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A FULL CHARACTERIZATION OF CONTINUOUS REPRESENTABILITY OF INTERVAL ORDERS

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Abstract: We characterize the representability of an interval order $\prec$ on a topological space (X,τ) through a pair (u,v) of continuous real-valued functions (in the sense that, for all $x,y \in X$, $x \prec y$ if and only if $v(x) < u(y)$). As a corollary of our main result, we prove a known characterization of the existence of a pair of continuous real-valued functions (u,v) representing an interval order $\prec$ on a topological space (X,τ), where u and v are utility functions for two weak orders naturally associated with $\prec$.

1. Introduction

In this paper we present a characterization of the existence of a pair (u,v) of continuous real-valued functions representing an interval order $\prec$ on a topological space (X,τ) (in the sense that, for all $x,y \in X$, $x \prec y$ if and only if $v(x) < u(y)$).

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Several authors were concerned with the existence of a pair of continuous or at least semicontinuous real-valued functions representing an interval order $\prec$ on a topological space (X, τ) (see e.g. Bosi [3], Bosi, Candeal and Induráin [7], Bosi and Isler [8], Bridges [12], Candeal, Induráin and M. Zudaire [15] and Chateauneuf [16]). Nevertheless, to the best of our knowledge a characterization of the existence of such a continuous representation (u, v) of an interval order $\prec$ on an arbitrary topological space (X, τ) is not available in the literature in case that we do not require that the functions u and v are utility functions for the weak orders $\prec^{**}$ and respectively $\prec^{*}$ defined as follows

$$[x \prec^{*} y \Leftrightarrow \exists \xi \in X : x \prec \xi \precsim y] \text{ and } [x \prec^{**} y \Leftrightarrow \exists \eta \in X : x \precsim \eta \prec y].$$

Recently, Bosi, Candeal and Induráin [5] adopted the biorder approach introduced by Doignon, Ducamp and Falmagne [17] in order to provide a characterization of a continuous representation of this kind. Another characterization of the existence of such a continuous representation of an interval order on a topological space is found in Bosi, Candeal, Campión and Induráin [6, Th. 4.1]. The authors also presented an example illustrating the fact that the existence of a continuous representation (u, v) of an interval order $\prec$ on a topological space (X, τ) does not imply continuity of the weak orders $\prec^{*}$ and $\prec^{**}$ intimately connected to $\prec$. Therefore, there exist interval orders which are continuously representable and such that the associated weak orders are not continuously representable. This consideration motivates the present paper.

In this paper, we therefore characterize the existence of a continuous representation of an interval order on a topological space in full generality. We use the notion of a *scale* in a topological space which was already used by Bosi and Isler [9] in order to discuss the existence of a continuous utility function for a preorder satisfying strong continuity assumptions. Th. 3.3 in this paper is a generalization of Th. 3.1 in Bosi [4] in which the notion of a *decreasing scale* in a preordered topological space is used (see e.g. Burgess and Fitzpatrick [14] and Herden [21]). As a corollary of our main result, we prove a known characterization of the existence of a pair of continuous real-valued functions (u, v) representing an interval order $\prec$ on a topological space (X, τ), where u and v are utility functions for two weak orders naturally associated with $\prec$ (see Th. 4.1 in Bosi, Candeal, Campión and Induráin [6] and Cor. 4.3 in Bosi, Candeal and Induráin [5]).

2. Notation and preliminaries

An *interval order* $\prec$ on an arbitrary nonempty set X is a binary relation on X which is *irreflexive* and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \prec z) \textit{ and } (y \prec w) \Rightarrow (x \prec w) \textit{ or } (y \prec z).$$

The *reflexive part* of a given interval order $\prec$ will be denoted by $\precsim$ (i.e., for all $x, y \in X$, $x \precsim y$ if and only if *not*$(y \prec x)$). Oloriz, Candeal and Induráin [22] showed that the reflexive part $\precsim$ of an interval order $\prec$ on a set X is necessarily total, in the sense that, for any two elements $x, y \in X$, either $x \precsim y$ or $y \precsim x$.

If $\prec$ is an interval order on a set X, then we may consider the binary relations $\prec^*$ and $\prec^{**}$ on X defined as follows:

$$x \prec^* y \Leftrightarrow \exists \xi \in X : x \prec \xi \precsim y \quad (x, y \in X),$$

$$x \prec^{**} y \Leftrightarrow \exists \eta \in X : x \precsim \eta \prec y \quad (x, y \in X).$$

The reflexive parts of the binary relations $\prec^*$ and $\prec^{**}$ will be denoted by $\precsim^*$ and respectively $\precsim^{**}$. Fishburn [19] proved that the binary relations $\prec^*$ and $\prec^{**}$ associated to any interval order $\prec$ on a set X are *weak orders* on X (i.e., they are *asymmetric* and *negatively transitive*). It is clear that, for any two elements $x, y \in X$, if either $x \precsim^* y$ or $x \precsim^{**} y$, then we have that $x \precsim y$.

Obviously every weak order $\prec$ on a set X is an interval order on X. In this case, we have that $\prec = \prec^* = \prec^{**}$. The importance of interval orders in economics lies on the fact that their reflexive parts are not necessarily transitive.

A real-valued function u on X is said to be a *utility function* for a weak order $\prec$ on a set X if, for all $x, y \in X$,

$$x \prec y \Leftrightarrow u(x) < u(y).$$

We say that a pair (u, v) of real-valued functions on X represents an interval order $\prec$ on X if, for all $x, y \in X$,

$$x \prec y \Leftrightarrow v(x) < u(y).$$

An interval order $\prec$ on a topological space (X, τ) is said to be *upper (lower) semicontinuous* if

$$L_\prec(x) = \{y \in X : y \prec x\} \quad (U_\prec(x) = \{y \in X : x \prec y\})$$

is an open subset of X for every $x \in X$. If $\prec$ is both upper and lower semicontinuous, then it is said to be *continuous*.

3. Continuous representability

Before presenting the main result of this section, we recall the definition of a scale in a topological space and a well known result, the proof of which may be found for example in the proof of the lemma on pages 43–44 in Gillman and Jerison (1960) (see also Th. 4.1 in Burgess and Fitzpatrick (1977)).

Definition 3.1. If (X,τ) is a topological space and $\mathbb{S}$ is a dense subset of $[0,1]$ such that $1\in\mathbb{S}$, then a family $\{G_r\}_{r\in\mathbb{S}}$ of open subsets of X is said to be a *scale* in (X,τ) if the following conditions hold:

(i) $G_1=X$; (ii) $\overline{G_{r_1}}\subseteq G_{r_2}$ for every $r_1,r_2\in\mathbb{S}$ such that $r_1<r_2$.

Lemma 3.2. *If $\{G_r\}_{r\in\mathbb{S}}$ is a scale in a topological space (X,τ), then the formula*

$$u(x)=\inf\{r\in\mathbb{S}:x\in G_r\}\quad (x\in X)$$

defines a continuous function on (X,τ) with values in $[0,1]$.

We are now ready to present a characterization of the existence of a pair (u,v) of continuous real-valued functions representing an interval order $\prec$ on a topological space (X,τ). We remark that u and v are not required to be utility functions for the associated weak orders $\prec^*$ and $\prec^{**}$.

Theorem 3.3. *Let $\prec$ be an interval order on a topological space (X,τ). Then the following conditions are equivalent:*

(i) *There exists a pair (u,v) of continuous real-valued functions on (X,τ) with values in $[0,1]$ representing the interval order $\prec$;*

(ii) *There exist two scales $\{G_r^*\}_{r\in\mathbb{S}}$ and $\{G_r^{**}\}_{r\in\mathbb{S}}$ in (X,τ) such that the family $\{(G_r^*,G_r^{**})\}_{r\in\mathbb{S}}$ satisfies the following conditions:*

(a) *$x\precsim y$ and $y\in G_r^*$ imply $x\in G_r^{**}$ for every $x,y\in X$ and $r\in\mathbb{S}$;*

(b) *for every $x,y\in X$ such that $x\prec y$ there exist $r_1,r_2\in\mathbb{S}$ such that $r_1<r_2<1$, $x\in G_{r_1}^*$, $y\notin G_{r_2}^{**}$.*

Proof. (i) $\Rightarrow$ (ii). If (u,v) is a continuous representation of the interval order $\prec$, then just define $\mathbb{S}=\mathbb{Q}\cap]0,1]$, $G_r^*=v^{-1}([0,r[)$, $G_r^{**}=u^{-1}([0,r[)$ for every $r\in\mathbb{S}$, and $G_1^*=G_1^{**}=X$ in order to immediately verify that $\{G_r^*\}_{r\in\mathbb{S}}$ and $\{G_r^{**}\}_{r\in\mathbb{S}}$ are two scales in (X,τ) such that the family $\{(G_r^*,G_r^{**})\}_{r\in\mathbb{S}}$ satisfies the above conditions (a) and (b).

(ii) $\Rightarrow$ (i). Conversely, assume that there exist two scales $\{G_r^*\}_{r\in\mathbb{S}}$ and $\{G_r^{**}\}_{r\in\mathbb{S}}$ such that the family $\{(G_r^*,G_r^{**})\}_{r\in\mathbb{S}}$ satisfies the above conditions (a) and (b). Then define two functions $u,v:X\to[0,1]$ as follows:

$$u(x) = \inf\{r \in \mathbb{Q} \cap]0,1] : x \in G_r^{**}\} \quad (x \in X),$$

$$v(x) = \inf\{r \in \mathbb{Q} \cap]0,1] : x \in G_r^{*}\} \quad (x \in X).$$

We claim that (u, v) is a pair of continuous functions on (X, τ) with values in $[0, 1]$ representing the interval order $\prec$.

From the definition of the functions u and v, it is clear that they both take values in $[0, 1]$. Let us first show that the pair (u, v) represents the interval order $\prec$. First consider any two elements $x, y \in X$ such that $x \prec y$. Then, by condition (b), there exist $r_1, r_2 \in \mathbb{S}$ such that $r_1 < r_2$, $x \in G_{r_1}^*$, $y \notin G_{r_2}^{**}$. Hence, we have that $v(x) \leq r_1 < r_2 \leq u(y)$, which obviously implies that $v(x) < u(y)$. Conversely, consider any two elements $x, y \in X$ such that $y \precsim x$, and observe that, for every $r \in \mathbb{Q} \cap$ $\cap]0, 1]$, if $x \in G_r^*$ then it must be $y \in G_r^{**}$ by the above condition (a). Hence, it must be $u(y) \leq v(x)$ from the definition of u and v.

Finally, let us observe that u and v are continuous real-valued functions on (X, τ) with values in $[0, 1]$ as an immediate consequence of Lemma 3.2. This consideration completes the proof. $\Diamond$

Remark 3.4. It should be noted that the real-valued functions u and v in the continuous representation (u, v) which has been characterized above are not required to be utility functions for the weak orders $\prec^{**}$ and respectively $\prec^*$. Nevertheless, it is clear that u and v are *weak utilities* for the weak orders $\prec^{**}$ and respectively $\prec^*$, in the sense that, for all $x, y \in X$,

$$x \prec^{**} y \Rightarrow u(x) < u(y), \qquad x \prec^* y \Rightarrow v(x) < v(y).$$

It is immediate to check that u and v are utility functions for the weak orders $\prec^{**}$ and $\prec^*$ respectively if and only if the scales $\{G_r^*\}_{r \in \mathbb{S}}$ and $\{G_r^{**}\}_{r \in \mathbb{S}}$ in (X, τ) satisfy the following properties for all $x, y \in X$ and $r \in \mathbb{S}$:

$$(x \precsim^* y) \text{ and } (y \in G_r^*) \Rightarrow x \in G_r^*,$$

$$(x \precsim^{**} y) \text{ and } (y \in G_r^{**}) \Rightarrow x \in G_r^{**}.$$

In this case the scales $\{G_r^*\}_{r \in \mathbb{S}}$ and $\{G_r^{**}\}_{r \in \mathbb{S}}$ in (X, τ) are *decreasing* with respect to the total preorders $\precsim^*$ and respectively $\precsim^{**}$. If we add this assumption to condition (ii) of the above Theorem 3.3, then we have that Th. 3.3 is equal to Th. 3.1 in Bosi [4].

Remark 3.5. We can observe that condition (ii) of Th. 3.3 implies that the interval order $\prec$ on the topological space (X, τ) is continuous.

In order to check that $\prec$ is upper semicontinuous consider any point $x \in X$. We claim that $L_{\prec}(x)$ is an open subset of X. Assume that $L_{\prec}(x)$ is not empty and consider any point $z \in L_{\prec}(x)$, i.e. $z \prec x$. Then from condition (b) there exist $r_1, r_2 \in \mathbb{S}$ such that $r_1 < r_2 < 1$, $z \in G^*_{r_1}$, $x \notin G^{**}_{r_2}$. We have that $G^*_{r_1} \subseteq L_{\prec}(x)$. Indeed, if there exists $w \in G^*_{r_1}$ such that $x \precsim w$ then from condition (a) we have that $x \in G^{**}_{r_1}$ which obviously implies that $x \in G^{**}_{r_2}$, a contradiction. Hence, $G^*_{r_1}$ is an open subset of X such that $z \in G^*_{r_1} \subseteq L_{\prec}(x)$. Analogously it may be shown that $\prec$ is lower semicontinuous. It should be noted that for every $x \in X$ we have that

$$L_{\prec^*}(x) = \bigcup_{\{\xi \in X,\, \xi \precsim x\}} L_{\prec}(\xi), \quad U_{\prec^{**}}(x) = \bigcup_{\{\eta \in X,\, x \precsim \eta\}} U_{\prec}(\eta)$$

and therefore continuity of the interval order $\prec$ on the topological space (X, τ) implies that the associated weak orders $\prec^*$ and $\prec^{**}$ on (X, τ) are respectively upper semicontinuous and lower semicontinuous.

In the following proposition we use Th. 3.3 in order to prove a proposition which provides a characterization of the existence of a pair of continuous real-valued functions (u, v) representing an interval order $\prec$ on a topological space (X, τ), where u and v are utility functions for the weak orders $\prec^{**}$ and respectively $\prec^*$ (see Th. 4.1 in Bosi, Candeal, Campión and Induráin [6]). In the proof we provide an explicit construction of the scales $\{G^*_r\}_{r \in \mathbb{S}}$ and $\{G^{**}_r\}_{r \in \mathbb{S}}$ which appear in the enunciate of the above theorem.

We recall that an interval order $\prec$ on a set X is said to be *i.o. separable* if there exists a countable subset D of X such that for all $x, y \in X$ with $x \prec y$ there exists $d \in D$ such that $x \prec d \precsim^{**} y$. In this case D is said to be an *i.o. order dense subset* of $(X, \prec)$.

If (X, τ) is a topological space, then we shall denote by $\overline{A}$ the topological closure of any subset A of X.

Proposition 3.6. *Let $\prec$ be an interval order on a topological space (X, τ). Then the following conditions are equivalent:*

(i) *There exists a pair (u, v) of continuous real-valued functions on (X, τ) with values in $[0, 1]$ representing the interval order $\prec$, where u is a utility function for the weak order $\prec^{**}$ and v is a utility function for the weak order $\prec^*$;*

(ii) *The following conditions hold:*

(a) *$\prec$ is i.o.-separable;*

(b) *$\prec$ is continuous, $\prec^*$ is lower semicontinuous and $\prec^{**}$ is upper semicontinuous.*

Proof. Let $\prec$ be an interval order on a topological space (X, τ).

(i) $\Rightarrow$ (ii). If $\prec$ is representable by a pair of real-valued functions, then $\prec$ is i.o. separable by Th. 1 in Oloriz, Candeal and Induráin [22]. In addition, if $\prec$ is representable by a pair (u, v) of continuous real-valued functions on (X, τ) with the indicated properties, then it is clear that the above condition (b) is verified.

(ii) $\Rightarrow$ (i). Assume that $\prec$ satisfies the above conditions (a) and (b) and observe that the weak orders $\prec^*$ and $\prec^{**}$ are actually continuous (see the considerations at the end of Remark 3.5). Let D be a countable i.o. order dense subset of $(X, \prec)$. We can assume without loss of generality that the weakly ordered set $(D, \prec^{**})$ is actually a *linearly ordered set* (i.e., $\prec^{**}$ on D is complete). A pair $(x, y) \in X \times X$ such that $x \prec y$ and $U_{\prec}(x) \cap L_{\prec^{**}}(y) = \emptyset$ is said to be a *hole* in $(X, \prec)$. For every $d \in D$ which is the right endpoint of some hole (x, d) in $(X, \prec)$ consider a copy $]0,1[_{\mathbb{Q}}^{(d)}$ of the rational numbers strictly between 0 and 1. Now consider any hole $(x, y) \in X \times X$ in $(X, \prec)$. Since D is an i.o. order dense subset of $(X, \prec)$, there exists $d \in D$ such that $d \precsim^{**} x \precsim^{**} d$. If $x' \in X$ is the left endpoint of any hole (x', y) in $(X, \prec)$ with the same endpoint y (observe that this happens if and only if $x' \precsim^{*} x \precsim^{*} x'$), interpose $]0,1[_{\mathbb{Q}}^{(d)}$ between x' and y, with the natural order on each of such sets $]0_n, 1_n[$ $(n \in \mathbb{N})$. Consider the interval order $\prec_0$ on $\hat{X} = X \cup \{]0_n, 1_n[\}_{n \in \mathbb{N}}$ which is now uniquely determined in a natural way. We just observe that if $(x, y) \in X \times X$ and $(z, w) \in X \times X$ are two different holes with $x \prec y \precsim z \prec w$ and interposed sets $]0,1[_{\mathbb{Q}}^{(d_y)}$ and $]0,1[_{\mathbb{Q}}^{(d_w)}$ respectively, and $r_1 < r_2$, $s_1 < s_2$ $(r_1, r_2 \in]0,1[_{\mathbb{Q}}^{(d_y)}$, $s_1, s_2 \in]0,1[_{\mathbb{Q}}^{(d_w)})$, then we must have that $\hat{x} \prec_0 \hat{r}_1 \prec_0 \hat{r}_2 \prec_0 \hat{y} \precsim_0 \hat{z} \prec_0 \hat{s}_1 \prec_0 \hat{s}_2 \prec_0 \hat{w}$, and $\hat{r} \prec_0 \hat{s}$ for every $r \in]0,1[_{\mathbb{Q}}^{(d_y)}$ and every $s \in]0,1[_{\mathbb{Q}}^{(d_w)}$. Denote by τ_0^{ord} the *natural order topology* on $\hat{X}$ which has

$$\mathcal{B}_0 = \big\{ \{L_{\prec_0}(\hat{x})\}_{\hat{x} \in \hat{X}} \cup \{U_{\prec_0}(\hat{x})\}_{\hat{x} \in \hat{X}} \cup \{L_{\prec_0^*}(\hat{x})\}_{\hat{x} \in \hat{X}} \cup \{U_{\prec_0^*}(\hat{x})\}_{\hat{x} \in \hat{X}} \\ \cup \{L_{\prec_0^{**}}(\hat{x})\}_{\hat{x} \in \hat{X}} \cup \{U_{\prec_0^{**}}(\hat{x})\}_{\hat{x} \in \hat{X}} \big\}$$

as a subbasis of open sets. Without loss of generality, we may assume that $\prec_0^{**}$ has neither a minimal nor a maximal element. Since D is a countable set and $\{]0_n, 1_n[\}_{n \in \mathbb{N}}$ is also a countable set, it is clear that there exists a countable i.o. order dense subset $\hat{D} = D \cup \{]0_n, 1_n[\}_{n \in \mathbb{N}}$ of $\hat{X}$ endowed with the interval order $\prec_0$. Observe in particular that for all $\hat{x}, \hat{y} \in \hat{X}$ such that $\hat{x} \prec_0 \hat{y}$ there exist $\hat{d}_1, \hat{d}_2 \in \hat{D}$ such that

$\hat{x} \prec_0 \hat{d}_1 \prec_0^{**} \hat{d}_2 \precsim_0^{**} \hat{y}$. Since $\hat{D}$ is dense in itself (i.e., for all $\hat{d}_1, \hat{d}_2 \in \hat{D}$ such that $\hat{d}_1 \prec_0^{**} \hat{d}_2$ there exists $\hat{d} \in \hat{D}$ such that $\hat{d}_1 \prec_0^{**} \hat{d} \prec_0^{**} \hat{d}_2$), from Birkhoff [2, Th. 23 on p. 200] we may conclude that there exists an order-preserving function $f : (\hat{D}, \prec_0^{**}) \to (]0,1[_{\mathbb{Q}}, <)$. Further, we may assume that the mapping f is onto. If $f^{-1}(r) = \hat{d}$ $(r \in]0,1[_{\mathbb{Q}})$, then define

$$G_{0r}^{*} = L_{\prec_0}(\hat{d}), \;\; G_{0r}^{**} = L_{\prec_0^{**}}(\hat{d}) \quad (r \in]0,1[_{\mathbb{Q}}),$$

and set $G_{01}^{*} = G_{01}^{**} = X$.

It is easy now to check that $\{G_{0r}^{*}\}_{r\in\mathbb{S}}$ and $\{G_{0r}^{**}\}_{r\in\mathbb{S}}$ are two scales in $(\hat{X}, \tau_0^{\mathrm{ord}})$ which satisfy condition (ii) of Th. 3.3 with $\mathbb{S} = \mathbb{Q} \cap]0,1]$.

It is clear that G_{0r}^{*} and G_{0r}^{**} are open subsets of $\hat{X}$ for every $r \in \in \mathbb{Q}\cap]0,1]$. In order to show that condition (a) of Th. 3.3 holds, just observe that, for every $\hat{d} \in \hat{D}$, $\hat{x} \precsim_0 \hat{y}$ and $\hat{y} \prec_0 \hat{d}$ imply $\hat{x} \prec_0^{**} \hat{d}$. In order to prove that $\{G_{0r}^{*}\}_{r\in\mathbb{Q}\cap]0,1]}$ and $\{G_{0r}^{**}\}_{r\in\mathbb{Q}\cap]0,1]}$ are actually two scales in $(\hat{X}, \tau_0^{\mathrm{ord}})$, observe that for all $\hat{d}_1, \hat{d}_2 \in \hat{D}$ such that $\hat{d}_1 \prec_0^{**} \hat{d}_2$ there exists $\hat{z} \in \hat{X}$ such that $\hat{d}_1 \precsim_0 \hat{z} \prec_0 \hat{d}_2$, and therefore, since the weak orders $\prec_0^{*}$ and $\prec_0^{**}$ are lower semicontinuous and respectively upper semicontinuous, we have that

$$L_{\prec_0}(\hat{d}_1) \subseteq \overline{L_{\prec_0}(\hat{d}_1)} \subseteq L_{\precsim_0^{*}}(\hat{z}) \subseteq L_{\prec_0}(\hat{d}_2),$$

$$L_{\prec_0^{**}}(\hat{d}_1) \subseteq \overline{L_{\prec_0^{**}}(\hat{d}_1)} \subseteq L_{\precsim_0^{**}}(\hat{d}_1) \subseteq L_{\prec_0^{**}}(\hat{d}_2),$$

where $L_{\precsim_0^{*}}(\hat{z}) = \{\hat{w} \in \hat{X} : \hat{w} \precsim_0^{*} \hat{z}\}$ and $L_{\precsim_0^{**}}(\hat{d}_1) = \{\hat{w} \in \hat{X} : \hat{w} \precsim^{**} \hat{d}_1\}$ are closed subsets of $\hat{X}$. In order to show that condition (b) of Th. 3.3 holds, observe that for all $\hat{x}, \hat{y} \in \hat{X}$ such that $\hat{x} \prec_0 \hat{y}$ there exist $\hat{d}_1, \hat{d}_2 \in \hat{D}$ such that $\hat{x} \prec_0 \hat{d}_1 \prec_0^{**} \hat{d}_2 \precsim_0^{**} \hat{y}$, and therefore $\hat{x} \in L_{\prec_0}(\hat{d}_1)$ and $\hat{y} \notin L_{\prec_0^{**}}(\hat{d}_2)$.

Further, it is easily seen that $\{G_{0r}^{*}\}_{r\in\mathbb{Q}\cap]0,1]}$ and $\{G_{0r}^{**}\}_{r\in\mathbb{Q}\cap]0,1]}$ satisfy the following conditions for all $\hat{x}, \hat{y} \in \hat{X}$ and $r \in \mathbb{Q}\cap]0,1[$: $[(\hat{x} \precsim_0^{*} \hat{y})$ and $(\hat{y} \in G_{0r}^{*}) \Rightarrow \hat{x} \in G_{0r}^{*}]$, $[(\hat{x} \precsim_0^{**} \hat{y})$ and $(\hat{y} \in G_{0r}^{**}) \Rightarrow \hat{x} \in \in G_{0r}^{**}]$.

Now it is not difficult to prove that the families $\{G_r^{*} = G_{0r}^{*} \cap \cap X\}_{r\in\mathbb{S}}$ and $\{G_r^{**} = G_{0r}^{**} \cap X\}_{r\in\mathbb{S}}$ are two scales in (X, τ) which satisfy condition (ii) of Th. 3.3 with $\mathbb{S} = \mathbb{Q}\cap]0,1]$ when we consider the interval order $\prec$ on X (see Lemma 3.1 and Constr. 3.3 in Alcantud et al. [1]). The considerations presented in Remark 3.4 guarantee the existence of a continuous representation (u, v) of the interval order $\prec$ with the indicated properties. So the proof is complete. $\Diamond$

Remark. Observe that if (X, τ) is a connected topological space then Chateauneuf's Theorem (see Chateauneuf's [16]) may be easily recovered as a consequence of Prop. 3.6. We recall that Chateauneuf's Theorem states that if $\prec$ is an interval order on a connected topological space (X, τ), then there exists a pair (u, v) of continuous real-valued functions on (X, τ) representing the interval order $\prec$, where u and v are utility functions for the weak orders $\prec^{**}$ and respectively $\prec^{*}$, provided that $\prec^{*}$ and $\prec^{**}$ are both continuous and $\prec$ is strongly separable (i.e., there exists a countable subset D of X such that for all $x, y \in X$ with $x \prec y$ there exist $d_1, d_2 \in D$ such that $x \prec d_1 \precsim d_2 \prec y$). It is clear that if $\prec$ is strongly separable, then it is i.o. separable. Further, the existence of a continuous representation (u, v) with the indicated properties implies that $\prec$ has no holes (i.e., for all $x, y \in X$ such that $x \prec y$ there exists $x' \in X$ such that $x \prec x' \prec^{**} y$). Hence, it is not hard to check that if there exists a continuous representation (u, v) of this kind for an interval order $\prec$ on a connected topological space (X, τ), then $\prec$ is actually strongly separable.

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