ON GAUSS LEMNISCATE FUNC-TIONS AND LEMNISCATIC MEAN

Edward **Neuman**

Department of Mathematics, Mailcode 4408, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901, USA

Abstract: This paper deals with Gauss lemniscate functions and the lemniscatic mean. They admit representations in terms of certain elliptic integrals which belong to a family of the *R*-hypergeometric functions. It has been proven that the lemniscatic mean and the Schwab–Borchardt mean are comparable. Four new means derived from the lemniscatic mean are discussed. Bounds for the Gauss functions and means under discussion are established. In particular, the Ky Fan inequalities involving new means and particular means derived from the Schwab–Borchardt mean are also included.

1. Introduction and definitions

Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine functions are defined by

$$(1.1) \qquad \operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1 - t^4}} \, , \quad |x| \le 1$$

and

$$(1.2) \qquad \operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1 + t^4}} \, ,$$

respectively (see [5, (2.5) and (2.6)], [1, p. 259], [13, Ch. 1]). First

---

E-mail address: edneuman@math.siu.edu

function has a simple geometric interpretation. It is well-known that the arc length s measured from the origin to a point with plane polar coordinates (r, θ) on the Bernoulli lemniscate $r^2 = \cos 2\theta$, is $s = \operatorname{arcsl} r$.

The lemniscate mean $LM(x, y) \equiv LM$ of $x > 0$ and $y \geq 0$ is defined as follows

$$(1.3) \quad \left[LM(x, y)\right]^{-1/2} = \begin{cases} (x^2 - y^2)^{-1/4} \operatorname{arcsl}\left(1 - \frac{y^2}{x^2}\right)^{1/4}, & y < x, \\[2mm] (y^2 - x^2)^{-1/4} \operatorname{arcslh}\left(\frac{y^2}{x^2} - 1\right)^{1/4}, & x < y, \\[2mm] x^{-1/2}, & x = y \end{cases}$$

(see [5, (2.7)], [1, (8.5.7) and (8.5.8)]). Other symbols used in mathematical literature to denote this mean are L_{13} (see [5, p. 500]) and C_{13} (see [1, p. 259]).

This paper is devoted to the discussion of Gauss lemniscate functions and the lemniscatic mean, with emphasis on bounds and inequalities, and is organized as follows. In Sec. 2 we introduce and present some results on a family of special functions called the R-hypergeometric functions. They are used in the subsequent sections to represent Gauss functions arcsl, arcslh and the mean LM. Two additional lemniscate functions are introduced in Sec. 3. They are used in Sec. 6 to define four new means in terms of LM and other elementary means of two variables. Bounds for the lemniscatic mean are established in Sec. 4. Inequalities for the Gauss functions arcsl and arcslh and for the lemniscatic mean LM are obtained in Sec. 5. Therein we shall prove that the Schwab–Borchardt mean studied in [10, 12] is comparable with the lemniscatic mean. Sec. 6 deals with the inequalities for particular means derived from the lemniscatic mean. The Ky Fan inequalities for these means are established. Some remarks about the first lemniscate constant and one of the means introduced in this section are also included.

2. The R-hypergeometric functions

In the subsequent sections of this paper we shall use some special functions of several variables which belong to the family of the R-hypergeometric functions of a negative order $-\alpha$. For the reader's convenience we recall definition of this important class of functions. In what follows, the symbols $\mathbb{R}_+$, $\mathbb{R}_>$ and $\mathbb{C}_>$ will stand for the nonnega-

tive semi-axis, the set of positive real numbers and the set of complex numbers in the open right half-plane.

Let $b = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ and let $X = (x_1, \dots, x_n) \in \mathbb{C}^n_>$. For $\alpha > 0$ we define $\alpha' = b_1 + \dots + b_n - \alpha$ and assume that $\alpha' > 0$. Following [6, (6.8-6)] the R-hypergeometric function $R_{-\alpha}(b; X)$ of order $-\alpha$ with parameters b and variables X is defined by

$$(2.1) \qquad R_{-\alpha}(b; X) = \frac{1}{B(\alpha, \alpha')} \int_0^\infty t^{\alpha'-1} \prod_{i=1}^n (t + x_i)^{-b_i} \, dt,$$

where $B(\cdot, \cdot)$ stands for the beta function. For particular values of α and the parameters b the assumption $X \in \mathbb{C}^n_>$ can be relaxed to admit vectors X with at most one variable to be equal to 0.

In this paper we shall use three particular R-hypergeometric functions which are denoted by R_B, R_F and R_C. They are defined as follows:

$$(2.2) \quad R_B(x, y) = R_{-1/4}\left(\frac{3}{4}, \frac{1}{2}; x, y\right) = \frac{1}{4} \int_0^\infty (t+x)^{-3/4}(t+y)^{-1/2} \, dt,$$

where $x > 0$ and $y \geq 0$ (see [5, (3.14)]),

$$(2.3) \quad \begin{aligned} R_F(x, y, z) &= R_{-1/2}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z\right) = \\ &= \frac{1}{2} \int_0^\infty \left[(t + x)(t + y)(t + z)\right]^{-1/2} \, dt, \end{aligned}$$

where at most one of the variables x, y, z is 0 (see [6, (9.2-1)]) and

$$(2.4) \quad R_C(x, y) = R_{-1/2}\left(\frac{1}{2}, 1; x, y\right) = \frac{1}{2} \int_0^\infty (t + x)^{-1/2}(t + y)^{-1} \, dt,$$

where $x \geq 0$ and $y > 0$ (see [6, (6.9-14)]).

In [6] the author used the notation R_A for the function R_B. We have made this change because the letter A will be used in this paper to denote the arithmetic mean of two numbers. Function R_F is the incomplete symmetric integral of the first kind and R_C is the degenerate case of R_F, i.e., $R_C(x, y) = R_F(x, y, y)$ and can be expressed in terms of elementary transcendental functions (see [6, (6.9-15)]).

For later use, let us record Carlson's inequality [3, Th. 3]

$$(2.5) \qquad \left[R_p(b; X)\right]^{1/p} \leq \left[R_q(b; X)\right]^{1/q}$$

$(p, q \neq 0, \, p \leq q, \, X \in \mathbb{R}^n_>)$. Equality holds in (2.5) if either $p = q$ or

$\min(X) = \max(X)$.

It has been demonstrated in [12, Prop. 2.1] that for $a > 0$ $R_{-\alpha}(b;X)$ is logarithmically-convex (log-convex) in its variables, i.e., the inequality

$$(2.6) \qquad R_{-\alpha}\big(b; \lambda X + (1 - \lambda)Y\big) \leq \big[R_{-\alpha}(b; X)\big]^{\lambda}\big[R_{-\alpha}(b; Y)\big]^{1-\lambda}$$

holds true for $0 \leq \lambda \leq 1$ and $X, Y \in \mathbb{R}^n_>$.

3. Lemniscate functions arctl and arctlh

Gauss' functions arcsl and arcslh admit representations in terms of the R-hypergeometric function R_B . Carlson [5, (4.1)] has shown that

$$(3.1) \qquad\qquad \mathrm{arcsl}\, x = x\, R_B(1, 1 - x^4)$$

and

$$(3.2) \qquad\qquad \mathrm{arcslh}\, x = x\, R_B(1, 1 + x^4).$$

In Sec. 6 we study certain means which are represented by these two functions and another pair of lemniscate functions denoted by arctl and arctlh. We define

$$(3.3) \qquad\qquad \mathrm{arctl}\, x = x\, R_B(1 + x^4, 1)$$

and

$$(3.4) \qquad\qquad \mathrm{arctlh}\, x = x\, R_B(1 - x^4, 1), \qquad |x| < 1.$$

They are closely related to the Gauss functions. We have

Proposition 3.1. *The following formulas*

$$(3.5) \qquad\qquad \mathrm{arctl}\, x = \mathrm{arcsl}\left(\frac{x}{\sqrt[4]{1 + x^4}}\right)$$

and

$$(3.6) \qquad\qquad \mathrm{arctlh}\, x = \mathrm{arcslh}\left(\frac{x}{\sqrt[4]{1 - x^4}}\right)$$

are valid.

Proof. For the proof of (3.5) we employ the first and third members of (2.2) to obtain

$$R_B(1+x^4,1) = \frac{1}{4} \int_0^\infty (t+1+x^4)^{-3/4}(t+1)^{-1/2} \, dt.$$

A change of variable $(t+1)/(t+1+x^4) = 1 - u^4$ gives

$$R_B(1+x^4,1) = \frac{1}{x} \int_0^{x(1+x^4)^{-1/4}} (1-u^4)^{-1/2} \, du =$$

$$= \frac{1}{x} \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}} \right),$$

where in the last step we have used (1.1). This in conjunction with (3.3) gives the desired result. Formula (3.6) is established in a similar fashion. Making use of (2.2) and next introducing a new variable u, where $(t+1)/(t+1-x^4) = 1 + u^4$, and utilizing (1.2) we obtain

$$R_B(1-x^4,1) = \frac{1}{4} \int_0^\infty (t+1-x^4)^{-3/4}(t+1)^{-1/2} \, dt =$$

$$= \frac{1}{x} \int_0^{x(1-x^4)^{-1/4}} (1+u^4)^{-1/2} \, du =$$

$$= \frac{1}{x} \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}} \right).$$

The assertion now follows because of (3.4). $\Diamond$

Gauss functions can also be expressed in terms of the incomplete symmetric elliptic integral R_F:

(3.7) $$\operatorname{arcsl} x = x \, R_F(1-x^2, 1+x^2, 1)$$

(see [6, Ex. 8.3-7]) and

(3.8) $$\operatorname{arcslh} x = x \, R_F(1-ix^2, 1+ix^2, 1),$$

where the last result is a special case of Ex. 8.1-5 in [6] with $\alpha = 1/2$, $\beta = \delta = -1/2$, $A = C = 1$ and $B = -D = -i$. We omit further details.

In [7] the author has proposed an algorithm for numerical computation of the integral R_F. This algorithm allows complex values for the variables of R_F. Thus all four lemniscate functions can be evaluated numerically using (3.7), (3.8), (3.5) and (3.6).

4. Bounds for the lemniscatic mean

The purpose of this section is to establish bounds for the lemniscatic mean LM (see (1.3)). We shall utilize the following formula

(4.1) $$\left[LM(x,y)\right]^{-1/2} = R_B(x^2, y^2)$$

(see [5, (3.10) and (3.14)], [2, (3.22)]).

The lemniscatic mean is the iterative mean, i.e.,

(4.2) $$LM(x,y) = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n,$$

where

(4.3) $$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = (x_{n+1} x_n)^{1/2},$$

$n \geq 0$ (see [5, Lemma 1]).

Some elementary properties of the mean under discussion follow immediately from (2.2), (4.1) and (4.3). We have

(i) The lemniscatic mean is a homogeneous function of degree 1 in its variables, i.e., $LM(\lambda x, \lambda y) = \lambda LM(x,y)$ $(\lambda > 0)$.

(ii) $LM(x,y)$ increases with an increase in either x or y.

(iii) $LM(x,y) \neq LM(y,x)$ provided $x \neq y$.

(iv) $LM(x,y) = LM(A, (Ax)^{1/2})$, where $A = (x+y)/2$. The last property is the invariance property of the lemniscatic mean.

In the proof of the main result of this section we shall utilize the following.

Lemma 4.1. *Let $x > 0$ and $y \geq 0$ $(x \neq y)$. Then*
(4.4)
$$(x^3 y^2)^{1/5} < (A^4 x)^{1/5} < LM(x,y) < \left(\frac{3A^2 + 2Ax}{5}\right)^{1/2} < \frac{3x + 2y}{5},$$

where $A = (x+y)/2$ is the arithmetic mean of x and y.

Proof. The first inequality in (4.4) is an immediate consequence of the inequality for the arithmetic and geometric means while the last one can be established using elementary algebraic computations. For the proof of the second and third inequalities we need the following one

(4.5) $$\left(\frac{3A^2 + 2Ax}{5}\right)^{-1/4} < R_B(A^2, Ax) < (A^4 x)^{-1/10}$$

which follows easily from (2.15) in [4, Th. 2] applied to

$$R_B(A^2, Ax) = R_{-1/4}\left(\frac{3}{4}, \frac{1}{2}; A^2, Ax\right)$$

(see (2.2)). Applying (4.1) and the invariance property (iv) to the second member of (4.5) we obtain the desired result. ◊

It is easy to show, using (4.3), that the sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$ converge linearly to LM and also that $(y_n - x_n)(x_{n+1} - y_{n+1}) > 0$ holds for every $n \geq 0$. This implies that sequences in question are not monotonic ones. For instance, if $x < y$, then

$$x_0 < y_1 < x_2 < \cdots < LM < \cdots < y_2 < x_1 < y_0.$$

The speed of convergence of both sequences can be improved. For the sake of presentation let us introduce two sequences $\{g_n\}_0^\infty$ and $\{a_n\}_0^\infty$, where

(4.6) $$g_n = (x_n^3 y_n^2)^{1/5}$$

and

(4.7) $$a_n = \frac{3x_n + 2y_n}{5}$$

($n \geq 0$). Thus g_n and a_n are the weighted geometric and arithmetic means, with weights $3/5$ and $2/5$, of x_n and y_n.

We are in a position to prove the following.

Theorem 4.2. *Let $x > 0$ and $y \geq 0$. Then for every $n \geq 0$*

(4.8) $$g_n < LM(x, y) < a_n$$

and

(4.9) $$\lim_{n \to \infty} g_n = \lim_{n \to \infty} a_n = LM(x, y).$$

Moreover, the sequence $\{g_n\}_0^\infty$ is strictly increasing and $\{a_n\}_0^\infty$ is strictly decreasing.

Proof. Using (4.4) we have

$$(x^3 y^2)^{1/5} < LM(x, y) < \frac{3x + 2y}{5}.$$

Replacing x by x_n, y by y_n, and next using (4.6), (4.7) and the invariance property of the lemniscatic mean $LM(x_n, y_n) = LM(x, y)$ we obtain the inequalities (4.8). Assertion (4.9) follows immediately from (4.6), (4.7) and (4.2). For the proof of monotonicity of $\{g_n\}_0^\infty$ we use (4.6) and the last formula of (4.3) followed by application of the inequality for the arithmetic and geometric means to obtain

$$g_{n+1} = (x_{n+1}^3 y_{n+1}^2)^{1/5} = (x_{n+1}^4 x_n)^{1/5} > \left[(x_n y_n)^2 x_n\right]^{1/5} = g_n \,.$$

Similarly, making use of (4.7), applying the inequality of the arithmetic and geometric means to y_{n+1} followed by use of the third formula of (4.3) we obtain

$$a_{n+1} = \frac{3x_{n+1} + 2y_{n+1}}{5} < \frac{4x_{n+1} + x_n}{5} = \frac{3x_n + 2y_n}{5} = a_n \,.$$

The proof is complete. $\Diamond$

5. Inequalities involving Gauss functions and the lemniscatic mean

In this section we shall establish several inequalities involving functions arcsl and arcslh and the mean LM.

For the sake of notation, let $p_1 = (1 - x^4)^{1/2}$ and $p_2 = (1 + x^4)^{1/2}$. Making use of (3.1), (3.2) and (4.1) we obtain

$$(5.1) \qquad\qquad\qquad \text{arcsl}\, x = x\left[LM(1, p_1)\right]^{-1/2}$$

and

$$(5.2) \qquad\qquad\qquad \text{arcslh}\, x = x\left[LM(1, p_2)\right]^{-1/2}.$$

It follows from (5.1) and (5.2) that the functions $f_1(x)$ and $f_2(x)$, where

$$(5.3) \qquad\qquad f_1(x) = \frac{\text{arcsl}\, x}{x}\,, \qquad f_2(x) = \frac{\text{arcslh}\, x}{x}$$

are positive on their domains, equal unity when $x = 0$, and satisfy $f_i(-x) = f_i(x)$ $(i = 1, 2)$. Bounds for these functions are proven in the following.

Theorem 5.1. *Let x_n and y_n $(n \geq 0)$ be defined recursively using (4.3) with $x_0 = 1$ and $y_0 = p_1$ when $i = 1$ and $y_0 = p_2$ if $i = 2$. Then the inequalities*

$$(5.4) \qquad\qquad \left(\frac{5}{3x_n + 2y_n}\right)^{1/2} < f_i(x) < (x_n^3 y_n^2)^{-1/10}$$

$(x \neq 0)$ are valid for every $n \geq 0$.

Proof. It follows from (5.3), (5.1) and (5.2) that

$$(5.5) \qquad\qquad\qquad f_i(x) = \left[LM(1, p_i)\right]^{-1/2}.$$

Application of (4.8) to the right side of (5.5) followed by use of (4.6)

and (4.7) completes the proof of inequalities (5.4). ◊

Letting $x = 1$, $y = p_i$ in (4.4) and utilizing (5.5) we obtain the chain of inequalities

$$(5.6) \quad \left(\frac{5}{3 + 2p_i}\right)^{1/2} < \left(\frac{5}{3q_i^2 + 2q_i}\right)^{1/4} < f_i(x) < (q_i)^{-2/5} < (p_i)^{-1/5},$$

where $q_i = (1 + p_i)/2$ and $x \neq 0$.

More bounds for the function f_1 can be obtained by using some of the results in [9] and [8] applied to (3.7). We omit further details.

Another mean used in this paper is the Schwab–Borchardt mean $SB(x, y)$ where $x \geq 0$ and $y > 0$. It is defined by [2, (3.21)]

$$(5.7) \qquad\qquad SB(x, y) = \left[R_C(x^2, y^2)\right]^{-1}.$$

Other symbols used in mathematical literature for this mean are L_{14} (see [5, p. 498]) and C_{14} (see [1, (8.5.3)]). The mean under discussion is the iterative mean, i.e.,

$$SB(x, y) = \lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n,$$

where

$$(5.8) \quad u_0 = x, \quad v_0 = y, \quad u_{n+1} = \frac{u_n + v_n}{2}, \quad v_{n+1} = (u_{n+1}v_n)^{1/2},$$

$n \geq 0$ (see [5, (2.3)] and [1, p. 257]). It is known that $SB(x, y)$ admit representations in terms of the inverse circular and the inverse hyperbolic functions:

$$(5.9) \quad SB(x, y) = \frac{\sqrt{x^2 - y^2}}{\operatorname{arcsinh}\sqrt{x^2/y^2 - 1}} = \frac{\sqrt{x^2 - y^2}}{\operatorname{arctanh}\sqrt{1 - y^2/x^2}}, \quad y < x,$$

and

$$(5.10) \quad SB(x, y) = \frac{\sqrt{y^2 - x^2}}{\arcsin\sqrt{1 - x^2/y^2}} = \frac{\sqrt{y^2 - x^2}}{\arctan\sqrt{y^2/x^2 - 1}}, \quad x < y,$$

(see, e.g., [10, (1.3) and (1.2)]). This mean has been studied recently in [10] and [12]

We shall prove that the means LM and SB are comparable.

Theorem 5.2. *Let $x > 0$, $y > 0$ with $x \neq y$ and let $A = (x + y)/2$. Then the following inequalities*

$$(5.11) \qquad SB(x, y) < LM(y, x) < A < LM(x, y) < SB(y, x)$$

$(x > y)$ and

(5.12) $\left[x^2 y\, LM(y,x)\right]^{1/4} < LM(x,y)$

hold true. Inequalities (5.11) are reversed if $x < y$.

Proof. We shall establish inequalities (5.11) when $x > y$. For the proof of the first one we shall employ the following result

(5.13) $\operatorname{arcsinh}(t^2) \geq (\operatorname{arcslh} t)^2$

$(t \in \mathbb{R})$. In order to establish (5.13) let us introduce a function

$$g(t) = \operatorname{arcsinh}(t^2) - (\operatorname{arcslh} t)^2.$$

Using (1.2) we obtain

$$g(t) = \int_0^{t^2} (1+u^2)^{-1/2}\, du - \left(\int_0^t (1+u^4)^{-1/2}\, du\right)^2.$$

Differentiation gives

(5.14) $g'(t) = 2(1+t^4)^{-1/2}(t - \operatorname{arcslh} t) = 2(1+t^4)^{-1/2} h(t),$

where $h(t) = t - \operatorname{arcslh} t$. Since $h(0) = 0$ and $h'(t) = 1 - (1+t^4)^{-1/2} \geq \; \geq 0$, $h(t) \geq 0$. This in conjunction with (5.14) and $g(0) = 0$ implies that the function $g(t)$ is nonnegative on $\mathbb{R}$. Hence (5.13) follows.

We are in a position to prove the first inequality in (5.11). To this aim we substitute $t = (x^2/y^2 - 1)^{1/4}$ in (5.13). Next we raise both sides to the power of -1 and multiply both sides by $(x^2 - y^2)^{1/2}$ to obtain, using (5.9) and (1.3), the desired result. For the proof of the second inequality in (5.11) we utilize the invariance property (iv) of the lemniscatic mean. We have
(5.15)
$$LM(y,x) = LM\left(A, (Ay)^{1/2}\right) < LM(A,A) < LM\left(A, (Ax)^{1/2}\right) =$$
$$= LM(x,y)$$

where both inequalities in (5.15) are consequences of the assumption that $y < x$ and the monotonicity property (ii). In the proof of the fourth inequality in (5.11) we shall use the following one

(5.16) $(\operatorname{arcsl} t)^2 \geq (\operatorname{arcsin} t)^2, \qquad 0 \leq t \leq 1.$

In order to establish (5.16) we shall prove first that the function

$$g(t) = (\operatorname{arcsl} t)^2 - (\operatorname{arcsin} t)^2$$

is nonnegative for $0 \leq t \leq 1$. Using (1.1) we have

$$g(t) = \left(\int_0^t (1 - u^4)^{1/2} \, du \right)^2 - \int_0^{t^2} (1 - u^2)^{-1/2} \, du.$$

Clearly $g(0) = 0$ and

$$g'(t) = 2(1 - t^4)^{-1/2}(\operatorname{arcsl} t - t) \geq 0.$$

The last inequality follows from the inequality $\operatorname{arcsl} t \geq t$ $(0 \leq t \leq 1)$ which in turn is obtained from (1.1) because $(1 - t^4)^{-1/2} \geq 1$ in the stated domain. Since $g(t) \geq 0$ on $[0, 1]$, the inequality (5.16) follows. To complete the proof of (5.11) we let $t = (1 - y^2/x^2)^{1/4}$ in (5.16), take reciprocals and multiply both sides by $(x^2 - y^2)^{1/2}$. Application of (1.3) and (5.10) gives the desired result.

In the proof of (5.12) we shall utilize the following result

$$(5.17) \qquad R_{-1}\left(\frac{3}{4}, \frac{1}{2}; x^2, y^2 \right) = \left[x^2 y \, LM(y, x) \right]^{-1/2}.$$

This can be established as follows. We use the transformation [6, (5.9-21)]

$$R_{-\alpha}(\beta, \beta'; x, y) = x^{\beta' - \alpha} y^{\beta - \alpha} R_{-\alpha'}(\beta', \beta; x, y)$$

which holds true provided $\alpha + \alpha' = \beta + \beta'$. Letting $\alpha = 1$, $\alpha' = 1/4$, $\beta = 3/4$, $\beta' = 1/2$, $x := x^2$ and $y := y^2$ we obtain

$$R_{-1}\left(\frac{3}{4}, \frac{1}{2}; x^2, y^2 \right) = (x^2 y)^{-1/2} R_{-1/4}\left(\frac{1}{2}, \frac{3}{4}; x^2, y^2 \right) =$$

$$= (x^2 y)^{-1/2} R_B(y^2, x^2) = \left[x^2 y \, LM(y, x) \right]^{-1/2}$$

where in the last step we have used (2.2) and (4.1). Making use of (5.17) and applying (2.5) we obtain

$$\left[x^2 y \, LM(y, x) \right]^{1/2} = \left[R_{-1}\left(\tfrac{3}{4}, \tfrac{1}{2}; x^2, y^2 \right) \right]^{-1} < \left[R_{-1/4}\left(\tfrac{3}{4}, \tfrac{1}{2}; x^2, y^2 \right) \right]^{-4} =$$

$$= \left[R_B(x^2, y^2) \right]^{-4} = \left[LM(x, y) \right]^2.$$

The proof is complete. $\Diamond$

We close this section with the following.

Theorem 5.3. *The following inequalities*

$$(5.18) \quad LM(x_1, x_2)LM(y_1, y_2) \leq \left[LM\left(\sqrt{\frac{x_1^2 + y_1^2}{2}}, \sqrt{\frac{x_2^2 + y_2^2}{2}} \right) \right]^2$$

$(x_1, y_1 > 0, \ x_2, y_2 \geq 0),$

(5.19)
$$\left(\frac{\mathrm{arcsl}\, z}{z}\right)^2 \le \frac{\mathrm{arcsl}\, x}{x}\, \frac{\mathrm{arcsl}\, y}{y}$$

$(|x| \le 1,\ |y| \le 1)$ *and*

(5.20)
$$\left(\frac{\mathrm{arcslh}\, z}{z}\right)^2 \le \frac{\mathrm{arcslh}\, x}{x}\, \frac{\mathrm{arcslh}\, y}{y}$$

$(x, y \in \mathbb{R})$, *where* $z^4 = (x^4 + y^4)/2$, *are valid.*

Proof. We use (2.6) with $\alpha = \lambda = 1/2$, $b = (3/4, 1/2)$, $X = (x_1^2, x_2^2)$, $Y = (y_1^2, y_2^2)$ and the first two members of (2.2) to obtain

$$\left[R_B\left(\frac{x_1^2 + y_1^2}{2}, \frac{x_2^2 + y_2^2}{2}\right)\right]^2 \le R_B(x_1^2, x_2^2) R_B(y_1^2, y_2^2).$$

Application of (4.1) completes the proof of (5.18). Inequality (5.19) follows from (5.18) by letting $x_1 = y_1 = 1$, $x_2 = (1 - x^4)^{1/2}$ and $y_2 = (1 - y^4)^{1/2}$. Making use of (3.1) we obtain the desired result. Inequality (5.20) can be established in an analogous manner. We use (5.18) again with $x_1 = y_1 = 1$, $x_2 = (1 + x^4)^{1/2}$, $y_2 = (1 + y^4)^{1/2}$ and next apply (3.2). The proof is complete. $\Diamond$

6. New means derived from the lemniscatic mean

In this section we discuss four new means of two variables. They are derived from the lemniscatic mean by using the geometric, arithmetic and the root-mean-square means. The Ky Fan and Ky Fan type inequalities involving these means are established.

For the sake of presentation, we shall rewrite formula (1.3) for LM using two lemniscate functions defined in Sec. 3. It follows from the first part of (1.3) and (3.5), written in the form

$$\mathrm{arcsl}\, x = \mathrm{arctl}\left(\frac{t}{(1 - t^4)^{1/4}}\right) \quad |t| < 1,$$

that

(6.1)
$$LM(x, y) = \frac{\sqrt{x^2 - y^2}}{(\mathrm{arcsl}\, \sqrt[4]{1 - y^2/x^2})^2} =$$
$$= \frac{\sqrt{x^2 - y^2}}{(\mathrm{arctl}\, \sqrt[4]{x^2/y^2 - 1})^2}, \quad y < x.$$

Similarly, using the second part of (1.3) together with (3.6), written as

$$\operatorname{arcslh} t = \operatorname{arctlh}\left(\frac{t}{(1+t^4)^{1/4}}\right) \quad (t \in \mathbb{R}),$$

we obtain

(6.2)
$$LM(x,y) = \frac{\sqrt{y^2 - x^2}}{(\operatorname{arcslh}\sqrt[4]{y^2/x^2 - 1})^2} =$$
$$= \frac{\sqrt{y^2 - x^2}}{(\operatorname{arctlh}\sqrt[4]{1 - x^2/y^2})^2}, \quad x < y.$$

In what follows, the symbols G, A and Q will stand for the geometric, arithmetic and the root-mean-square means of x and y, respectively, i.e.,

(6.3)
$$G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad Q = \sqrt{\frac{x^2 + y^2}{2}}.$$

The new means derived from the lemniscatic mean are defined as follows

(6.4)
$$U(x,y) \equiv U = LM(G,A), \qquad V(x,y) \equiv V = LM(A,G),$$
$$R(x,y) \equiv R = LM(A,Q), \qquad S(x,y) \equiv S = LM(Q,A).$$

It is easy to see that they are symmetric in x and y and homogeneous of degree 1 in their variables. Without loss of generality, we may assume that $x > y > 0$. Also, let

(6.5)
$$z = \frac{x-y}{x+y}.$$

Application of (6.1) and (6.2) to (6.4) gives explicit formulas for U, V, R and S in terms of four lemniscate functions

(6.6)
$$U(x,y) = \frac{x-y}{2(\operatorname{arctlh}\sqrt{z})^2}, \qquad V(x,y) = \frac{x-y}{2(\operatorname{arcsl}\sqrt{z})^2},$$

(6.7)
$$R(x,y) = \frac{x-y}{2(\operatorname{arcslh}\sqrt{z})^2}, \qquad S(x,y) = \frac{x-y}{2(\operatorname{arctl}\sqrt{z})^2}.$$

For later use, let us recall definitions of four means, studied recently in [10] and [12], and derived from the Schwab–Borchardt mean as follows

$$L(x,y) = SB(A,G) = \frac{x-y}{2\operatorname{arctanh} z} = \frac{x-y}{\ln x - \ln y},$$

$$P(x,y) = SB(G,A) = \frac{x-y}{2\arcsin z},$$

(6.8)

$$M(x,y) = SB(Q,A) = \frac{x-y}{2\operatorname{arcsinh} z},$$

$$T(x,y) = SB(A,Q) = \frac{x-y}{2\arctan z}$$

(see [10, (2.3)–(2.6)]). Here L is the well-known logarithmic mean and the means P and T have been introduced by Seiffert in [14] and [15], respectively. Comparison of (6.6) and (6.7) with (6.8) shows that U, V, R and S can be regarded as the lemniscate counterparts of L, P, M and T. It has been shown [10, (2.10)] that the latter means are comparable and they satisfy the inequalities

(6.9) $$L < P < A < M < T.$$

Using the well-known inequality $G < A < Q$, (6.4), (5.11), (6.9) and the monotonicity property (ii) of LM one easily obtains

(6.10) $$L < U < V < P < A < M < R < S < T.$$

Thus the means introduced in this section are comparable.

Several inequalities involving these means can be easily obtained using some results established in Sec. 5.

We have

Proposition 6.1. *The following inequalities*

(6.11) $$A^2 GU < V^4, \quad G^2 AV < U^4,$$

(6.12) $$A^2 QS < R^4, \quad Q^2 AR < S^4,$$

and

(6.13) $$US < A^2, \quad\quad VR < A^2$$

hold true.

Proof. Inequalities (6.11) follow from (5.12) by letting $x := A$, $y := G$ and $x := G$ and $y := A$, respectively, followed by use of (6.4). For the proof of inequalities (6.12) we employ (5.12) again with $x := A$, $y := Q$ and $x := Q$ and $y := A$, respectively and utilize (6.4). In order to establish the first inequality in (6.13) we substitute $x_1 = G$, $y_1 = Q$ and $x_2 = y_2 = A$ in (5.18) and utilize (6.4). The second inequality in (6.13) can be proven in a similar fashion. We let $x_1 = y_1 = A$, $x_2 = G$ and $y_2 = Q$ in (5.18) and next apply (6.4). ◊

In the proof of the next result we shall utilize a result of Vamana-murthy and Vuorinen [17, Lemma 1.1]:

Lemma A. *Let f and g be continuous functions on $[c,d]$. Assume that they are differentiable and $g'(t) \neq 0$ on (c,d). If f'/g' is strictly increasing (decreasing) on (c,d), then so are*

$$\frac{f(t)-f(c)}{g(t)-g(c)} \quad \text{and} \quad \frac{f(t)-f(d)}{g(t)-g(d)}.$$

We shall now deal the the Ky Fan and the Ky Fan type inequalities for the means under discussion. In what follows, we will assume that $0 < x,\, y \leq 1/2\ (x \neq y)$. With $x' = 1 - x$ and $y' = 1 - y$ we shall write L' for $L(x',y')$, U' for $U(x',y')$, etc. It has been shown in [10, Prop. 2.2] that

$$(6.14) \qquad \frac{L}{L'} < \frac{P}{P'} < \frac{A}{A'} < \frac{M}{M'} < \frac{T}{T'}.$$

Refinements of these inequalities are established in the following.

Theorem 6.2. *We have*

$$(6.15) \qquad \frac{L}{L'} < \frac{U}{U'} < \frac{V}{V'} < \frac{P}{P'} < \frac{A}{A'} < \frac{M}{M'} < \frac{R}{R'} < \frac{S}{S'} < \frac{T}{T'}.$$

Proof. It follows from (6.14) that we need to establish the first three and the last three inequalities in (6.15). For the proof of the first one we define a function $\phi = L/U$. Using (6.6), (6.8), (3.4) and the known formula [6, Ex. 6.9–16]

$$\operatorname{arctanh} z = z\, R_C(1, 1 - z^2)$$

we obtain

$$(6.16) \qquad \phi(z) = \frac{(\operatorname{arctlh} \sqrt{z})^2}{\operatorname{arctanh} z} = \frac{R_B(1 - z^2, 1)^2}{R_C(1, 1 - z^2)},$$

where z is defined in (6.5). It follows from (6.16) that $\phi(0) = 1$ and $\phi(-z) = \phi(z)$. Our goal is to demonstrate that $\phi(z)$ is strictly decreasing on $[0, 1)$. To this aim we shall utilize Lemma A with $f(t) = (\operatorname{arctlh} \sqrt{t})^2$ and $g(t) = \operatorname{arctanh} t$. Using a differentiation formula $(\operatorname{arctlh} t)' = (1 - t^4)^{-3/4}$, which follows from (3.6) and (1.2), we have

$$f'(t) = \frac{\operatorname{arctlh} \sqrt{t}}{\sqrt{t}} (1 - t^2)^{-3/4} = R_B(1 - t^2, 1)(1 - t^2)^{-3/4},$$

where in the last step we have used (3.4). Hence

$$\frac{f'(t)}{g'(t)} = R_B(1 - t^2, 1)(1 - t^2)^{1/4} = R_B\left(1, \frac{1}{1 - t^2}\right)$$

because R_B is a homogeneous function of degree $-1/4$ in its variables (see (2.2)). If t increases from 0 to 1, then $R_B\left(1, 1/(1 - t^2)\right)$ decreases. Since $f(0) = g(0) = 0$, Lemma A implies that the function $\phi(z) = f(z)/$ $/g(z)$ is strictly decreasing on $[0, 1)$. Now let $z' = (x' - y')/(x' + y')$. We have [10, (2.13)]

(6.17) $$0 < |z'| < |z| < 1, \qquad zz' < 0.$$

Assume that $y < x \le 1/2$. It follows from (6.17) that $0 < -z' < z < < 1$. This in turn implies that $\phi(z) < \phi(-z')$ or what is the same as $L/U < L'/U'$. One can prove that the last inequality is also valid if $x < y \le 1/2$. This completes the proof of the first inequality in (6.15). The second inequality in (6.15) can be established in a similar manner. We introduce a function $\phi = U/V$. Application of (6.6), (3.1) and (3.4) gives

$$\phi(z) = \left(\frac{\text{arcsl}\sqrt{z}}{\text{arctlh}\sqrt{z}}\right)^2 = \left(\frac{R_B(1, 1 - z^2)}{R_B(1 - z^2, 1)}\right)^2.$$

It follows that $\phi(0) = 1$ and $\phi(-z) = \phi(z)$. We shall prove that $\phi(z)$ is a decreasing function on $[0, 1)$. To this aim we define two functions $f(t) = \text{arcsl}\sqrt{t}$ and $g(t) = \text{arctlh}\sqrt{t}$. Making use of (1.1), (3.6) and (1.2) we obtain

$$\frac{f'(t)}{g'(t)} = (1 - t^2)^{1/2}.$$

Lemma A implies that the function $f(t)/g(t)$ is strictly decreasing on $[0, 1)$ and so is the function $\phi = (f/g)^2$. The remaining part of the proof of the second inequality in (6.15) goes along the lines introduced in the proof of the first inequality in (6.15). The remaining inequalities in the chain (6.15) can be established in a similar way. We omit further details. The proof is complete. $\Diamond$

Before we will state the next result of this paper, let us recall the so-called Ky Fan rules, which are utilized when proving the Ky Fan type inequalities, and they are contained in the following [11, Lemma 2.1].

Lemma B. *Let a, a', b and b' be positive numbers.*

(i) *If $a < b$ and $a/a' < b/b' < 1$ or if $1 < a/a' < b/b'$, then*

$$\frac{1}{a'} - \frac{1}{a} < \frac{1}{b'} - \frac{1}{b}.$$

(ii) *If $a' < b'$ and $a/a' < b/b'$, then*

$$aa' < bb'.$$

Theorem 6.3. *Let $0 < x, y \leq 1/2$ ($x \neq y$). Then*

$$\frac{1}{U'} - \frac{1}{U} < \frac{1}{V'} - \frac{1}{V} < \frac{1}{A'} - \frac{1}{A} < \frac{1}{R'} - \frac{1}{R} < \frac{1}{S'} - \frac{1}{S}$$

and

$$UU' < VV' < AA' < RR' < SS'.$$

Proof. Apply Lemma B using (6.10) and (6.15). $\diamond$

We close this section with a result about the first lemniscate constant, which we denote by a. It is defined by

$$a = \mathrm{arcsl}(1)$$

(see [16, p. 14]). We shall show that the constant in question can be expressed in terms of the mean V.

Proposition 6.4. *We have*

(6.18) $$a = \bigl[V(0,2)\bigr]^{-1/2}.$$

Proof. It follows from (1.3), (3.7) and (3.8) that

$$\bigl[LM(x,y)\bigr]^{-1/2} = R_F\bigl(x - \sqrt{x^2 - y^2}, x + \sqrt{x^2 - y^2}, x\bigr).$$

Letting $x := A$ and $y := G$ and next using (6.4) we obtain

$$\bigl[V(x,y)\bigr]^{-1/2} = R_F(x, y, A).$$

Formula (6.18) now follows by letting $x = 0$ and $y = 2$ followed by application of (3.7) with $x = 1$. $\diamond$

The elliptic integral $R_F(x, y, A)$ is also called the general case of the first lemniscate constant.

References

[1] BORWEIN, J. M. and BORWEIN, P. B.: Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity, Wiley, New York, 1987.

[2] BRENNER, J. L. and CARLSON, B. C.: Homogeneous mean values: Weights and asymptotics, *J. Math. Anal. Appl.* **123** (1987), 265–280.

[3] CARLSON, B. C.: A hypergeometric mean value, *Proc. Amer. Math. Soc.* **16** (1965), 759–766.

[4] CARLSON, B. C.: Some inequalities for hypergeometric functions, *Proc. Amer. Math. Soc.* **17** (1966), 32–39.

[5] CARLSON, B. C.: Algorithms involving arithmetic and geometric means, *Amer. Math. Monthly* **78** (1971), 496–505.

[6] CARLSON, B.C.: Special Functions of Applied Mathematics, Academic Press, New York, 1977.

[7] CARLSON, B. C.: Numerical computation of real and complex elliptic integrals, *Numer. Algorithms* **10** (1995), 13–26.

[8] KAZI, H. and NEUMAN, E.: Inequalities and bounds for elliptic integrals, submitted.

[9] NEUMAN, E.: Bounds for symmetric elliptic integrals, *J. Approx. Theory* **122** (2003), 249–259.

[10] NEUMAN, E. and SÁNDOR, J.: On the Schwab–Borchardt mean, *Math. Pannonica* **14** (2003), 253–266.

[11] NEUMAN, E. and SÁNDOR, J.: On the Ky Fan inequality and related inequalities II, *Bull. Austral. Math. Soc.* **72** (2005), 87–107.

[12] NEUMAN, E. and SÁNDOR, J.: On the Schwab–Borchardt mean II, *Math. Pannonica* **17** (2006), 49–59.

[13] SIEGEL, C. L.: Topics in Complex Function Theory, vol. 1, Wiley, New York, 1969.

[14] SEIFFERT, H.-J.: Problem 887, *Nieuw. Arch. Wisk.* **11** (1993), 176.

[15] SEIFFERT, H.-J.: Aufgabe 16, *Wurzel* **29** (1995), 87.

[16] TODD, J.: The lemniscate constants, *Comm. ACM* **18** (1975), 14–19. For corrigendum see same journal **18** (1975), 462.

[17] VAMANAMURTHY, M. K. and VUORINEN, M.: Inequalities for means, *J. Math. Anal. Appl.* **183** (1994), 155–166.