Mathematica Pannonica
18/1 (2007), 101–124

SYMMETRIC SHIFT RADIX SYSTEMS AND FINITE EXPANSIONS

Shigeki **Akiyama**

Department of Mathematics, Faculty of Science, Niigata University, Ikarashi 2-8050, Niigata 950-2181, Japan

Klaus **Scheicher**

Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria

Received: January 2006

MSC 2000: 11 A 63, 37 B 10

Keywords: Pisot number, number system, beta expansion.

Abstract: Shift radix systems provide a unified notation to study several important types of number systems. However, the classification of such systems is already hard in two dimensions. In this paper, we consider a symmetric version of this concept which turns out to be easier: the set of such number systems with finite expansions can be completely classified in dimension two.

1. Introduction

Shift radix systems, defined in [4], provide a unified notation for canonical number systems (for short CNS) as well as β-expansions. Both concepts are generalisations of the well-known b-ary expansions of integers.

Let $d \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. With $\mathbf{r}$ we associate a mapping $\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ in the following way: if $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{Z}^d$ then let

E-mail addresses: akiyama@math.sc.niigata-u.ac.jp, klaus.scheicher@oeaw.ac.at

$$\tilde{\tau}_{\mathbf{r}}(\mathbf{z}) = (z_2, \dots, z_d, -\lfloor \mathbf{r}\mathbf{z} \rfloor), \tag{1.1}$$

where $\mathbf{r}\mathbf{z} = r_1 z_1 + \dots + r_d z_d$ is the inner product of the vectors $\mathbf{r}$ and $\mathbf{z}$. Then $(\mathbb{Z}^d, \tilde{\tau}_{\mathbf{r}})$ is called *a shift radix system* (for short SRS) on $\mathbb{Z}^d$. From (1.1) it follows that $\tilde{\tau}_{\mathbf{r}}(\mathbf{z}) = (z_2, \dots, z_{d+1})$ if and only if

$$0 \le r_1 z_1 + r_2 z_2 + \dots + r_d z_d + z_{d+1} < 1. \tag{1.2}$$

It is an important problem for CNS and β-expansions to determine whether or not each number admits a finite expansion. In SRS language, it translates to one question:

For which $\mathbf{r}$*, do all orbits of* $(\mathbb{Z}^d, \tilde{\tau}_{\mathbf{r}})$ *end in* $\mathbf{0} = (0, \dots, 0)$*?*

Unfortunately, already for $d = 2$, this seems to be a hard problem, though in [5], a partial answer is given. A main difficulty arises when the polynomial $x^d + r_d x^{d-1} + \dots + r_1$ has a root close to the unit circle.

In the present paper, we will study a 'symmetric' version of SRS. We consider a mapping

$$\tau_{\mathbf{r}}(\mathbf{z}) = (z_2, \dots, z_d, -\lfloor \mathbf{r}\mathbf{z} + \tfrac{1}{2} \rfloor) \tag{1.3}$$

instead of (1.1). Therefore, we obtain a condition

$$-\tfrac{1}{2} \le r_1 z_1 + r_2 z_2 + \dots + r_d z_d + z_{d+1} < \tfrac{1}{2} \tag{1.4}$$

instead of (1.2). Then $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ is called a *symmetric shift radix system* (for short SSRS) on $\mathbb{Z}^d$. Let

$\mathcal{D}_d := \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d$, the sequence $(\tau_{\mathbf{r}}^k(\mathbf{z}))_{k=0}^{\infty}$ is eventually periodic$\}$,

$\mathcal{D}_d^0 := \{\mathbf{r} \in \mathbb{R}^d : \forall\ \mathbf{z} \in \mathbb{Z}^d,\ \exists\ k > 0 : \tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}\}.$

Our aim is to describe as precisely as possible the sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$. For $d = 1$, we have:

$$\mathcal{D}_1 = [-1, 1] \quad \text{and} \quad \mathcal{D}_1^0 = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right]. \tag{1.5}$$

Since $\tau_{\mathbf{r}}(\mathbf{x}) = -\lfloor \mathbf{r}\mathbf{x} + \frac{1}{2} \rfloor$, both equalities are almost trivial. In fact, to deduce the second one, we just have to look at orbits $\tau_{\mathbf{r}}^n(\pm 1)$, $n = 1, 2, \dots$.[1]

Curiously the minor change from (1.1) to (1.3) affects substantially the behaviour of the system. The above *finiteness problem* for $d = 2$ is satisfactory settled in Th. 5.2 by giving an exact shape (Fig. 1) of $\mathcal{D}_2^0$, which forms the main result of this paper. The difficulty in $d = 2$

[1] This can also be seen as a corollary of Th. 5.2.

disappears in this case. An essential reason is that the corresponding roots stay far inside the unit circle when we deal with $\mathcal{D}_2^0$. Ths. 2.2 and 3.8 illustrate the strength of this result.

On the other hand the change from (1.1) to (1.3) gives rise to changes of the digit sets of the number systems. They are symmetric canonical number systems and symmetric β-expansions. By considering a ternary expansion using digits $\{-1, 0, 1\}$ instead of $\{0, 1, 2\}$, we can easily imagine this change in the case of canonical number systems. We shall discuss basic facts on the symbolic system associated with this expansion in §3. See [10, 17, 22, 23] for results on usual β-expansions.

Note that

$$\tau'_{\mathbf{r}}(\mathbf{z}) = (z_2, \dots, z_d, -\lceil \mathbf{rz} - \tfrac{1}{2} \rceil) \tag{1.6}$$

defines a slightly different system. However, it is isomorphic to the one obtained from $\tau_{\mathbf{r}}$ through an involution correspondence $\mathbb{Z}^d \ni z \mapsto -z \in \in \mathbb{Z}^d$. Obviously, the sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ are identical for both systems.

2. SSRS and symmetric CNS

The only difference between symmetric CNS and usual CNS is the set of digits. Let

$$P(x) = x^d + b_{d-1}x^{d-1} + \dots + b_0 \in \mathbb{Z}[x]$$

with $b_0 \neq 0$ and $\mathcal{R} = \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$. Define a digit set $\mathcal{N} = = \left[-\frac{|b_0|}{2}, \frac{|b_0|}{2}\right) \cap \mathbb{Z}$ which consists of $|b_0|$ consecutive integers.

We say that an element of $\mathcal{R}$ has a finite representation, if it admits a representation of the form

$$\ell_0 + \ell_1 x + \dots + \ell_h x^h$$

with $\ell_j \in \mathcal{N}$ for $0 \leq j \leq h$. This representation, if it exists, is unique since $\mathcal{N}$ forms a complete residue system of $\mathcal{R}/x\mathcal{R}$. Actually the digits are determined from ℓ_0 to ℓ_h by the so called backward division algorithm: letting

$$\mathbf{z} = \mathbf{z}^{(0)} = z_0 + z_1 x + \dots + z_{d-1}x^{d-1} \in \mathcal{R}, \quad z_j \in \mathbb{Z},$$

we first get $\ell_0 \in \mathcal{N}$ by $z_0 \equiv \ell_0 \pmod{b_0}$. Then ℓ_0 is a unique choice in $\mathcal{N}$ such that $\mathbf{z}^{(0)} - \ell_0 \equiv 0 \pmod{x}$ and one put $\mathbf{z}^{(1)} := (\mathbf{z}^{(0)} - \ell_0)/x$. Iterate this process and define $\mathbf{z}^{(n)} \in \mathcal{R}$ by $\mathbf{z}^{(n)} := (\mathbf{z}^{(n-1)} - \ell_{n-1})/x$ with $z_{n-1} \equiv \ell_{n-1} \pmod{b_0}$. Then we obtain

$$\mathbf{z}^{(0)} = \ell_0 + \ell_1 x + \cdots + \ell_{n-1} x^{n-1} + x^n \mathbf{z}^{(n)}.$$

Thus, $\mathbf{z}$ admits a finite representation, if and only if there is an n with $\mathbf{z}^{(n)} = 0$.

The numbers $\ell_j = \ell_j(\mathbf{z})$, $j \geq 0$, are called the digits of $\mathbf{z}$ with respect to $(P(x), \mathcal{N})$. The pair $(P(x), \mathcal{N})$ is called *symmetric canonical number system* (for short SCNS) in $\mathcal{R}$, if each $\mathbf{z} \in \mathcal{R}$ has a finite representation. In other words, a SCNS requires, that for each initial value $\mathbf{z} \in \mathcal{R}$, the backward division algorithm terminates in finitely many steps.

Theorem 2.1 (cf. [4, Th. 3.1])**.** Let $P(x) := x^d + b_{d-1}x^{d-1} + \cdots + b_0 \in \mathbb{Z}[x]$ and $\mathcal{N} = [-\frac{|b_0|}{2}, \frac{|b_0|}{2}) \cap \mathbb{Z}$.[2] Then $(P(x), \mathcal{N})$ is a SCNS if and only if

$$(r_1, \ldots, r_d) := \left(\tfrac{1}{b_0}, \tfrac{b_{d-1}}{b_0}, \ldots, \tfrac{b_1}{b_0}\right) \in \mathcal{D}_d^0.$$

We omit the proof since it is identical to [4, Th. 3.1]. When b_0 is negative, the corresponding system comes from (1.6) instead of (1.4).

Generalising a result of Kátai [18] for Gaussian integers, we have

Theorem 2.2. 1. *$(x + a, \mathcal{N})$ is a SCNS if and only if $a \geq 2$ or $a < -2$.*

2. *$(x^2 + Ax + B, \mathcal{N})$ is a SCNS if and only if one of the following condition holds*

(a) *$|A| < 1 + B/2$ and $|B| \geq 2$,*

(b) *$A = 1 + B/2$ and $|B| > 2$.*

This is a consequence of Ths. 5.2 and 2.1 for $d \in \{1, 2\}$. For instance, if $d = 2$ we have

$$-\frac{1}{B} - \frac{1}{2} < \frac{A}{B} \leq \frac{1}{B} + \frac{1}{2} \text{ and } \frac{1}{B} \leq \frac{1}{2}.$$

Example 2.3. $(x^2 + 2x + 2, \{-1, 0\})$ is a SCNS while $(x^2 - 2x + 2, \{-1, 0\})$ is not. These correspond to $(\frac{1}{2}, 1) \in \mathcal{D}_2^0$ and $(\frac{1}{2}, -1) \notin \mathcal{D}_2^0$ in Fig. 1. For example, compare

$$1 - x \equiv -1 - x - x^2 - x^3 - x^4 \pmod{x^2 + 2x + 2}$$

and

$$1 - x \equiv -1 - x - \cdots - x^{n-1} + x^n(1 - x) \pmod{x^2 - 2x + 2}.$$

[2] Instead, one may take $(-\frac{|b_0|}{2}, \frac{|b_0|}{2}] \cap \mathbb{Z}$ which corresponds to the isomorphic system by the involution $z \mapsto -z$.

3. SSRS and symmetric β-expansions

β-expansions of real numbers were introduced by Rényi [23]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors (cf. for instance [1, 2, 9, 10, 15, 22, 26]). For a recent survey, we refer to Ch. 7 in Lothaire [21]. In the present paper, we consider a symmetric version of this concept.

Let $\beta > 1$ be a fixed real number. Define[3]

$$T_\beta : [-\tfrac{1}{2}, \tfrac{1}{2}) \to [-\tfrac{1}{2}, \tfrac{1}{2}) \quad \text{by}$$
$$x \mapsto \beta x - \lfloor \beta x + \tfrac{1}{2} \rfloor .$$

For every $x \in [-\frac{1}{2}, \frac{1}{2})$, T_β generates an expansion

$$x = \sum_{j=1}^{\infty} \frac{d_j(x)}{\beta^j},$$

where $d_j = d_j(x) = \lfloor \beta T_\beta^{j-1}(x) + \frac{1}{2} \rfloor$ for every $j \geq 1$. We shall consider the digits d_j as functions of x. When the argument is clear, we shall write d_j instead of $d_j(x)$. We will call this expansion a *symmetric β-expansion* of x in base β. Each $d_j(x)$ is contained in $\mathcal{N} = (-\frac{\beta+1}{2}, \frac{\beta+1}{2}) \cap$ $\cap\, \mathbb{Z}$, the digit set. Define a map

$$\mathbf{d} : [-\tfrac{1}{2}, \tfrac{1}{2}) \to \mathcal{N}^{\mathbb{N}} \quad \text{by}$$
$$x \mapsto d_1(x) d_2(x) \cdots .$$

By considering formally an orbit starting from $\frac{1}{2}$, we also define an expansion $\mathbf{d}\left(\frac{1}{2}\right)$.

It should be noticed that when $1 < \beta < 2$, the dynamical system $([-\frac{1}{2}, \frac{1}{2}), T_\beta)$ does not have an invariant measure which is absolutely continuous to the Lebesgue measure, since each orbit of $x \neq 0$ eventually falls into $[-\frac{1}{2}, \frac{\beta}{2} - 1) \cup [1 - \frac{\beta}{2}, \frac{1}{2})$. Hence no $x \neq 0$ has a finite expansion. We do not address to the ergodic study of this system, e.g., the construction of the invariant measure for $\beta \geq 2$, in the present paper.

For $\beta > 1$ and $x \in [-\frac{1}{2}, \frac{1}{2})$, we will call an infinite series of the form $x = \sum_{j=1}^{\infty} s_j/\beta^j$, $s_k \in \mathcal{N}$ a *representation* of x in base β. In general, there exist infinitely many representations of x in base β.

---

[3] An isomorphic system using (1.6) is defined by $x \mapsto \beta x + \lfloor -\beta x + \frac{1}{2} \rfloor$ which acts on $(-\frac{1}{2}, \frac{1}{2}]$.

Let us treat hereafter $\mathcal{W} = \mathcal{N}^{\mathbb{N}}$ as a shift space, endowed with the lexicographical order $<_{\text{lex}}$ and the product topology. Let σ be the one sided shift on $\mathcal{W}$, that is, $\sigma(s_1 s_2 \cdots) = (s_2 s_3 \cdots)$. The language $L(\mathcal{U})$ of a subshift $\mathcal{U}$ is the set of all words which appear as subwords of an element of $\mathcal{U}$.

A sequence $(s_i)_{i=1}^{\infty} \in \mathcal{W}$ is called *admissible*, if there exists a number $x \in [-\frac{1}{2}, \frac{1}{2})$ such that $s_i = d_i(x)$ holds for each $i \geq 1$. This means, that $(s_i)_{i=1}^{\infty}$ is realized as a symmetric β-expansion $x = \sum_{j=1}^{\infty} s_j/\beta^j$.

Let $\mathcal{A}$ be the set of all admissible sequences. The *symmetric β-shift* $\mathcal{S}$ is defined to be the closure of $\mathcal{A}$ by the topology of $\mathcal{W}$. Then $\mathcal{S}$ is a subshift of the full shift $\mathcal{W}$.

The mapping $\mathbf{d} : [-\frac{1}{2}, \frac{1}{2}) \to \mathcal{A}$ is continuous and fulfils $\mathbf{d} \circ T_\beta = \sigma \circ \mathbf{d}$, which shows that the following diagram is commutative:

$$\begin{array}{ccc} [-\frac{1}{2}, \frac{1}{2}) & \xrightarrow{T_\beta} & [-\frac{1}{2}, \frac{1}{2}) \\ \mathbf{d}\Big\downarrow & & \Big\downarrow\mathbf{d} \\ \mathcal{A} & \xrightarrow[\sigma]{\qquad} & \mathcal{A}. \end{array}$$

An infinite sequence $s = (s_i)_{i=1}^{\infty} \in \mathcal{W}$ is admissible (i.e., $s \in \mathcal{A}$) if and only if

$$-\frac{1}{2\beta^{n-1}} \leq \sum_{i=n}^{\infty} \frac{s_i}{\beta^i} < \frac{1}{2\beta^{n-1}} \quad \text{for every } n \geq 1. \tag{3.1}$$

It is easy to see that $\mathbf{d}$ is an order-preserving map:

$$x < y \Longleftrightarrow \mathbf{d}(x) <_{\text{lex}} \mathbf{d}(y)$$

for $x, y \in [-\frac{1}{2}, \frac{1}{2})$. For $s = (s_i)_{i=1}^{\infty} \in \mathcal{W}$, we write $-s = (-s_i)_{i=1}^{\infty}$.

Theorem 3.1. *An infinite sequence $s = (s_i)_{i=1}^{\infty} \in \mathcal{W}$ is admissible (i.e., $s \in \mathcal{A}$) if and only if*

$$\mathbf{d}(-\tfrac{1}{2}) \leq_{\text{lex}} \sigma^n(s) <_{\text{lex}} -\mathbf{d}(-\tfrac{1}{2})$$

for all $n = 0, 1, \cdots$.

Proof. Since $\mathbf{d}$ is order-preserving, the necessity of this inequality is obvious. Let us prove that $\sigma^m(s) <_{\text{lex}} -\mathbf{d}(-\frac{1}{2})$ for $m \geq n \geq 1$ implies

$$\frac{s_n}{\beta} + \frac{s_{n+1}}{\beta^2} + \cdots < \frac{1}{2}.$$

Let $-\mathbf{d}(-\frac{1}{2}) = \bar{t}_1 \bar{t}_2 \cdots$. Decompose $\sigma^n(s)$ into *admissible blocks* in the following manner: $\sigma^n(s) = s_{n+1} s_{n+2} \cdots = w_1^{(n)} w_2^{(n)} \cdots$ with $w_i^{(n)} \in \mathcal{N}^*$ such that $w_i^{(n)} = \bar{t}_1 \cdots \bar{t}_\ell v$ with $v \in \mathcal{N}$ and $v <_{\text{lex}} \bar{t}_{\ell+1}$. In other words, each word $w_i^{(n)}$ coincides with the prefix of $-\mathbf{d}(-\frac{1}{2})$ apart from the last digit of $w_i^{(n)}$ and all blocks are chosen to have maximal length with this property. By definition of the symmetric β-expansion, we have

$$-\frac{1}{2\beta^n} < \frac{1}{2} - \sum_{i=1}^{n} \frac{\bar{t}_i}{\beta^i} \le \frac{1}{2\beta^n}.$$

This implies that

$$\sum_{v=1}^{k_i} \frac{c_{i,v}}{\beta^v} \le \sum_{v=1}^{k_i} \frac{\bar{t}_i}{\beta^v} - \frac{1}{\beta^{k_i}} < \frac{1}{2}\left(1 - \frac{1}{\beta^{k_i}}\right)$$

for $w_i^{(n)} = c_{i,1} \cdots c_{i,k_i}$. Therefore

$$\sum_{i=1}^{\infty} \frac{s_{i+n}}{\beta^i} < \frac{1}{2}\left(1 - \frac{1}{\beta^{k_1}}\right) + \frac{1}{2\beta^{k_1}}\left(1 - \frac{1}{\beta^{k_2}}\right) + \\ + \frac{1}{2\beta^{k_1+k_2}}\left(1 - \frac{1}{\beta^{k_3}}\right) + \cdots = \frac{1}{2}.$$

In a similar manner, we prove that $\mathbf{d}(-\frac{1}{2}) \le_{\text{lex}} \sigma^m(s)$ for $m \ge n \ge 1$ implies

$$-\frac{1}{2} \le \frac{s_n}{\beta} + \frac{s_{n+1}}{\beta^2} + \cdots . \quad \Diamond$$

A word $w_1 \cdots w_m \in \mathcal{N}^*$ is *admissible* if it is contained in the language $L(\mathcal{S})$. The next corollary gives a combinatorial criterion for words to be admissible.

Corollary 3.2. *Let* $\mathbf{d}(-\frac{1}{2}) = t_1 t_2 \cdots$ *and* $-\mathbf{d}(-\frac{1}{2}) = \bar{t}_1 \bar{t}_2 \cdots$. *A word* $s_1 \cdots s_m \in \mathcal{N}^*$ *is admissible if and only if*

$$t_1 \cdots t_{m-n+1} \le_{\text{lex}} s_n \cdots s_m \le_{\text{lex}} \bar{t}_1 \cdots \bar{t}_{m-n+1} \tag{3.2}$$

for $1 \le n \le m$.

Proof. If the word $s_1 \cdots s_m$ is admissible, then by definition, there exists an admissible sequence $v_1 v_2 \cdots \in \mathcal{A}$, such that $s_1 \cdots s_m v_1 v_2 \cdots$ is admissible. Thus the necessity of (3.2) follows from

$$\mathbf{d}(-\tfrac{1}{2}) \le_{\text{lex}} \sigma^n(s_1 \cdots s_m v_1 v_2 \cdots) <_{\text{lex}} -\mathbf{d}(-\tfrac{1}{2}) \quad \text{for } n \ge 0.$$

Now we prove the sufficiency of (3.2). Similarly as in the proof of Th. 3.1, one can decompose $s_1 \cdots s_m$ into admissible blocks in two ways

$$s_1 \cdots s_m = w_1 w_2 \cdots w_p = w'_1 w'_2 \cdots w'_q$$

by $\mathbf{d}(-\frac{1}{2})$ and $-\mathbf{d}(-\frac{1}{2})$ respectively. In other words, each proper prefix of w_i (resp. w'_i) is a prefix of $t_1t_2\cdots$ (resp. $\bar{t}_1\bar{t}_2\cdots$) and the w_i (resp. w'_i) are chosen to have maximal length with this property. Only the last w_p and w'_q can coincide with a prefix of $t_1t_2\cdots$ (resp. $\bar{t}_1\bar{t}_2\cdots$). We wish to find an extension $v_1v_2\cdots\in\mathcal{A}$ such that $s_1\cdots s_mv_1v_2\cdots$ is admissible.

If w_p (resp. w'_q) is not a prefix of $t_1t_2\cdots$ (resp. $\bar{t}_1\bar{t}_2\cdots$) simultaneously, then $t_1\cdots t_m <_{\text{lex}} s_1\cdots s_m <_{\text{lex}} \bar{t}_1\cdots\bar{t}_m$. Thus any $v_1v_2\cdots\in$ $\in\mathcal{A}$ works, since $t_1t_2\ldots \leq_{\text{lex}} \sigma^n(s_1\cdots s_mv_1v_2\cdots) <_{\text{lex}} \bar{t}_1\bar{t}_2\cdots$ for any $n\geq 0$.

Now assume that $w_p = t_1\cdots t_k$ and w'_q is not a prefix of $\bar{t}_1\bar{t}_2\cdots$. Then $w'_q = \bar{t}_1\cdots\bar{t}_{k'-1}\ s_m$ with $s_m < \bar{t}_{k'}$. Thus $s_1\cdots s_mv_1v_2\cdots <$ $<_{\text{lex}} \bar{t}_1\bar{t}_2\cdots$ for any $v_1v_2\cdots\in\mathcal{A}$. In this case we take $v_1v_2\cdots =$ $= t_{k+1}t_{k+2}\cdots\in\mathcal{A}$. Then $s_1\cdots s_mv_1v_2\cdots$ is admissible. Similarly if $w'_q = \bar{t}_1\cdots\bar{t}_k$ and w_p is not a prefix of $t_1t_2\cdots$, we take $v_1v_2\cdots =$ $= \bar{t}_{k+1}\bar{t}_{k+2}\cdots$.

Finally if $w_p = t_1\cdots t_k$ and $w'_q = \bar{t}_1\cdots\bar{t}_{k'}$, then without loss of generality we assume that $k\geq k'$. Then w'_q is a suffix of w_p. In this case we take $v_1v_2\cdots = t_{k+1}t_{k+2}\cdots$. Then, $s_1\cdots s_mv_1v_2\cdots$ must be admissible since $t_1t_2\cdots$ and $\bar{t}_1\cdots\bar{t}_{k'}t_{k+1}t_{k+2}\cdots$ are suffixes of $\mathbf{d}(-\frac{1}{2})$. $\diamondsuit$

The sequence $\mathbf{d}(x)$ associated with x is said to be *finite* if there exists $m\geq 0$, such that $d_j = 0$ for $j\geq m$. It is called *eventually periodic* if there exist $m\geq 0$ and $p>0$, such that $d_{j+p} = d_j$ for $j\geq m$. If $\mathbf{d}(x)$ is finite, we can write $\mathbf{d}(x) = d_1\cdots d_{m-1}$ for simplicity.

An eventually periodic word in $\mathcal{N}^{\mathbb{N}}$ is denoted by

$$b_1\cdots b_k\,(b_{k+1}\cdots b_{k+\ell})^{\omega}$$

with the period $b_{k+1},\ldots,b_{k+\ell}$. Obviously, if $\mathbf{d}(\frac{1}{2})$ is finite or eventually periodic, then β must be an algebraic integer. Note that $-T_\beta(x) =$ $= T_\beta(-x)$ holds when $T_\beta(x)\neq -\frac{1}{2}$. Therefore, if $T^n_\beta(x)\neq -\frac{1}{2}$ for all $n\geq 0$, then $\mathbf{d}(x) = -\mathbf{d}(-x)$.

An important consequence is that there exists $m\geq 0$ such that

$$\mathbf{d}(-\tfrac{1}{2}) = \pm\sigma^m\mathbf{d}(\tfrac{1}{2}).$$

Especially, $\mathbf{d}(-\frac{1}{2})$ is finite if and only if $\mathbf{d}(\frac{1}{2})$ is finite. The same is valid for eventual periodicity. Furthermore, there is a direct relation between $\mathbf{d}(\frac{1}{2})$ and $\mathbf{d}(-\frac{1}{2})$, even when $T^m_\beta(\frac{1}{2}) = -\frac{1}{2}$ holds for some $m>0$:

Lemma 3.3. *Let* $\mathbf{d}(-\frac{1}{2}) = t_1t_2\cdots$ *and* $-\mathbf{d}(-\frac{1}{2}) = \bar{t}_1\bar{t}_2\cdots$. *The following statements are equivalent:*

(1) *There exists an $m > 0$ such that $T_\beta^m\left(\frac{1}{2}\right) = -\frac{1}{2}$.*
(2) $\mathbf{d}\left(-\frac{1}{2}\right)$ *is purely periodic, i.e.,* $\mathbf{d}\left(-\frac{1}{2}\right) = (t_1 \cdots t_\ell)^\omega$.
(3) $\mathbf{d}\left(\frac{1}{2}\right)$ *has the form* $\mathbf{d}\left(\frac{1}{2}\right) = \bar{t}_1 \cdots \bar{t}_{\ell-1}(\bar{t}_\ell + 1)\,(t_1 \cdots t_\ell)^\omega$.

Proof. The expansion $\mathbf{d}(\frac{1}{2})$ can not be purely periodic. In fact, $\mathbf{d}(\frac{1}{2}) = (a_1 \cdots a_p)^\omega$ implies $T_\beta^p(\frac{1}{2}) = \frac{1}{2}$ which is impossible. Therefore (2) implies $\mathbf{d}\left(-\frac{1}{2}\right) \neq -\mathbf{d}\left(\frac{1}{2}\right)$ and consequently (1). Assume (1). Take the smallest m, such that $T_\beta^m(\frac{1}{2}) = -\frac{1}{2}$. This means that $\beta T_\beta^{m-1}(\frac{1}{2}) \in \frac{1}{2} + \mathbb{Z}$. Then $T_\beta^m(-\frac{1}{2}) = T_\beta(T_\beta^{m-1}(-\frac{1}{2})) = T_\beta(-T_\beta^{m-1}(\frac{1}{2})) = -\frac{1}{2}$, which shows (2). Suppose (2) is valid. Take the smallest m such that $T_\beta^m(-\frac{1}{2}) = -\frac{1}{2}$. Then $T_\beta^m(\frac{1}{2}) = T_\beta(T_\beta^{m-1}(\frac{1}{2})) = T_\beta(-T_\beta^{m-1}(-\frac{1}{2})) = -\frac{1}{2}$ which shows (3) (and also (1)). The expansion of (3) gives $\beta^\ell\left(\frac{1}{2} - \sum_{i=1}^{\ell} \bar{t}_i \beta^{-i}\right) = \frac{1}{2}$ which shows (2). $\lozenge$

Remark 3.4. Notice that if $\mathbf{d}(-\frac{1}{2})$ is purely periodic, then the norm of β is odd. Therefore, if the norm of β is even, then $-\mathbf{d}(\frac{1}{2}) = \mathbf{d}(-\frac{1}{2})$.

Now we discuss basic symbolic dynamical properties of the symmetric β-shift. Recall that a subshift $\mathcal{U}$ of the full shift $\mathcal{W}$ is of *finite type* if it can be described by a finite set of forbidden blocks. A subshift $\mathcal{U}$ is called *sofic* if each element is recognised by a finite automaton (cf. [20, 21]).

Theorem 3.5. *The symmetric β-shift $\mathcal{S}$ is sofic if and only if* $\mathbf{d}(-\frac{1}{2})$ *is eventually periodic.*

Proof. We follow the classical technique described in [14]. For an introduction to automata theory, we refer to [13, 24]. The proof of Th. 3.1 also implies that $s \in \mathcal{S}$ if and only if

$$\mathbf{d}(-\tfrac{1}{2}) \leq_{\mathrm{lex}} \sigma^n(s) \leq_{\mathrm{lex}} -\mathbf{d}(-\tfrac{1}{2}) \tag{3.3}$$

holds for all $n \geq 0$.

Assume that $\mathbf{d}(-\frac{1}{2}) = t_1 \cdots t_N\,(t_{N+1} \cdots t_{N+p})^\omega$, such that $t_{k+p} = t_k$ for all $k \geq N+1$. Construct an automaton S_1 as follows: The set of states is given by $\{s_1, \ldots, s_{N+p}\}$ and the labels are taken from $\mathcal{N}$. The initial state is s_1. For $j < N+p$, draw an arrow from s_j to s_{j+1} labeled by t_j, while arrows with greater labels lead to s_1. Draw an arrow from s_{N+p} to s_{N+1} labeled by t_{N+p}, while arrows with greater labels lead back to s_1. This automaton can check if a given sequence $s \in \mathcal{W}$ fulfils $\mathbf{d}(-\frac{1}{2}) \leq_{\mathrm{lex}} \sigma^n(s)$ for all $n \geq 0$. Analogously, we construct an automaton S_2 that checks if $\sigma^n(s) \leq_{\mathrm{lex}} -\mathbf{d}(-\frac{1}{2})$ holds for all $n \geq 0$. Finally, using S_1 and S_2, we construct a product automaton $S_1 \times S_2$

that checks if $\mathbf{d}(-\frac{1}{2}) \leq_{\text{lex}} \sigma^n(s) \leq_{\text{lex}} -\mathbf{d}(-\frac{1}{2})$ holds for all $n \geq 0$. This shows that $\mathcal{S}$ is sofic.

Recall that we denote by $L(\mathcal{S})$ the language of $\mathcal{S}$. Suppose that $\mathbf{d}(-\frac{1}{2}) = t_1 t_2 \cdots$ is not eventually periodic. Then the sequences $t_k t_{k+1} t_{k+2} \cdots$ are pairwise different for all $k \geq 1$. Therefore, for all pairs $j, \ell \geq 1$, $j \neq \ell$, there exist $p \geq 0$, such that $t_{j+p} \neq t_{\ell+p}$ and $w := t_j \cdots t_{j+p-1} = t_\ell \cdots t_{\ell+p-1}$ (with the convention that, if $p = 0$, then w is equal to the empty word). Without loss of generality, we assume that $t_{j+p} > t_{\ell+p}$. Then $t_1 \cdots t_{j-1} w\, t_{\ell+p} <_{\text{lex}} \mathbf{d}(-\frac{1}{2})$, and therefore $t_1 \cdots t_{j-1} w\, t_{\ell+p} \notin L(\mathcal{S})$ and $t_1 \cdots t_{\ell-1} w\, t_{\ell+p} \in L(\mathcal{S})$. Thus, the number of right congruence classes modulo $L(\mathcal{S})$ is infinite. Therefore, $L(\mathcal{S})$ is not recognisable by a finite automaton. $\Diamond$

Recall that a subshift $\mathcal{U}$ is *M-step* if it can be described by a collection of forbidden blocks all of which have length $M + 1$. If a subshift $\mathcal{U}$ is of finite type, then there is an $M \geq 0$ such that $\mathcal{U}$ is M-step (cf. [20, Prop. 2.1.7]).

Theorem 3.6. *The symmetric β-shift $\mathcal{S}$ is of finite type if and only if $\mathbf{d}(-\frac{1}{2})$ is purely periodic.*[4]

Proof. Suppose that $\mathbf{d}(-\frac{1}{2}) = (t_1 \cdots t_m)^\omega$. Then the set

$$U = \bigcup_{1 \leq i \leq m} \{\pm u \in \mathcal{N}^i : u <_{\text{lex}} t_1 \cdots t_i\}$$

is a finite set of forbidden words. From (3.3), it is easy to show that $s \in \mathcal{W}$ is an element of $\mathcal{S}$ if no subword of s is contained in U.

If $\mathcal{S}$ is of finite type, then it is sofic. Thus by Th. 3.5, $\mathbf{d}(-\frac{1}{2})$ must be eventually periodic. Set $\bar{t}_i = -t_i$ and assume that there exists $N \geq 1$ that

$$-\mathbf{d}(-\tfrac{1}{2}) = \bar{t}_1 \cdots \bar{t}_N \, (\bar{t}_{N+1} \cdots \bar{t}_{N+p})^\omega$$

where $\bar{t}_{k+p} = \bar{t}_k$ for all $k \geq N + 1$ and $\bar{t}_N \neq \bar{t}_{N+p}$. Since $\mathbf{d}(-\frac{1}{2})$ is not purely periodic, we have $-T_\beta^k(-\frac{1}{2}) < \frac{1}{2}$ and $\sigma^k(-\mathbf{d}(-\frac{1}{2})) <_{\text{lex}} -\mathbf{d}(-\frac{1}{2})$ for $k \geq 1$. By the admissible block decomposition as in the proof of Th. 3.1, for each positive integer s, there exists a $k \in \{1, 2, \ldots, p\}$ such that

$$\sigma^n((\bar{t}_{N+1} \cdots \bar{t}_{N+p})^s \bar{t}_{N+1} \cdots \bar{t}_{N+k-1}(\bar{t}_{N+k} + 1)) \leq_{\text{lex}} \bar{t}_1 \cdots \bar{t}_{sp+N+k-n}$$

for each $n \leq sp + N + k - 1$. Here we used an abusive terminology of

[4] Unlike usual β-expansion, if $\mathbf{d}(-\frac{1}{2})$ is finite, then it is *not* of finite type. This fact follows from the proof below with $t_{N+1} \cdots t_{N+p} = 0^p$.

σ acting on finite words by $\sigma(w_1 \cdots w_m) = w_2 \cdots w_m$. By Cor. 3.2, we have

$$(\bar{t}_{N+1} \cdots \bar{t}_{N+p})^s \bar{t}_{N+1} \cdots \bar{t}_{N+k-1}(\bar{t}_{N+k} + 1) \in L(\mathcal{S}).$$

Since $\mathcal{S}$ is of finite type, it must be M-step with some $M \geq 0$. Take a positive integer s such that $sp \geq \max(N+p, M)$. Then the two words $\bar{t}_1 \cdots \bar{t}_N(\bar{t}_{N+1} \cdots \bar{t}_{N+p})^s$ and $(\bar{t}_{N+1} \cdots \bar{t}_{N+p})^s \bar{t}_{N+1} \cdots \bar{t}_{N+k-1}(\bar{t}_{N+k}+1)$ are both in $L(\mathcal{S})$. However

$$\bar{t}_1 \cdots \bar{t}_N(\bar{t}_{N+1} \cdots \bar{t}_{N+p})^p \bar{t}_{N+1} \cdots \bar{t}_{N+k-1}(\bar{t}_{N+k} + 1) \notin L(\mathcal{S}).$$

This is a contradiction. $\Diamond$

For a real number x, there exists $m \geq 0 \in \mathbb{Z}$ that $\beta^{-m}x \in [-\frac{1}{2}, \frac{1}{2})$ with $\mathbf{d}(\beta^{-m}x) = x_1 x_2 \cdots$. Following the usual convention to express real numbers, we write:

$$x = x_1 \cdots x_m . x_{m+1} x_{m+2} \cdots \tag{3.4}$$

using the 'decimal' point. For the integer part $x_1 \cdots x_m$, the leading zeros may be omitted.

The sequence (3.4) is called a *symmetric β-expansion* of x. In the sequel, we will use the following notations:

$$\mathbf{Per}(\beta) = \{x \in \mathbb{R} : x \text{ has eventually periodic symmetric } \beta\text{-expansion}\},$$

$$\mathbf{Fin}(\beta) = \{x \in \mathbb{R} : x \text{ has finite symmetric } \beta\text{-expansion}\}.$$

Recall that a *Pisot number* is an algebraic integer $\beta > 1$ for which all algebraic conjugates γ with $\gamma \neq \beta$ satisfy $|\gamma| < 1$. If β is a Pisot number, then $\mathbf{Per}(\beta) = \mathbb{Q}(\beta)$ which follows by a similar proof as in [9, 26]. Especially $\mathcal{S}$ is sofic when β is a Pisot number.

Following [15], we say that β has the *symmetric finiteness property* if

$$\mathbf{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]. \tag{SF}$$

A similar proof as in [15] allows us to show, that if β has the property (SF), then β is a Pisot number. Moreover the weaker condition $\mathbb{Z} \cap \cap [0, \infty) \subset \mathbf{Fin}(\beta)$ implies the same fact (cf. [3]). For symmetric SRS, we can show:

Theorem 3.7. *Let β be a Pisot number with the minimal polynomial $X^d - a_1 X^{d-1} - \cdots - a_d$. Set*

$$r_{d-j+1} = \frac{a_j}{\beta} + \frac{a_{j+1}}{\beta^2} + \cdots + \frac{a_d}{\beta^{d-j+1}} \qquad \text{for } 2 \leq j \leq d.$$

Then β has the property (SF) *if and only if $\mathbf{r} = (r_1, \ldots, r_{d-1}) \in \mathcal{D}_{d-1}^0$.*

This is merely a reformulation of [4, Th. 2.1] originally due to Hollander [16]. Using (1.5) and Ths. 5.2 and 3.7, we can show:

Theorem 3.8. 1. *The quadratic Pisot number β with the minimal polynomial $X^2 - aX - b$ has the property* (SF), *if and only if*

$$-\frac{1}{2} < \frac{b}{\beta} \leq \frac{1}{2}.$$

2. *The cubic Pisot number β with the minimal polynomial $X^3 - AX^2 - BX - C$ has the property* (SF), *if and only if*

$$-\frac{C}{\beta} - \frac{1}{2} < \frac{B}{\beta} + \frac{C}{\beta^2} \leq \frac{C}{\beta} + \frac{1}{2} \quad \textit{and} \quad \frac{C}{\beta} < \frac{1}{2}.$$

The first statement is an application of (1.5) and Th. 3.7 with $d = 2$, $r_1 = b/\beta$. The second statement follows from Ths. 5.2 and 3.7 with $d = 3$, $r_1 = C/\beta$ and $r_2 = B/\beta + C/\beta^2$. Note that $C/\beta \neq \frac{1}{2}$ since C is an integer.

In both quadratic and cubic cases of Th. 3.8, each right inequality can not be an equality, taking the degree of β into consideration.

Example 3.9. $(x^3 - 3x^2 - 2x - 1, \{-2, -1, 0, 1, 2\})$ has the property (SF) since $(0.276, 0.627) \in \mathcal{D}_2^0$ in Fig. 1. We have $\mathbf{d}(-1/2) = ((-2)1(-1)(-1)111)^\omega$ and for example

$$\mathbf{d}(\beta^2 - 4\beta + 1) = (-1)(-1).$$

On the other hand, $(x^3 - 3x^2 - 3x - 1, \{-2, -1, 0, 1, 2\})$ does not have the property (SF), since $(0.260, 0.847) \notin \mathcal{D}_2^0$. We have $\mathbf{d}(-1/2) = ((-2)011)^\omega$ and

$$\mathbf{d}(\beta^2 - 4\beta + 1) = (2(-2))^\omega .$$

Both of them are of finite type by Th. 3.6.

Th. 3.8 is substantially stronger than the one in [3] where the cubic Pisot *units* with (F) are classified. The characterisation problem of cubic Pisot numbers with (F) is still open for the usual β-expansion. Some partial answers for this problem are given in [5].

4. Basic properties of symmetric SRS

Let us first study $\mathcal{D}_d$. For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$, let

$$R(\mathbf{r}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & -r_{d-1} & -r_d \end{pmatrix}.$$

For $M \in \mathbb{R}^{d\times d}$, denote by $\varrho(M)$ the spectral radius of M, i.e., the maximum absolute value of all eigenvalues of M. For any $\delta > \varrho(M)$, we can fix a vector norm $\|\cdot\|_{M,\delta}$ with $\|Mv\|_{M,\delta} \le \delta\|v\|_{M,\delta}$. For instance, if $|||\cdot|||$ denotes the euclidean norm, then

$$\|\mathbf{x}\|_{M,\delta} := \max_{k=0}^{\infty}\left\{\frac{1}{\delta^k}|||M^k\mathbf{x}|||\right\} \tag{4.1}$$

has the desired property. This maximum exists since $\lim\limits_{k\to\infty}|||M^k\mathbf{x}|||/\delta^k = 0$. For a similar definition, see the formula (3.2) of [19]. Note that $\|\cdot\|_{M,\delta}$ depends on M and δ. For simplicity, we write

$$\varrho(\mathbf{r}) := \varrho(R(\mathbf{r})) \quad \text{and} \quad \|\mathbf{x}\|_{\mathbf{r},\delta} := \|\mathbf{x}\|_{R(\mathbf{r}),\delta}. \tag{4.2}$$

Let $\mathcal{E}_d = \{\mathbf{r}\in\mathbb{R}^d \ : \ \varrho(\mathbf{r}) < 1\}$. It is known that the closure $\overline{\mathcal{E}}_d$ is a regular set, i.e., the set coincides with the closure of its interior (cf. [4, Lemma 4.3]). Furthermore, $\overline{\mathcal{E}}_d$ coincides with $\{\mathbf{r}\in\mathbb{R}^d \ : \ \varrho(\mathbf{r}) \le 1\}$. Thus

$$\mathcal{E}_d \subset \mathcal{D}_d \subset \overline{\mathcal{E}}_d.$$

It seems to be a hard problem to characterise the set $\mathcal{D}_d \cap \partial\mathcal{E}_d$ (cf. [5]). In the non symmetric case (1.1), some partial results have been proved by Akiyama et al. [6].

Now we turn to the study of $\mathcal{D}_d^0$. Similarly as in [4], we can deduce a bound for the periodic orbits. Let $\mathbf{r}\in\mathbb{R}^d$ with $\varrho(\mathbf{r}) < \delta < 1$. If $\mathbf{a}$ is periodic for $\tau_{\mathbf{r}}$, then

$$\|\mathbf{a}\|_{\mathbf{r},\delta} \le \frac{1}{2(1-\delta)} =: K.$$

Simply by testing all $\mathbf{a}\in\mathbb{Z}^d$ with $\|\mathbf{a}\|_{\mathbf{r},\delta} \le K$, this estimate provides an algorithm to determine whether $\mathbf{r}\in E$ is contained in $\mathcal{D}_d^0$ or not.

However, we do not use this method in the present paper because we develop an efficient alternative way. Let $\mathbf{e}_i = (0,\dots,0,1,0,\dots,0)$ be the i-th unit vector. For a given $\mathbf{r} = (r_1,\dots,r_d)$, we say that a finite set $\mathcal{V}(\mathbf{r}) \subset \mathbb{Z}^d$ is a *set of witnesses for* $\mathbf{r}$, if $\{\mathbf{e}_i, -\mathbf{e}_i\} \subset \mathcal{V}(\mathbf{r})$ for

$1 \le i \le d$, and for each $(z_1, \dots, z_d) \subset \mathcal{V}(\mathbf{r})$, the element $(z_2, \dots z_{d+1})$ belongs to $\mathcal{V}(\mathbf{r})$ provided that

$$-1 < r_1 z_1 + \cdots + r_d z_d + z_{d+1} < 1. \tag{4.3}$$

If $\rho(\mathbf{r}) < 1$, then $\mathcal{V}(\mathbf{r})$ can be constructed by successive addition of new elements $(z_2, \dots, z_{d+1})$. For simplicity, we write $\mathcal{V} = \mathcal{V}(\mathbf{r})$ when $\mathbf{r}$ is fixed. Let $\mathcal{G}(\mathcal{V})$ be a directed graph with vertices $\mathcal{V}$ and edges defined by $(z_1, \dots, z_d) \to (z_2, \dots, z_{d+1})$ if and only if

$$-\tfrac{1}{2} \le r_1 z_1 + \cdots + r_d z_d + z_{d+1} < \tfrac{1}{2}. \tag{4.4}$$

Note that the set of vertices $\mathcal{V}$ is exactly the same as in [4, Th. 5.1]. However, the edges are defined in a different manner. By definition, for each vertex there exists exactly one outgoing edge.

Theorem 4.1. *Let $\mathbf{r} \in \mathbb{R}^d$. If every walk in the graph $\mathcal{G}(\mathcal{V}(\mathbf{r}))$ falls into the trivial cycle $\mathbf{0} \to \mathbf{0}$, then $\mathbf{r}$ belongs to $\mathcal{D}_d^0$.*

Proof. The proof is a generalisation of ideas from [4, 7, 8, 11, 12]. Let $\mathbf{r} = (r_1, \dots , r_d)$. We say that $\mathbf{a} \in \mathbb{Z}^d$ is SSRS-finite, if there is a $k \ge 0$ such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$.

Suppose that $\mathbf{a}$ is SSRS-finite and $\mathbf{b} \in \mathcal{V}$. We will prove that $\mathbf{a} + \mathbf{b}$ is SSRS-finite. Denote by $\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots , a_{d+1})$ and $\tau_{\mathbf{r}}(\mathbf{b}) = (b_2, \dots , b_{d+1})$. Since $a_{d+1} = -\lfloor \mathbf{ra} + \frac{1}{2} \rfloor$, $b_{d+1} = -\lfloor \mathbf{rb} + \frac{1}{2} \rfloor$, we obtain

$$-\tfrac{1}{2} \le r_1 a_1 + \cdots + r_d a_d + a_{d+1} < \tfrac{1}{2} \quad \text{and} \tag{4.5}$$

$$-\tfrac{1}{2} \le r_1 b_1 + \cdots + r_d b_d + b_{d+1} < \tfrac{1}{2}. \tag{4.6}$$

Thus

$$-1 \le \underbrace{r_1(a_1 + b_1) + \cdots + r_d(a_d + b_d) + a_{d+1} + b_{d+1}}_{=:\ \xi} < 1.$$

We distinguish between three cases.

(i) If $\frac{1}{2} \le \xi$, then $0 < r_1 b_1 + \cdots + r_d b_d + b_{d+1}$ implies

$$-1 < r_1 b_1 + \cdots + r_d b_d + b_{d+1} - 1 < 1$$

and

$$-\tfrac{1}{2} \le r_1(a_1 + b_1) + \cdots + r_d(a_d + b_d) + a_{d+1} + b_{d+1} - 1 < \tfrac{1}{2}.$$

Thus take $\eta = (b_2, \dots , b_d, b_{d+1} - 1) \in \mathcal{V}$.

(ii) If $\xi < -\frac{1}{2}$, then $r_1 b_1 + \cdots + r_d b_d + b_{d+1} < 0$ implies

$$-1 < r_1 b_1 + \cdots + r_d b_d + b_{d+1} + 1 < 1$$

and

$$-\tfrac{1}{2} \le r_1(a_1 + b_1) + \cdots + r_d(a_d + b_d) + a_{d+1} + b_{d+1} + 1 < \tfrac{1}{2}.$$

In this case, we take $\eta = (b_2, \ldots, b_d, b_{d+1} + 1) \in \mathcal{V}$.

(iii) If $-\frac{1}{2} \le \xi < \frac{1}{2}$, then take $\eta = (b_2, \ldots, b_d, b_{d+1}) \in \mathcal{V}$.

Therefore, in any case, there exists $\eta \in \mathcal{V}$ such that $\tau_{\mathbf{r}}(\mathbf{a} + \mathbf{b}) = \tau_{\mathbf{r}}(\mathbf{a}) + \eta$ with $\eta =: \eta^{(1)} \in \mathcal{V}$. Repeating this argument, we find that

$$\tau_{\mathbf{r}}^k(\mathbf{a} + \mathbf{b}) = \tau_{\mathbf{r}}^k(\mathbf{a}) + \eta^{(k)}$$

with $\eta^{(k)} \in \mathcal{V}$. Since $\mathbf{a}$ is SSRS-finite, there is a k such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = \mathbf{0}$. By the assumption of the theorem, we conclude that $\mathbf{a} + \mathbf{b}$ is SSRS-finite. Since $\mathcal{V}$ contains the i-th unit vectors, the proof is finished. $\Diamond$

The set $\mathcal{D}_d^0$ can be constructed from $\mathcal{D}_d$ by cutting out countable many families of convex polyhedra. The following technique was developed in [4, 16, 25]. Consider a finite sequence $a_1, \ldots, a_L$ of integers. Define a_j $(j \in \mathbb{Z})$ by periodicity $a_j = a_{j+L}$. This gives a periodic bi-infinite word designated by $\pi = [a_1, \ldots, a_L]^\infty \in \mathbb{Z}^{\mathbb{Z}}$ which is called a *cycle* of length L. We define the set:

$$P(\pi) = \bigl\{\mathbf{r} \in \mathbb{R}^d : -\tfrac{1}{2} \le r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < \tfrac{1}{2} \text{ for } j \in \mathbb{Z}\bigr\}. \tag{4.7}$$

Since each of these inequalities gives an upper (resp. lower) halfspace in $\mathbb{R}^d$, the system (4.7) defines a (possibly degenerated) convex polyhedron. A cycle $\pi = [a_1, \ldots, a_d]^\infty$ of period L is called *primitive,*[5] if the vectors $(a_{1+j}, \ldots, a_{d+j})$ are pairwise different for $j = 0, \ldots, L-1$. Using this terminology, we trivially have

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi} P(\pi),$$

where π runs through all non zero primitive cycles $\pi = [a_1, \ldots, a_L]^\infty$ of arbitrary length. Unfortunately since the set of periods is infinite, this expression is far from being practical.

As an analogy of Th. 5.2 in [4], the next theorem gives an efficient algorithm for a closed set $H \subset \mathcal{E}_d$ to construct $H \cap \mathcal{D}_d^0$. The basic idea is to collect and merge all possible graphs $\mathcal{G}(\mathcal{V}(\mathbf{r}))$ which correspond

[5] Remark that this definition depends on the dimension d.

to points $\mathbf{r} \in H$. This allows us to apply Th. 4.1 not only for a single point $\mathbf{r}$ but also for the set H. The proof given below is rewritten from the one in [4] in order to make clearer the convergence of the algorithm.

Theorem 4.2. *Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathcal{D}_d$ and let H be the convex hull of $\mathbf{r}_1, \dots, \mathbf{r}_k$. Assume that H is contained in the interior of $\mathcal{D}_d$ and sufficiently small in diameter. Then there exists an algorithm to construct a finite directed graph $G = (V, E)$ with vertices $V \subset \mathbb{Z}^d$ and edges $E \subset V \times V$ which satisfy*

(i) *$\pm\mathbf{e}_i \in V$ for all $i = 1, \dots, d$,*
(ii) *$\mathcal{G}(\mathcal{V}(\mathbf{s}))$ is a subgraph of G for all $\mathbf{s} \in H$,*
(iii) *$H \cap \mathcal{D}_d^0 = H \setminus \bigcup_\pi P(\pi)$, where π are taken over all nonzero primitive cycles of G.*

Proof. For $\mathbf{z} \in \mathbb{Z}^d$, let

$$m(\mathbf{z}) = \min_{1\le i\le k} \{-\lfloor \mathbf{r}_i\mathbf{z}\rfloor\} \qquad M(\mathbf{z}) = \max_{1\le i\le k} \{-\lfloor \mathbf{r}_i\mathbf{z}\rfloor\},$$
$$\xi_1(\mathbf{z}) = \min\{m(\mathbf{z}), -M(-\mathbf{z})\}, \quad \xi_2(\mathbf{z}) = \max\{-m(-\mathbf{z}), M(\mathbf{z})\}.$$

Set $V_1 = \{\pm\mathbf{e}_i : i = 1, \dots, n\}$ and define inductively[6]

$$V_{i+1} = V_i \cup \big\{(z_2, \dots, z_d, j) : \mathbf{z} = (z_1, \dots, z_d) \in V_i \text{ and } j \in [\xi_1(\mathbf{z}), \xi_2(\mathbf{z})] \cap \mathbb{Z}\big\}.$$

Assume for the moment that there exists a finite limit set $V = \bigcup_i V_i$, i.e., there exists i with $V_{i+1} = V_i := V$. Draw edges

$$\mathbf{z} = (z_1, \dots, z_d) \to (z_2, \dots, z_d, j)$$

according to the SSRS algorithm, i.e.,

$$\min_{1\le i\le k} \big\{-\lfloor \mathbf{r}_i\mathbf{z} + \tfrac{1}{2}\rfloor\big\} \le j \le \max_{1\le i\le k} \big\{-\lfloor \mathbf{r}_i\mathbf{z} + \tfrac{1}{2}\rfloor\big\}.$$

We claim that this finite graph $G := (V, E)$ has the desired properties.

In fact, the condition (i) is trivial. For $\mathbf{s} \in H$, the graph $\mathcal{G}(\mathcal{V}(\mathbf{s}))$ was given by the algorithm:

$$\mathcal{V}_1(\mathbf{s}) = \{\mathbf{e}_i, -\mathbf{e}_i : i = 1, \dots, n\}$$
$$\mathcal{V}_{i+1}(\mathbf{s}) = V_i(\mathbf{s}) \cup \big\{(z_2, \dots, z_d, j) : \mathbf{z} = (z_1, \dots, z_d) \in \mathcal{V}_i(\mathbf{s}) \text{ and } -1 < \mathbf{s}\mathbf{z} + j < 1\big\}$$

together with the corresponding edges. The inequality $-1 < \mathbf{s}\mathbf{z} + j < 1$ only gives

[6] In fact, $\xi_1(\mathbf{z}) = -M(-\mathbf{z})$ and $\xi_2(\mathbf{z}) = M(\mathbf{z})$ always hold. See [4].

$$j \in [\xi_1(\mathbf{z}), \xi_2(\mathbf{z})] \cap \mathbb{Z}$$

since $\mathbf{s}$ is a convex linear combination of $\mathbf{r}_i$'s. Hence we see that $\mathcal{G}(\mathcal{V}(\mathbf{s}))$ is a subgraph of G which shows the property (ii). Now let us prove (iii). Obviously

$$H \setminus \bigcup_{\pi} P(\pi) \supset H \cap \mathcal{D}_d^0.$$

Take $\mathbf{r} \in H \setminus \mathcal{D}_d^0$. In view of Th. 4.1 since $H \subset \mathcal{D}_d$, there exists a non zero primitive cycle π in $\mathcal{G}(\mathcal{V}(\mathbf{r}))$ and $\mathbf{r} \in P(\pi)$. By using (ii), π is a non zero primitive cycle of (V, E) as well. This shows that

$$\mathbf{r} \notin H \setminus \bigcup_{\pi} P(\pi)$$

which proves the claim.

Finally it remains to show the existence of i with $V_{i+1} = V_i$, i.e., the convergence of our procedure, provided the diameter of H is sufficiently small. According to (4.1), we start with an arbitrary point $\mathbf{r} \in \mathcal{E}_d$ with $\rho(\mathbf{r}) < \delta < 1$ and choose a norm $\|\cdot\|_{\mathbf{r},\delta}$, such that $R(\mathbf{r})$ is contractive. For a matrix M, the same symbol $\|M\|_{\mathbf{r},\delta}$ stands for the operator norm, that is,

$$\|M\|_{\mathbf{r},\delta} = \sup_{\mathbf{v} \neq 0} \frac{\|M\mathbf{v}\|_{\mathbf{r},\delta}}{\|\mathbf{v}\|_{\mathbf{r},\delta}}.$$

Take H small enough such that $\mathbf{r} \in H$ and for any $\mathbf{s} \in H$, we have

$$\|R(\mathbf{s}) - R(\mathbf{r})\|_{\mathbf{r},\delta} < \delta_1 < 1 - \delta.$$

This is possible since $\|\cdot\|_{\mathbf{r},\delta}$ is continuous. Put $\delta_2 = \delta_1 + \delta < 1$. Then, by induction, for any $\mathbf{s}_i \in H$ and $\mathbf{v} \in \mathbb{R}^d$, we have

$$\|R(\mathbf{s}_k) \cdots R(\mathbf{s}_2) R(\mathbf{s}_1) \mathbf{v}\|_{\mathbf{r},\delta} < \delta_2^k \|\mathbf{v}\|_{\mathbf{r},\delta}. \tag{4.8}$$

Since $j \in [\xi_1(\mathbf{z}), \xi_2(\mathbf{z})] \cap \mathbb{Z}$, the mapping $(z_1, \ldots, z_d) \to (z_2, \ldots, z_d, j)$ is given by

$$(z_2, \ldots, z_d, j)^t = R(\mathbf{s})(z_1, \ldots, z_d)^t + \mathbf{u}$$

with $\|\mathbf{u}\| \leq \frac{1}{2}$ and $\mathbf{s} \in H$. Successive applications of this mapping yield elements of the form

$$R(\mathbf{s}_k) \cdots R(\mathbf{s}_1)(z_1, \ldots, z_d)^t + R(\mathbf{s}_k) \cdots R(\mathbf{s}_2)\mathbf{u}_1 + \\ + R(\mathbf{s}_k) \cdots R(\mathbf{s}_3)\mathbf{u}_2 + \cdots + \mathbf{u}_k$$

with bounded $\mathbf{u}_i$'s. By using the estimate (4.8), the set V_i $(i = 1, 2, \ldots)$ must be uniformly bounded. Thus there exists i such that $V_i = V_{i+1}$. $\Diamond$

This algorithm is quite sensitive to the choice of the initial convex hull H. Let us denote by $\mathcal{G}(H)$ the corresponding graph by this algorithm. (It might be an infinite graph.) If the convex hull H is subdivided into several convex hulls $\bigcup_i H_i$, then $\mathcal{G}(H) \supset \bigcup_i \mathcal{G}(H_i)$ but it is usually not equal. For example, let $d = 2$ and H_i $(i = 1, 2)$ be segments from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, 1)$ and from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, -1)$. Then $\mathcal{G}(H_1 \cup H_2)$ is infinite[7] since one can easily show that $(n-1, n) \in V_{n+1}$ for $n \geq 2$, but both $\mathcal{G}(H_i)$ are finite graphs with 19 vertices. Especially $\mathcal{G}(H)$ is usually larger than $\bigcup_{\mathbf{r} \in H} G(\mathcal{V}(\mathbf{r}))$. Therefore subdivision of the initial convex hull H is meaningful not only by the requirement of Th. 4.2 but also from the computational point of view. Even if we get a finite graph $\mathcal{G}(H)$ given by the algorithm, further subdivision of H may drastically help us to create smaller graphs. This is practically important since it is hard to list up all the primitive cycles out of a large graph.

Since for each compact set $A \subset \mathcal{E}_d$ we can find a finite covering of A by sufficiently small convex hulls containing open balls, Th. 4.2 theoretically gives an algorithm to construct $A \cap \mathcal{D}_d^0$. The clue of convergence is the smallness of H with respect to the norm $\|\cdot\|_{\mathbf{r},\delta}$. This criterion is not easily checked. However from the proof, H must be smaller if it is located closer to the boundary $\partial\mathcal{E}_d$ since δ becomes larger. When implementing Th. 4.2 in a computer language equipped with interruption, there is no need to care on the smallness of H. Perform 'trial and error'. If we get a finite graph $\mathcal{G}(H)$, then we are ready. If the graph grows too large, then interrupt and restart with a smaller H.

5. Two dimensional symmetric SRS

In this section, we give a complete description of $\mathcal{D}_2^0$. We already know that

$$\mathcal{D}_2 \subset \Delta := (\mathcal{E}_2)^\omega = \{(x, y) \in \mathbb{R}^2 \ : \ x \leq 1, |y| \leq x + 1\}.$$

First we show

Proposition 5.1. $\mathcal{D}_2^0 \subset \frac{\Delta}{2} := \left\{(x, y) \in \mathbb{R}^2 \ : \ x \leq \frac{1}{2}, |y| \leq x + \frac{1}{2}\right\}$.

Proof. We give cycle π's so that $F = \Delta \setminus \frac{\Delta}{2}$ is completely covered by the cutout $P(\pi)$'s. The purpose is almost fulfilled by the following five cycles.

- $\pi_1 = [-1]^\infty$ gives $P(\pi_1) = \left\{(x, y) \ : \ -\frac{3}{2} < y + x \leq -\frac{1}{2}\right\}$.

[7] This also shows the necessity of subdivision in the proof of Th. 5.2.

- $\pi_2 = [1,-1]^\infty$ gives $P(\pi_2) = \{(x,y) \ : \ \frac{1}{2} < y-x < \frac{3}{2}\}$.
- $\pi_3 = [-1,0,1]^\infty$ gives
  $P(\pi_3) = \{(x,y) \ : \ \frac{1}{2} < x \le \frac{3}{2},\ \frac{1}{2} \le y < \frac{3}{2},\ -\frac{1}{2} < y-x \le \frac{1}{2}\}$.
- $\pi_4 = [0,-1,0,1]^\infty$ gives
  $P(\pi_4) = \{(x,y) \ : \ \frac{1}{2} < x < \frac{3}{2},\ -\frac{1}{2} < y < \frac{1}{2}\}$.
- $\pi_5 = [1,1,0,-1,-1,0]^\infty$ gives
  $P(\pi_5) = \{(x,y) \ : \ \frac{1}{2} < x < \frac{3}{2}, -\frac{3}{2} < y < -\frac{1}{2}, -\frac{1}{2} < y+x < \frac{1}{2}\}$.

Then

$$F \setminus \Big(\bigcup_{i=1}^{5} P(\pi_i)\Big) = \{(1,\tfrac{1}{2}),(1,\tfrac{3}{2})\} \cup \underbrace{\{(x,y) \ : \ \tfrac{1}{2} < x \le 1, y = -\tfrac{1}{2}\}}_{=:M}.$$

It is easily proved that

$$\begin{aligned}(1,\tfrac{1}{2}) &= P([0,1,-1,-1,1,0,-1]^\infty),\\ (1,\tfrac{3}{2}) &= P([0,1,-2,2,-1,-1,2,-2,1,0,-1,1,-1]^\infty),\\ M &= P([0,1,0,-1,-1]^\infty). \qquad \Diamond\end{aligned}$$

Theorem 5.2. *Define two segments by* $L_1 = \{(x,y) : |x| \le \frac{1}{2}, y = -x-\frac{1}{2}\}$ *and* $L_2 = \{(\frac{1}{2},y) : \frac{1}{2} < y < 1\}$. *Then* $\mathcal{D}_2^0 = \frac{\Delta}{2} \setminus (L_1 \cup L_2)$.

Proof. In light of Prop. 5.1, we subdivide the triangle $\frac{\Delta}{2}$ into small ones and apply Th. 4.2. Let $T(a,b,c)$ be the triangle of vertices a, b and c. Applying Th. 4.2 for $\Delta_1 = T((-\frac{1}{2},0),(0,-\frac{1}{2}),(0,\frac{1}{2}))$, the associated graph (V_1,E_1) is given by

$V_1 = \{(0,1),(1,0),(0,-1),(-1,0),(-1,-1),(1,-1),(0,0),(-1,1),(1,1)\}$

and

$$\begin{aligned}E_1 = \{&(-1,-1) \to (-1,-1), (-1,-1) \to (-1,0), (0,-1) \to (-1,-1),\\ &(0,-1) \to (-1,0), (1,-1) \to (-1,-1), (1,-1) \to (-1,0),\\ &(-1,0) \to (0,-1), (-1,0) \to (0,0), (0,0) \to (0,0),\\ &(1,0) \to (0,0), (-1,1) \to (1,-1), (-1,1) \to (1,0), (0,1) \to (1,-1),\\ &(0,1) \to (1,0), (1,1) \to (1,-1), (1,1) \to (1,0)\}.\end{aligned}$$

This graph has exactly one non-trivial[8] strongly connected component (cf. Fig. 2):

$$(-1,-1) \leftarrow (-1,-1) \to (-1,0) \leftrightarrow (0,-1) \to (-1,-1).$$

This yields three primitive cycles: $\pi_1 = [-1]^\infty$, $\theta_1 = [0,-1]^\infty$ and $\theta_2 = [-1,-1,0]^\infty$. Since

[8]The 0-cycle always forms a single component.

$$P(\pi_1) = \{(x,y) \ : \ -\tfrac{3}{2} < x+y \le -\tfrac{1}{2}\},$$
$$P(\theta_1) = \{(x,y) \ : \ -\tfrac{1}{2} < y \le \tfrac{1}{2}, -\tfrac{3}{2} < x \le -\tfrac{1}{2}\} \quad \text{and}$$
$$P(\theta_2) = \emptyset,$$

we have shown that $\Delta_1 \cap \mathcal{D}_2^0 = \Delta_1 \setminus L_1$.

We proceed in a similar manner with

$$\begin{aligned}
\Delta_2 &= T((\tfrac{1}{2},0),(0,-\tfrac{1}{2}),(0,\tfrac{1}{2}))\\
\Delta_3 &= T((\tfrac{1}{2},0),(0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2}))\\
\Delta_4 &= T((\tfrac{1}{2},1),(0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2}))\\
\Delta_5 &= T((\tfrac{1}{2},0),(0,-\tfrac{1}{2}),(\tfrac{1}{2},-\tfrac{1}{2}))\\
\Delta_6 &= T((\tfrac{1}{2},-1),(0,-\tfrac{1}{2}),(\tfrac{1}{2},-\tfrac{1}{2})).
\end{aligned}$$

The corresponding graphs (V_i, E_i) $(i = 2, \dots, 6)$ are depicted in Figures 3, 4, 5, 6 and 7.

The triangle Δ_2 only gives π_1 which shows $\Delta_2 \cap \mathcal{D}_2^0 = \Delta_2 \setminus L_1$. The triangle Δ_3 gives rise to the primitive cycle $[-1,-1,1]^\infty$. Since $P([-1,-1,1]^\infty) = \emptyset$, we have $\Delta_3 \subset \mathcal{D}_2^0$. The triangle Δ_4 only gives $\theta_3 = [0,-1,1]^\infty$ and $P(\theta_3) = \{(x,y) \ : \ \tfrac{1}{2} \le x < \tfrac{3}{2}, \ \tfrac{1}{2} < y \le \tfrac{3}{2}, \ -\tfrac{1}{2} \le$ $\le y - x < \tfrac{1}{2}\}$. Thus we have $\Delta_4 \cap \mathcal{D}_2^0 = \Delta_4 \setminus L_2$. Both Δ_5 and Δ_6 only give π_1. Thus $(\Delta_5 \cup \Delta_6) \setminus L_1 \subset \mathcal{D}_2^0$. Summing up, we have shown the result. $\diamondsuit$

Remark 5.3. The referee pointed out that in the last proof, several regions could be merged like $\Delta_3 \cup \Delta_4$ and $\Delta_1 \cup \Delta_2 \cup \Delta_5 \cup \Delta_6$ to obtain the same result. This comment is correct and we will have two larger graphs with 19 vertices. As we discussed the end of §4, this algorithm is rather sensitive to the choice of subdivisions and under this choice we have to study all primitive cycles in these new graphs. This may be finished automatically by computer. However by our choice, the graphs are smaller and all computation can be confirmed possibly by hand. The reader may easily list up all primitive cycles by observing the graphs listed below. In this sense, we believe that our choice is more handy than the suggestion by the referee.

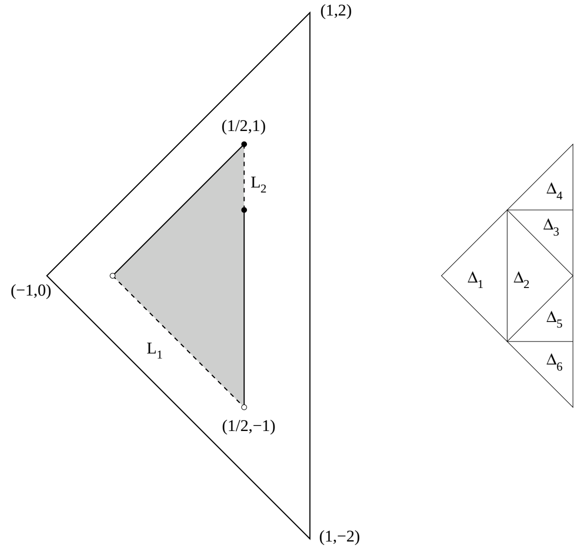

Figure 1. The sets $\mathcal{D}_2^0$ and $\overline{\mathcal{E}}_2$

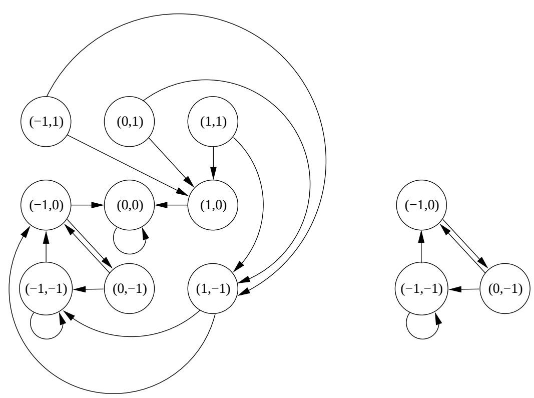

Figure 2. The graph (V_1, E_1)

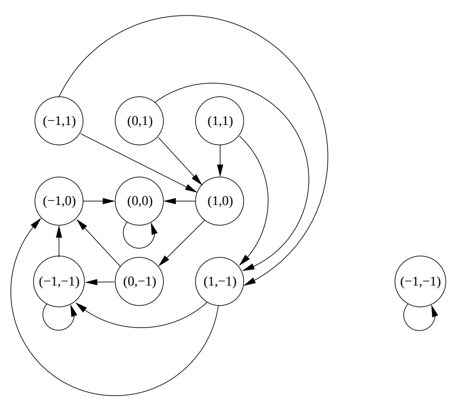

Figure 3. The graph (V_2, E_2)

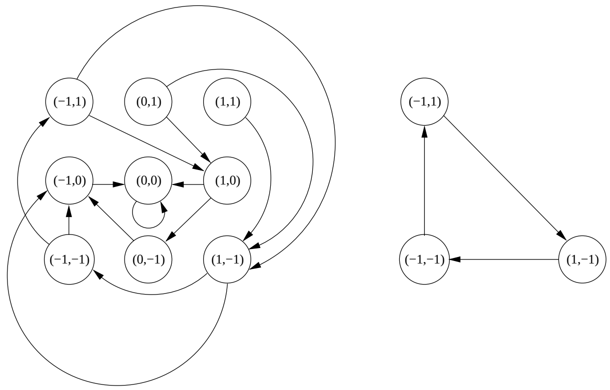

Figure 4. The graph (V_3, E_3)

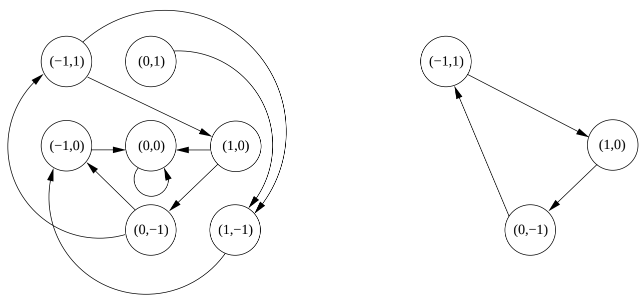

Figure 5. The graph (V_4, E_4)

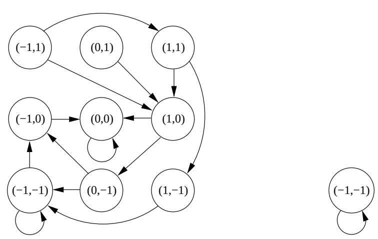

Figure 6. The graph (V_5, E_5)

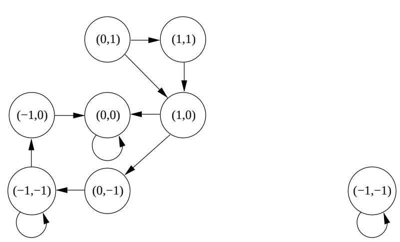

Figure 7. The graph (V_6, E_6)

Acknowledgement. The authors thank the referee for many valuable suggestions and remarks.

References

[1] AKIYAMA, S.: Pisot numbers and greedy algorithm, in: Number theory (Eger, 1996), pp. 9–21, de Gruyter, Berlin, 1998.

[2] AKIYAMA, S.: Self affine tilings and Pisot numeration systems, in: K. Győry and S. Kanemitsu, editors, Number Theory and its Applications, pp. 1–17, Kluwer, 1999.

[3] AKIYAMA, S.: Cubic Pisot units with finite beta expansions, in: F. Halter-Koch and R. F. Tichy, editors, Algebraic Number Theory and Diophantine Analysis, de Gruyter, 2000.

[4] AKIYAMA, S., BORBÉLY, T., BRUNOTTE, H., PETHŐ, A. and THUSWALDNER, J. M.: Generalized radix representations and dynamical systems I, *Acta Math. Hungar.* **108** (2005), 207–238.

[5] AKIYAMA, S., BRUNOTTE, H., PETHŐ, A. and THUSWALDNER, J. M.: Generalized radix representations and dynamical systems II, *Acta Arith.* **121** (1) (2006), 21–61.

[6] AKIYAMA, S., BRUNOTTE, H., PETHŐ, A. and STEINER, W.: Remarks on a conjecture on certain integer sequences, *Period. Math. Hungar.* **52** (1) (2006), 1–17.

[7] AKIYAMA, S. and RAO, H.: New criteria for canonical number systems, *Acta Arith.* **111** (1) (2004), 5–25.

[8] AKIYAMA, S. RAO, H. and STEINER, W.: A certain finiteness property of Pisot number systems, *J. Number Theory* **107**/1 (2004), 135–160.

[9] BERTRAND, A.: Développements en base de Pisot et répartition modulo 1, *C. R. Acad. Sci. Paris Sér. A-B* **285** (6) (1977), A419–A421.

[10] BLANCHARD, F.: β-expansions and symbolic dynamics, *Theoret. Comput. Sci.* **65** (2) (1989), 131–141.

[11] BRUNOTTE, H.: On trinomial bases of radix representations of algebraic integers, *Acta Sci. Math. (Szeged)* **67** (3-4) (2001), 521–527.

[12] BRUNOTTE, H.: Characterization of CNS trinomials, *Acta Sci. Math. (Szeged)* **68** (2002), 673–679.

[13] EILENBERG, S.: Automata, languages, and machines, Vol. A, Pure and Applied Mathematics, Vol. 58, Academic Press, New York, 1974.

[14] FROUGNY, CH. and SOLOMYAK, B.: On representation of integers in linear numeration systems, in: Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser. 228, pp. 345–368. Cambridge Univ. Press, Cambridge, 1996.

[15] FROUGNY, CH. and SOLOMYAK, B.: Finite beta-expansions, *Ergodic Theory Dyn. Sys.* **12** (1992), 713–723.

[16] HOLLANDER, M.: *Linear numeration systems, finite beta expansions, and discrete spectrum of substitution dynamical systems*, PhD thesis, University of Washington, 1996.

[17] ITO, SH. and TAKAHASHI, Y.: Markov subshifts and realization of β-expansions, *J. Math. Soc. Japan* **26** (1974), 33–55.

[18] KÁTAI, I.: Generalized number systems and fractal geometry, Lecture note at Pécs, Hungary, 1995.

[19] LAGARIAS, J. and WANG, Y.: Integral self-affine tiles in $\mathbb{R}^n$ II. lattice tilings, *J. Fourier Anal. Appl.* **3** (1997), 83–102.

[20] LIND, D. and MARCUS, B.: An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

[21] LOTHAIRE, M.: Numeration systems, in: Algebraic combinatorics on words, Encyclopedia of Mathematics and its Applications 90, Ch. 7, Cambridge University Press, Cambridge, 2002.

[22] PARRY, W.: On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.

[23] RÉNYI, A.: Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.

[24] SAKAROVITCH, J.: Éléments de théorie des automates, Vuibert Informatique, Vuibert, Paris, 2003.

[25] SCHEICHER, K. and THUSWALDNER, J. M.: On the Characterization of Canonical Number Systems, *Osaka J. Math.* **41** (2004), 327–351.

[26] SCHMIDT, K.: On periodic expansions of Pisot numbers and Salem numbers, *Bull. London Math. Soc.* **12** (1980), 269–278.