since

$$\log\left(1+\frac{u_{i_2\dots i_m}(x)-u_{i_2\dots i_m}(0)}{y+i_1+u_{i_2\dots i_m}(0)}\right) \leq$$
$$\leq \frac{|(q_{m-1}x+q_m)/(p_{m-1}x+p_m)-q_m/p_m|}{q_m/p_m+y} =$$
$$= \frac{x}{(p_{m-1}x+p_m)(p_my+q_m)} \leq \frac{1}{p_m(p_{m-1}+p_m)}$$

for all $x, y \in I$ it follows that

$$S_5 \leq 2 \sum_{i_1,\dots,i_m} \frac{1}{p_m(p_{m-1}+p_m)} = 2.$$

Thus the upper bound announced follows from $|S_4| + S_5 \leq 4 + 2 = 6$.

(ii) *The lower bound*. It follows from the result just established that

$$H_m(1,y) = G(y) = \frac{1}{\log 2}\log(y+1),\ m \in \mathbb{N}_+,\ y \in I.$$

(This can be also checked by direct computation.) Then, for any $a \in I$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_+$ we have

$$\sup_{x,y\in I}\left|\ \gamma_a(\tau^{n+m} \leq x, s_n^a \leq y) - H_m(x,y)\right| \geq$$
$$\geq \sup_{y\in I}\left|\ \gamma_a(\tau^{n+m} \leq 1, s_n^a \leq y) - H_m(1,y)\right| =$$
$$= \sup_{y\in I}|\gamma_a(s_n^a \leq y) - G(y)| \geq \frac{a+1}{2(F_n+aF_{n-1})(F_{n+1}+aF_n)}. \qquad \Diamond$$

Remark. The upper and lower bounds in (12) are of order $O(g^{2n})$ as $n \to \infty$.

4. Approximating the limiting distribution

Using some mixing properties of the sequence $(a_n)_{n\in\mathbb{N}_+}$ under γ_a, $a \in I$, we shall now provide an approximation of the limiting distribution H_m, see Th. 1. We use the notation in Subsec. 1.3.6 from [6].

For any $k \in \mathbb{N}_+$, let $\mathcal{B}_1^k = \sigma(a_1,\dots,a_k)$ and $\mathcal{B}_k^\infty = \sigma(a_k, a_{k+1},\dots)$

denote the σ-algebras generated by the random variables $a_1, \dots, a_k$, respectively, $a_k, a_{k+1}, \dots$.

For any γ_a, $a \in I$, consider the ψ-mixing coefficients

$$\psi_{\gamma_a}(n) = \sup \left| \frac{\gamma_a(A \cap B)}{\gamma_a(A)\gamma_a(B)} - 1 \right|, \; n \in \mathbb{N}_+, \tag{15}$$

where the supremum is taken over all $A \in \mathcal{B}_1^k$ and $B \in \mathcal{B}_{k+n}^{\infty}$ such that $\gamma_a(A)\gamma_a(B) \neq 0$, and $k \in \mathbb{N}_+$.

By Prop. 2.3.7 from [6], the sequence $(a_n)_{n\in\mathbb{N}_+}$ is ψ-mixing under any γ_a, $a \in I$, that is, $\lim_{n\to\infty} \psi_{\gamma_a}(n) = 0$. For any $a \in I$ we have $\psi_{\gamma_a}(1) \leq 0.61231\dots$ and

$$\psi_{\gamma_a}(n) \leq \frac{\varepsilon_2 \lambda_0^{n-2}(1+\lambda_0)}{1-\varepsilon_2\lambda_0^{n-1}}, \; n \geq 2, \tag{16}$$

where $\varepsilon_2 = 0.14018\dots$ and $\lambda_0 = 0.303633002898732658559\dots$.

We first notice that putting $A = \{s_n^a \leq y\} \in \mathcal{B}_1^n$ and $B = \{\tau^{n+m} \leq x\} \in \mathcal{B}_{n+m}^{\infty}$ by (15) we have

$$|\gamma_a(A \cap B) - \gamma_a(A)\gamma_a(B)| \leq \psi_{\gamma_a}(m)\gamma_a(A)\gamma_a(B) \tag{17}$$

for any $a \in I$ and $n, m \in \mathbb{N}_+$. Recall (see Sec. 3) that

$$\lim_{n\to\infty} \gamma_a(A) = \gamma([0,y]) = \frac{\log(y+1)}{\log 2}, \; a, y \in I. \tag{18}$$

Also, by Th. 1.3.12 from [6] we have

$$\lim_{n\to\infty} \gamma_a(B) = \gamma([0,x]) = \frac{\log(x+1)}{\log 2}, \; a, x \in I. \tag{19}$$

Finally, notice that

$$\gamma_a(A \cap B) = \gamma_a(\tau^{n+m} \leq x, s_n^a \leq y) \tag{20}$$

for any $a, x, y \in I$ and $n, m \in \mathbb{N}_+$. Now, by (18), (19), (20) and Th. 1, letting $n \to \infty$ in (17) yields

$$\left| H_m(x,y) - \frac{\log(x+1)\log(y+1)}{(\log 2)^2} \right| \leq \psi_{\gamma_a}(m) \frac{\log(x+1)\log(y+1)}{(\log 2)^2}$$

for any $m \in \mathbb{N}_+$ and $a, x, y \in I$. In conjunction with (15), the last inequality provides a good approximation of $H_m(x,y)$ for moderately large values of $m \in \mathbb{N}_+$.

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