Mathematica Pannonica
18/1 (2007), 63–76

A NOTE ON SUBSEMIGROUPS OF $\mathbf{Sl}(\boldsymbol{n}, \mathbb{R})$

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Received: May 2006

MSC 2000: 17 B 20, 22 E 15, 22 E 46, 22 E 60

Keywords: Parabolic subgroup, root system, semisimple Lie algebra, semisimple Lie group, the special linear group $\mathrm{Sl}(n, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(n, \mathbb{R})$.

Abstract: Let $\mathfrak{g} = \mathfrak{k}\mathfrak{a}\mathfrak{n}$ be the Iwasawa decomposition of the semisimple real Lie algebra $\mathfrak{g}$, G a Lie group with Lie algebra $\mathfrak{g}$, $G = KAN$ the corresponding Iwasawa decomposition on the group level, Φ the root system of the pair $(\mathfrak{g}, \mathfrak{a})$, and Π the base of Φ that corresponds to $\mathfrak{n}$. In the structure theory of semisimple Lie groups one attaches to every subset Θ of Π a a parabolic subgroup $P(\Theta)$ of G, and a semisimple Lie subgroup $G(\Theta)$ of $P(\Theta)$. We describe in terms of set of matrices, for every $\Theta \subseteq \Pi$, the structure of the subsemigroups of $P(\Theta)$ which contain both N and $G(\Theta)$ in the case of the special linear group $G = \mathrm{Sl}(n, \mathbb{R})$.

1. Prerequisites

1.1. Actions of semigroups on ordered sets. Let S be a monoid and $(X, \leq)$ be an ordered set. A function $\varphi\colon S \times X \to X$ is called an action of S on $(X, \leq)$ if

(i) $\varphi(s_1, \varphi(s_2, x)) = \varphi(s_1 s_2, x)$, for every $s_1, s_2 \in S$, and every $x \in X$,
(ii) $\varphi(1_S, s) = s$, for every $s \in S$,
(iii) $\varphi(s, x) \leq \varphi(s, y)$, whenever $x \leq y$ in X and $s \in S$.

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We denote by $s \cdot x := \varphi(s, x)$. For every submonoid M of S and every $x \in X$ let

$$M_x := \{s \in M \mid s \cdot x \leq x\}.$$

It is clear that M_x is a submonoid of S whose group of units $H(M_x)$ is a subset of the stabilizer of x, i.e., $H(M_x) \subseteq \{s \in S \mid s \cdot x = x\}$.

1.2. Notations. Let n be a natural number. In what follows S will be the semigroup $\mathrm{End}(\mathbb{R}^n)$ of endomorphisms (linear maps) $\mathbb{R}^n \to \mathbb{R}^n$ (endowed with the composition of functions), $(X, \leq) = (\mathcal{S}(\mathbb{R}^n), \subseteq)$, where $(\mathcal{S}(\mathbb{R}^n), \subseteq)$ is the lattice of all vector subspaces of $\mathbb{R}^n$ augmented by the empty set and ordered by inclusion, and the action of $\mathrm{End}(\mathbb{R}^n)$ on $\mathcal{S}(\mathbb{R}^n)$ is the natural action

$$(f, V) \in \mathrm{End}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longmapsto f(V) \in \mathcal{S}(\mathbb{R}^n).$$

The $\mathbb{R}$ vector space $\mathrm{End}(\mathbb{R}^n)$ endowed with the Lie bracket $[f, g] = f \circ \circ g - g \circ f$ is a Lie algebra and is denoted with $\mathfrak{gl}(n, \mathbb{R})$. The general linear group $\mathrm{Gl}(n, \mathbb{R})$ of automorphisms (bijective endomorphisms) of $\mathbb{R}^n$ is a Lie group with Lie algebra $\mathfrak{gl}(n, \mathbb{R})$.

For $V \in \mathcal{S}(\mathbb{R}^n)$ denote for simplicity

$$\mathfrak{gl}_V := \mathrm{End}(\mathbb{R}^n)_V, \quad \mathrm{Gl}_V := \mathrm{Gl}(n, \mathbb{R})_V.$$

Observe that Gl_V is the stabilizer of V in S.

If G is a closed subgroup of $\mathrm{Gl}(n, \mathbb{R})$ then $L(G) \subseteq \mathfrak{gl}(n, \mathbb{R})$ denotes the Lie algebra of G. For a subspace $\mathfrak{h}$ of $\mathfrak{gl}(n, \mathbb{R})$ let $N_G(\mathfrak{h}) = \{g \in \in G \mid \mathrm{Ad}(g)(\mathfrak{h}) \subseteq \mathfrak{h}\}$ be the normalizer of $\mathfrak{h}$ in G. ($\mathrm{Ad}\colon G \to \mathrm{Aut}(L(G))$ denotes the adjoint representation of G.) Similarly, if $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ then $N_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

1.3. Lemma. *Let G be a closed subgroup of $\mathrm{Gl}(n, \mathbb{R})$ with Lie algebra $L(G) = \mathfrak{g}$, and consider $V \in \mathcal{S}(\mathbb{R}^n)$. Then G_V is a closed subgroup of $\mathrm{Gl}(n, \mathbb{R})$ with $L(G_V) = \mathfrak{g}_V$.*

Proof. It is clear that G_V is a closed subgroup of $\mathrm{Gl}(n, \mathbb{R})$. The equality $L(\mathrm{Gl}_V) = \mathfrak{gl}_V$ is a consequence of the properties of the exponential function. (Recall that $f(v) = \lim_{t \to 0} \frac{e^{tf}(v) - v}{t}$, for every $f \in \mathfrak{gl}(n, \mathbb{R})$ and $v \in \mathbb{R}^n$.) Since $G_V = G \cap \mathrm{Gl}_V$ and $L(G_V) = L(G) \cap L(\mathfrak{gl}_V)$ the assertion follows. $\Diamond$

1.4. Further notations. Let $\{e^1, \ldots, e^n\}$ be the canonical base of $\mathbb{R}^n$. For every natural number $k \in \{1, \ldots, n-1\}$ put

$$V_k := \mathrm{span}\{e^1, \ldots, e^k\}.$$

For a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n, \mathbb{R})$, a closed subgroup G of $\mathrm{Gl}(n, \mathbb{R})$, and a nonempty subset $I \subseteq \{1, \ldots, n-1\}$ denote by

$$\mathfrak{g}_I := \bigcap_{i \in I} \mathfrak{g}_{V_i}, \quad G_I := \bigcap_{i \in I} G_{V_i}.$$

If $I = \emptyset$ put $\mathfrak{g}_I := \mathfrak{g}$ and $G_I = G$. For simplicity let again

$$\mathfrak{gl}_I := \bigcap_{i \in I} \mathfrak{gl}_{V_i}, \quad \mathrm{Gl}_I := \bigcap_{i \in I} \mathrm{Gl}_{V_i}.$$

1.5. Remark. If $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n-1$ then we recall that the nested sequence of subspaces

$$\{0\} \subset V_{i_1} \subset \ldots \subset V_{i_k} \subset \mathbb{R}^n$$

is called a *flag* in $\mathbb{R}^n$. Hence Gl_I can be seen as the stabilizer of the above flag.

1.6. Corollary. *Let G be a closed subgroup of* $\mathrm{Gl}(n, \mathbb{R})$ *with Lie algebra* $\mathfrak{g}$, *and $I \subseteq \{1, \ldots, n-1\}$. Then G_I is a closed subgroup of* $\mathrm{Gl}(n, \mathbb{R})$ *with $L(G_I) = \mathfrak{g}_I$.*

Proof. The assertion follows from 1.3 taking into account the fact that the Lie algebra of the intersection of a family of closed subgroups of $\mathrm{Gl}(n, \mathbb{R})$ is the intersection of the family of the Lie algebras of these subgroups. $\Diamond$

1.7. Convention. Throughout this paper we identify every element of $\mathrm{End}(\mathbb{R}^n)$ with its matrix relative to the canonical base of $\mathbb{R}^n$. Thus, according to the context we are working within, we regard an element of $\mathrm{End}(\mathbb{R}^n)$ either as a linear map or as a $n \times n$ matrix with real entries.

1.8. Remarks. 1) For every $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define the matrix $E_{ij} \in \mathfrak{gl}(n, \mathbb{R})$ be 1 in the $(i,j)^{\text{th}}$ place and 0 elsewhere. If $i > j$ and $k \in \{1, \ldots, n-1\}$ then it follows easily that $E_{ij} \in \mathfrak{gl}_{V_k}$ if and only if $k < j$ or $k \geq i$.

2) If $I = \emptyset$ then $\mathfrak{gl}_I = \mathfrak{gl}(n, \mathbb{R})$, and if $I = \{1, \ldots, n-1\}$ then $\mathfrak{gl}_I$ consists of all upper triangular matrices. If $I \subseteq \{1, \ldots, n-1\}$ then clearly every upper triangular matrix belongs to $\mathfrak{gl}_I$.

3) Let $I = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n-1$. Put $i_0 = 0$ and $i_{k+1} = n$. Then every element $X \in \mathfrak{gl}_I$ [$X \in \mathrm{Gl}_I$] is a matrix of the following type: Along the main diagonal there are $k+1$ block matrices, denoted by $X_1, \ldots, X_{k+1}$, such that for every $j \in \{1, \ldots, k+1\}$ the matrix X_j is an element of $\mathfrak{gl}(i_j - i_{j-1}, \mathbb{R})$ [$\mathrm{Gl}(i_j - i_{j-1}, \mathbb{R})$]. [Note that the determinant of X is the product of the determinants of these block matrices along the main diagonal. Thus $\det X \neq 0$ if and only if $\det X_j \neq 0$, for every $j \in \{1, \ldots, k+1\}$.] The entries above these blocks are reals, and those below the blocks are all equal to zero. Hence

$$X = \begin{pmatrix} X_1 & * & * & \dots & * \\ 0 & X_2 & * & \dots & * \\ 0 & 0 & X_3 & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & X_{k+1} \end{pmatrix},$$

where the stars $*$ stay for real matrices of suitable dimension. Define the map

$$\mathrm{pr}\colon \mathrm{Gl}_I \to \prod_{j=1}^{k+1} \mathrm{Gl}(i_j - i_{j-1}, \mathbb{R}) \quad \text{by} \quad \mathrm{pr}(X) = (X_1, \dots, X_{k+1}),$$

where $X \in \mathrm{Gl}_I$ is a matrix of the type described above. It is clear that pr is a group homomorphism.

For the next proposition we recall that $\mathrm{Sl}(n, \mathbb{R})$ is the special linear group of $n \times n$ matrices with real entries and with determinant 1. Its Lie algebra is $\mathfrak{sl}(n, \mathbb{R}) := \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{tr}(X) = 0\}$.

1.9. Proposition. *Let $G \in \{\mathrm{Gl}(n, \mathbb{R}), \mathrm{Sl}(n, \mathbb{R})\}$ and $\mathfrak{g} := L(G)$. If $I \subseteq \{1, \dots, n-1\}$ then the following assertions hold:*

a) *$\mathfrak{g}_I$ is a self-normalizing Lie subalgebra of $\mathfrak{g}$, i.e., $N_{\mathfrak{g}}(\mathfrak{g}_I) = \mathfrak{g}_I$.*

b) *$G_I = N_G(\mathfrak{g}_I)$.*

Proof. The assertions trivially hold if $I = \emptyset$. Assume that $I \neq \emptyset$.

a) We already know from Cor. 1.6 that $\mathfrak{g}_I$ is a Lie subalgebra of $\mathfrak{g}$. Suppose now that there is an endomorphism $f \in N_{\mathfrak{g}}(\mathfrak{g}_I) \setminus \mathfrak{g}_I$. Then we find indices $k, j \in I$ with $k < j$ such that $f(e^k) \notin V_j$. Let $f(e^k) = = t_1 e^1 + \dots + t_n e^n$ with $t_1, \dots, t_n \in \mathbb{R}$ According to the choice of k there exists an index $j_0 > j$ such that $t_{j_0} \neq 0$. Define $g \in \mathrm{End}(\mathbb{R}^n)$ by

$$g(e^\ell) = \begin{cases} 0_n & \text{if } \ell \in \{1, \dots, n\} \setminus \{k, j_0\} \\ -e^\ell & \text{if } \ell = k \\ e^\ell & \text{if } \ell = j_0. \end{cases}$$

It is clear that $g \in \mathfrak{sl}(n, \mathbb{R})_I$. Also,

$$[f, g](e^k) = f(g(e^k)) - g(f(e^k)) =$$
$$= -f(e^k) + t_k e^k - t_{j_0} e^{j_0} = -2t_{j_0} e^{j_0} + \sum_{i \neq k, j_0} t_i e^i \notin V_j,$$

which contradicts the fact that $f \in N_{\mathfrak{g}}(\mathfrak{g}_I)$. Hence $\mathfrak{g}_I$ is self-normalizing.

b) We argue again by contradiction. Suppose that there is an automorphism $f \in N_G(\mathfrak{g}_I) \setminus G_I$. Then we find an index $j \in I$ and an element $v \in V_j$ such that $f(v) \notin V_j$. Let $f(v) = t_1 e^1 + \cdots + t_n e^n$, $t_1, \ldots, t_n \in \mathbb{R}$, and $j_0 > j$ be so that $t_{j_0} \neq 0$. Since f is an automorphism, the condition $f(V_j) \not\subseteq V_j$ implies that $V_j \setminus f(V_j) \neq \emptyset$. Fix a vector $w \in V_j \setminus f(V_j)$, and define $h \in \mathrm{End}(\mathbb{R}^n)$ by

$$h(e^k) = \begin{cases} 0_n & \text{if } k \in \{1, \ldots, n\} \setminus \{j_0\} \\ w & \text{if } k = j_0 \end{cases}.$$

Then $h \in \mathfrak{sl}(n,\mathbb{R})_I$ and $(f^{-1} \circ h \circ f)(v) = t_{j_0} f^{-1}(w) \notin V_j$, hence $f \notin N_G(\mathfrak{g}_I)$. The contradiction we have obtained yields the asserted equality. $\Diamond$

2. Semisimple Lie algebras and semisimple Lie groups

In this section we recall some basic facts concerning the structure of semisimple Lie groups and their Lie algebras (for details we refer, for ex., to [1], [2] or [3]).

2.1. Some notation. Throughout this section $\mathfrak{g}$ will denote a semisimple real Lie algebra and G a connected Lie group with finite center having $\mathfrak{g}$ as Lie algebra. As usual, $\kappa\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ denotes the Killing form of $\mathfrak{g}$, and $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$. Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be a Cartan involution with the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, where $\mathfrak{k}$ and $\mathfrak{s}$ are the $+1$, resp., -1 eigenspaces of τ. (Note that $\mathfrak{k}$ is a subalgebra and $\mathfrak{s}$ is a vector subspace of $\mathfrak{g}$.) In what follows $\mathfrak{g}$ is assumed to be equipped with the scalar product $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined as

$$\langle X, Y\rangle = -\kappa(X, \tau(Y)), \text{ for all } X, Y \in \mathfrak{g}.$$

2.2. The root space decomposition of $\mathfrak{g}$. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$. The definition of $\langle\cdot,\cdot\rangle$ implies that the set $\{\mathrm{ad}(H) \mid H \in \mathfrak{a}\}$ is a commuting family of self-adjoint (hence diagonable) transformations of $\mathfrak{g}$. Thus $\mathfrak{g}$ can be written as the (orthogonal) direct sum of simultaneous eigenspaces

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha, \quad \text{where } \alpha \in \mathfrak{a}^*,$$
$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

and $\Phi = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\}$. Any $\alpha \in \Phi$ is called a *root of* $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}^\alpha$ is the corresponding *root space*, and Φ is the *root system of the pair* $(\mathfrak{g}, \mathfrak{a})$. In fact, Φ is a root system in $\mathfrak{a}^*$ (when $\mathfrak{a}^*$ is equipped with the scalar product obtained by transferring to $\mathfrak{a}^*$ the restriction $\langle\cdot,\cdot\rangle|_{\mathfrak{a}\times\mathfrak{a}}$).

2.3. The Iwasawa decompositions. Choose a base Π for Φ. Let Φ^+, respectively, Φ^- be the set of positive, respectively, negative roots relative to Π, and define $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+}\mathfrak{g}^\alpha$. Then $\mathfrak{n}$ and $\mathfrak{a}\oplus\mathfrak{n}$ are subalgebras of $\mathfrak{g}$ with $\mathfrak{n}$ nilpotent and $\mathfrak{a}\oplus\mathfrak{n}$ solvable, and the following so-called Iwasawa decomposition holds for $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}.$$

Let K, A, and N be the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$. Then K is compact, and A and N are simply connected. The diffeomorphism $(k,a,n)\in K\times A\times N\mapsto kan\in G$ gives rise to the decomposition $G = KAN$, called the Iwasawa decomposition of G.

2.4. Parabolic subalgebras and parabolic subgroups. Let Θ be a subset of Π and consider the parabolic set $\mathcal{P}(\Theta)$ in Φ defined by

$$\mathcal{P}(\Theta) := \Phi^+\cup\Phi(\Theta),\ \text{ where } \Phi(\Theta) := \Phi\cap\operatorname{span}(\Theta).$$

(We recall that $\operatorname{span}(\Theta)$ stays for the intersection of all vector subspaces of $\mathfrak{a}^*$ containing Θ.) Define now

$$\mathfrak{p}(\Theta) := \mathfrak{g}^0\oplus\bigoplus_{\alpha\in\mathcal{P}(\Theta)}\mathfrak{g}^\alpha,\quad P(\Theta) := N_G(\mathfrak{p}(\Theta)).$$

Note that $\mathfrak{p}(\emptyset) = \mathfrak{g}^0\oplus\mathfrak{n}$ and $\mathfrak{p}(\Phi) = \mathfrak{g}$. For every $\Theta\subseteq\Pi$, the vector space $\mathfrak{p}(\Theta)$ is a self-normalizing subalgebra of $\mathfrak{g}$, and $P(\Theta)$ is a closed subgroup of G with Lie algebra $\mathfrak{p}(\Theta)$. The subalgebras $\mathfrak{p}(\Theta)$, $\Theta\subseteq\Pi$, are the parabolic subalgebras of $\mathfrak{g}$, and $P(\Theta)$, $\Theta\subseteq\Pi$, the standard parabolic subgroups of G.

2.5. The subalgebras $\mathfrak{g}(\Theta)$ of $\mathfrak{g}$ and the subgroups $G(\Theta)$ of G. Let Θ be a subset of Π. Write $\mathfrak{g}(\Theta)$ for the Lie algebra generated by the root spaces $\mathfrak{g}^\alpha$, $\alpha\in\Phi(\Theta)$, where $\Phi(\Theta)$ is the set defined in the above paragraph. An easy computation yields that

$$\mathfrak{g}(\Theta) = \Big(\bigoplus_{\alpha\in\Phi(\Theta)}\mathfrak{g}^\alpha\Big)\oplus\Big(\sum_{\alpha\in\Phi(\Theta)}[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}]\Big).$$

The Lie algebra $\mathfrak{g}(\Theta)$ is semisimple and the corresponding analytic subgroup $G(\Theta)$ is closed in G.

3. Applications in the case of $\mathfrak{sl}(n,\mathbb{R})$ and $\mathrm{Sl}(n,\mathbb{R})$

$\mathrm{Sl}(n,\mathbb{R})$ is a connected semisimple Lie group with finite center and with Lie algebra $\mathfrak{sl}(n,\mathbb{R})$. Our first task is to identify within $\mathfrak{sl}(n,\mathbb{R})$

and Sl$(n, \mathbb{R})$ the elements presented in the previous section for arbitrary semisimple Lie algebras and Lie groups. Throughout this section the letter $\mathfrak{g}$ will stay for the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$, and G for the Lie group Sl$(n, \mathbb{R})$. We start by specifying a Cartan involution in $\mathfrak{g}$: Define $\tau\colon \mathfrak{g} \to \mathfrak{g}$ by $\tau(X) = -X^t$, where X^t is the transpose of X. For showing that τ is a Cartan involution we need a little preparation.

3.1. Lemma. *The map $\sigma\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $\sigma(X, Y) = \mathrm{tr}(XY^t)$ has the following properties:*

(i) *σ is a scalar product on $\mathfrak{g}$.*

(ii) *$(\mathrm{ad}\, X)^* = \mathrm{ad}(X^t)$, where $(\mathrm{ad}\, X)^*$ denotes the adjoint of $\mathrm{ad}\, X$ relative to the scalar product σ.*

Proof. Assertion (i) is a direct consequence of the properties of the trace function tr: its linearity, $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, $\mathrm{tr}(X) = \mathrm{tr}(X^t)$, and $\mathrm{tr}(XX^t) = 0$ if and only if $X = 0$.

(ii) The following equalities hold for every $X, Y, Z \in \mathfrak{g}$

$$\sigma(\mathrm{ad}\, X(Y), Z) = \mathrm{tr}(XYZ^t - YXZ^t) = \mathrm{tr}(YZ^tX - YXZ^t) = \\ = \mathrm{tr}(Y(X^tZ - ZX^t)^t) = \sigma(Y\, \mathrm{ad}(X^t)(Z)).$$

Thus $(\mathrm{ad}\, X)^* = \mathrm{ad}(X^t)$. $\Diamond$

3.2. Lemma. *Let $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form of $\mathfrak{g}$. The map $\kappa_\tau\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $\kappa_\tau(X, Y) = -\kappa(X, \tau(Y))$, for every $X, Y \in \mathfrak{g}$, is a scalar product.*

Proof. The bilinearity of κ_τ follows from that of κ. Since

$$\kappa(\varphi(X), \varphi(Y)) = \kappa(X, Y) \quad \text{for every} \quad X, Y \in \mathfrak{g}$$

and every Lie algebra automorphism φ of $\mathfrak{g}$, we get that

$$\kappa_\tau(Y, X) = -\kappa(Y, \tau(X)) = -\kappa(\tau(Y), \tau(\tau(X))) = \\ = -\kappa(\tau(Y), X) = -\kappa(X, \tau(Y)) = \kappa_\tau(X, Y)$$

for every $X, Y \in \mathfrak{g}$. Thus κ_τ is symmetric. Pick now an arbitrary $X \in \mathfrak{g}$. According to Lemma 3.1 we have that

$$\kappa_\tau(X, X) = \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}(X^t)) = \mathrm{tr}(\mathrm{ad}\, X (\mathrm{ad}\, X)^*).$$

Hence $\kappa_\tau(X, X) \geq 0$ and $\kappa_\tau(X, X) = 0$ if and only if $X = 0$, showing that κ_τ is a scalar product. $\Diamond$

3.3. Corollary. *The map $\tau\colon \mathfrak{g} \to \mathfrak{g}$, $X \mapsto -X^t$, is a Cartan involution.* $\Diamond$

The Cartan decomposition corresponding to τ is

$$\mathfrak{g} = \mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{s},$$

where $\mathfrak{so}(n, \mathbb{R}) = \{X \in \mathfrak{g} \mid X + X^t = 0\}$, $\quad \mathfrak{s} = \{X \in \mathfrak{g} \mid X = X^t\}$. The subset $\mathfrak{a} \subseteq \mathfrak{s}$ consisting of all diagonal matrices of trace 0 is a maximal abelian subspace of $\mathfrak{s}$. (Note that every matrix $X \in \mathfrak{g}$ having the property that $[X, \mathfrak{a}] = \{0\}$ belongs to $\mathfrak{a}$.) The dimension of $\mathfrak{a}$ is $n-1$. For simplicity we write $(d_1, \ldots, d_n)$, $d_1, \ldots, d_n \in \mathbb{R}$, for the diagonal matrix $X = (x_{ij})$ with $x_{ii} = d_i$, $i = \overline{1, n}$. For $i \in \{1, \ldots, n\}$ let $f_i \in \mathfrak{a}^*$ be defined by

$$f_i(d_1, \ldots, d_n) = d_i.$$

For each $H \in \mathfrak{a}$ and every $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we have that

$$\operatorname{ad} H(E_{ij}) = (f_i(H) - f_j(H)) E_{ij},$$

so E_{ij} is a simultaneous eigenvector for all $\operatorname{ad} H$, $H \in \mathfrak{a}$. It follows that the root system of the pair $(\mathfrak{g}, \mathfrak{a})$ is

$$\Phi = \{f_i - f_j \mid i, j \in \{1, \ldots, n\}, i \neq j\}.$$

The corresponding root spaces are $\mathfrak{g}^{f_i - f_j} = \mathbb{R} E_{ij}$. Also, $\mathfrak{g}^0 = \mathfrak{a}$. We thus get the following root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{i \neq j} \mathbb{R} E_{ij}.$$

For every $i \in \{1, ..., n-1\}$ let $\alpha_i := f_i - f_{i+1}$. The set $\Pi = \{\alpha_1, ..., \alpha_{n-1}\}$ is a base of Φ. Indeed, Π is a vector space base of $\mathfrak{a}^*$ since it consists of $n-1$ linearly independent elements. Also, if $i, j \in \{1, \ldots, n\}$ are so that $i < j$ then

$$(*) \qquad f_i - f_j = \sum_{k=i}^{j-1} \alpha_k.$$

It follows that

$$\Phi^+ = \big\{f_i - f_j \mid i, j \in \{1, \ldots, n\}, i < j\big\},$$
$$\Phi^- = \big\{f_i - f_j \mid i, j \in \{1, \ldots, n\}, i > j\big\}.$$

So, $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha$ is the subspace of $\mathfrak{g}$ consisting of all strictly upper triangular matrices, and

$$\mathfrak{g} = \mathfrak{so}(n, \mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is the Iwasawa decomposition of $\mathfrak{g}$. The corresponding Iwasawa decomposition of G is $G = KAN$, where

$$K = \mathrm{SO}(n, \mathbb{R}) = \{g \in G \mid gg^t = 1\},$$
$$A = \{(d_1, \ldots, d_n) \in G \mid d_i > 0, i = \overline{1, n}\},$$

and N consists of all upper triangular matrices with real entries and with 1 on the main diagonal.

3.4. Lemma. *For a nonempty set* $J \subseteq \{1,\dots,n-1\}$ *the following assertions hold:*

(i) *There exist nonzero real numbers* t_j, $j \in J$, *such that* $\sum_{j\in J} t_j\alpha_j \in \in \Phi$ *if and only if* J *consists of consecutive natural numbers.*

(ii) *If* $\sum_{j\in J} t_j\alpha_j \in \Phi$ *for some nonzero real numbers* t_j, $j \in J$, *then either* $t_j = 1$ *for all* $j \in J$, *or* $t_j = -1$ *for all* $j \in J$.

Proof. The assertions follow from $(*)$ and from the fact that $\Pi = = \{\alpha_1,\dots,\alpha_{n-1}\}$ is a base of Φ. $\Diamond$

3.5. Notation. For every nonempty subset $J \subseteq \{1,\dots,n-1\}$ denote by $\Theta_J := \{\alpha_j \mid j \in J\}$ and by $\Phi(J) := \Phi(\Theta_J) = \Phi \cap \mathrm{span}(\Theta_J)$. Put $\Phi(\emptyset) := \emptyset$. Also, let $\mathfrak{p}(J)$ $[P(J)]$ stay for the parabolic subalgebra [subgroup] $\mathfrak{p}(\Theta_J)$ $[P(\Theta_J)]$. Similarly, $\mathfrak{g}(J)$ $[G(J)]$ denotes the set $\mathfrak{g}(\Theta_J)$ $[G(\Theta_J)]$ defined in 2.5.

3.6. Definition. Let $\emptyset \neq I \subseteq \mathbb{N}$. A partition $I = \cup_{k=1}^{p} I_k$ of I into p disjoint nonempty subsets is called the *partition of* I *into maximal sets of consecutive numbers* if the following conditions are satisfied:

(i) For every $k \in \{1,\dots,p\}$ the set I_k consists of consecutive natural numbers.

(ii) For every $k \in \{1,\dots,p-1\}$ the inequality $\min(I_{k+1}) - \max(I_k) \geq \geq 2$ holds.

3.7. Example. If $I = \{1,2,3,7,9,10\}$ then $I = \{1,2,3\} \cup \{7\} \cup \{9,10\}$ is the partition of I into maximal sets of consecutive numbers.

3.8. Proposition. *Let* $\emptyset \neq J \subseteq \{1,\dots,n-1\}$ *and* $J = \cup_{k=1}^{p} J_k$ *be the partition of* J *into maximal sets of consecutive numbers. Then the following equality holds*

$$\Phi(J) = \bigcup_{k=1}^{p} \{f_i - f_j \mid i,j \in [\min(J_k), \max(J_k)+1] \cap \mathbb{N}, i \neq j\}.$$

Proof. Consider an element $\alpha = f_i - f_j \in \Phi(J)$, $i,j \in \{1,\dots,n\}$, $i \neq j$. Then there exist a subset Z of J and nonzero real numbers t_z, $z \in Z$, such that $\alpha = \sum_{z\in Z} t_z\alpha_z$. According to Lemma 3.4, we find an index $k \in \{1,\dots,p\}$ such that $Z \subseteq J_k$ and $i,j \in [\min(J_k), \max(J_k)+1]$. This proves the inclusion

$$\Phi(J) \subseteq \bigcup_{k=1}^{p} \{f_i - f_j \mid i,j \in [\min(J_k), \max(J_k)+1] \cap \mathbb{N}, i \neq j\}.$$

For the converse inclusion pick $k \in \{1,\dots,p\}$ and $i,j \in \in [\min(J_k), (\max J_k)+1]$ with $i < j$. Then $f_i - f_j = \alpha_i + \dots + \alpha_{j-1} \in \in \Phi(J)$ and $f_j - f_i \in \Phi(J)$. $\Diamond$

3.9. Corollary. *Let $\emptyset \neq J \subseteq \{1,\dots,n-1\}$ and $i,j \in \{1,\dots,n-1\}$ with $i > j$. Then $f_i - f_j \in \Phi(J)$ if and only if $\mathbb{N} \cap [j, i-1] \subseteq J$.* $\Diamond$

3.10. Lemma. *Let J be a nonempty and proper subset of $\{1,...,n-$ $-1\}$, $J = \cup_{\ell=1}^{p} J_\ell$ the partition of J into maximal sets of consecutive numbers, and $I = \{i_1,\dots,i_k\} = \{1,\dots,n-1\} \setminus J$ with $i_1 < \dots < i_k$. Put $i_0 = 0$ and $i_{k+1} = n$. Then the following assertions hold:*

(i) *For every $\ell \in \{1,\dots,p\}$ there exists exactly one index $m \in$ $\in \{1,\dots,k+1\}$ such that $\min(J_\ell) - 1 = i_{m-1}$ and $\max(J_\ell) +$ $+ 1 = i_m$.*

(ii) *If $m \in \{1,\dots,k+1\}$ is such that $i_m - i_{m-1} \geq 2$ then there exists an index $\ell \in \{1,\dots,p\}$ such that $\min(J_\ell) - 1 = i_{m-1}$ and $\max(J_\ell) + 1 = i_m$.*

(iii) *The equality*

$$\sum_{\ell=1}^{p} \big((\max(J_\ell) - \min(J_\ell) + 2)^2 - 1\big) = \sum_{m=1}^{k=1} \big((i_m - i_{m-1})^2 - 1\big)$$

holds.

Proof. (i) Pick $\ell \in \{1,\dots,p\}$.

Case 1. If $\min(J_\ell) = 1$ then obviously $\ell = 1$ and $\max(J_\ell)+1 = i_1$. Thus, in this case, we can choose $m = 1$.

Case 2. If $\min(J_\ell) > 1$ and $\max(J_\ell) < n-1$, then we find an index $m \in \{2,\dots,k-1\}$ such that $\min(J_\ell)-1 = i_{m-1}$ and $\max(J_\ell)+1 = i_m$.

Case 3. If $\max(J_\ell) = n-1$ then $i_k = \min(J_\ell) - 1$. Hence $m =$ $= k+1$.

(ii) If $m = 1$ then $J_1 = \{1,\dots,i_1 - 1\}$, thus $\ell = 1$ satisfies the required conditions. If $2 \leq m \leq k$ then there exists an index $\ell \in$ $\in \{1,\dots,p\}$ such that $J_\ell = \{i_{m-1}+1,\dots,i_m - 1\}$. Finally, if $m = k+$ $+1$ then $J_p = \{i_k + 1,\dots,n-1\}$, hence $\ell = p$ satisfies the required conditions.

The equality (iii) follows from (i) and (ii). $\Diamond$

In the next theorem we first establish the connection between the parabolic subalgebras $\mathfrak{p}(J)$ [parabolic subgroups $P(J)$] of $\mathfrak{g}$ [G] an the subalgebras [subgroups] introduced in the first section. Afterwards we characterize $\mathfrak{g}(J)$ [$G(J)$] as a suitable set $\mathrm{diag}_{\mathfrak{g}}(I)$ [$\mathrm{diag}_G(I)$].

3.11. Notation. For a subset $I \subseteq \{1,\dots,n-1\}$ we define the set $\mathrm{diag}_{\mathfrak{g}}(I)$ [$\mathrm{diag}_G(I)$] in the following way: If $I = \emptyset$ then $\mathrm{diag}_{\mathfrak{g}}(I) = \mathfrak{g}$ [$\mathrm{diag}_G(I) = G$]. If $I \neq \emptyset$ let $I = \{i_1,\dots,i_k\}$ with $1 \leq i_1 < \dots < i_k \leq$ $\leq n-1$. Put $i_0 = 0$, $i_{k+1} = n$, and define $\mathrm{diag}_{\mathfrak{g}}(I)$ [$\mathrm{diag}_G(I)$] to be the set consisting of all matrices

$$\begin{pmatrix} X_1 & 0 & 0 & \dots & 0 \\ 0 & X_2 & 0 & \dots & 0 \\ 0 & 0 & X_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & X_{k+1} \end{pmatrix},$$

where $X_j \in \mathfrak{sl}(i_j - i_{j-1}, \mathbb{R})$ [$X_j \in \mathrm{Sl}(i_j - i_{j-1}, \mathbb{R})$], for every $j \in$ $\in \{1, \dots, k+1\}$. It is clear that $\mathrm{diag}_{\mathfrak{g}}(I)$ is a subalgebra of $\mathfrak{g}$, and that $\mathrm{diag}_G(I)$ is a subgroup of G. Note that for $I = \{1, \dots, n-1\}$ the set $\mathrm{diag}_{\mathfrak{g}}(I)$ [$\mathrm{diag}_G(I)$] consists only of the zero [identity] matrix.

In the next theorem we first establish the connection between the parabolic subalgebras $\mathfrak{p}(J)$ [parabolic subgroups $P(J)$] of $\mathfrak{g}$ [G] an the subalgebras [subgroups] introduced in the first section. Afterwards we characterize $\mathfrak{g}(J)$ [$G(J)$] as a suitable set $\mathrm{diag}_{\mathfrak{g}}(I)$ [$\mathrm{diag}_G(I)$].

3.12. Theorem. *Let* $J \subseteq \{1, \dots, n-1\}$ *and* $I = \{1, \dots, n-1\} \setminus J$. *Then the following assertions hold:*

(i) $p(J) = \mathfrak{g}_I$. (iii) $\mathfrak{g}(J) = \mathrm{diag}_{\mathfrak{g}}(I)$.

(ii) $P(J) = G_I$. (iv) $G(J) = \mathrm{diag}_G(I)$.

Proof. (i) If $J = \emptyset$ then $\mathfrak{p}(J) = \mathfrak{a} \oplus \mathfrak{n}$, and if $J = \{1, \dots, n-1\}$ then $\mathfrak{p}(J) = \mathfrak{g}$. Thus, in both of these cases, according to 2) of 1.8, the equality $\mathfrak{p}(J) = \mathfrak{g}_I$ holds. Assume now that J is a nonempty and proper subset of $\{1, \dots, n-1\}$. By definition,

$$\mathfrak{p}(J) = (\mathfrak{a} \oplus \mathfrak{n}) + (\oplus_{\alpha \in \Phi(J)} \mathfrak{g}^{\alpha}).$$

We prove first that $\mathfrak{p}(J) \subseteq \mathfrak{g}_I$. Since $\mathfrak{a} \oplus \mathfrak{n} \subseteq \mathfrak{g}_I$ (by assertion 2) of 1.8), we have only to prove that $E_{ij} \in \mathfrak{sl}(n, \mathbb{R})_I$ whenever $i, j \in \{1, \dots, n-$ $-1\}$ are so that $i > j$ and $f_i - f_j \in \Phi(J)$. Pick $k \in I$ and indices $i, j \in \{1, \dots, n-1\}$ so that $i > j$ and $f_i - f_j \in \Phi(J)$. It follows from Cor. 3.9 that $\mathbb{N} \cap [j, i-1] \subseteq J$. Hence $k < j$ or $k \geq i$, so $E_{ij} \in V_k$ by assertion 1) of 1.8. This proves the inclusion $\mathfrak{p}(J) \subseteq \mathfrak{g}_I$. For the converse inclusion pick an element $X \in \mathfrak{g}_I$. According to the root space decomposition of $\mathfrak{g}$ we find $X_0 \in \mathfrak{a}$ and $X_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Phi$, such that $X = X_0 + \sum_{\alpha \in \Phi} X_\alpha$. We show that $X_\alpha = 0$ for every $\alpha \in \Phi^- \setminus \Phi(J)$. Pick an element $\alpha \in \Phi^- \setminus \Phi(J)$. Let $i, j \in \{1, \dots, n-1\}$, $i > j$, be so that $\alpha = f_i - f_j$, and $t \in \mathbb{R}$ so that $X_\alpha = tE_{ij}$. Applying once again Cor. 3.9, we find an index $\ell \in (\mathbb{N} \cap [j, i-1]) \setminus J$. It follows that $\ell \in I$, thus $X \in \mathfrak{g}_{V_\ell}$, so $X(e^j) \in V_\ell$. On the other hand,

$$X(e^j) \in \mathbb{R}e^j + te^i + \sum_{k>j, k\neq i} \mathbb{R}e^k,$$

hence $t = 0$ (note that $\ell < i$), i.e., $X_\alpha = 0$. Since $\alpha \in \Phi^- \setminus \Phi(J)$ was arbitrary, we finally obtain the converse inclusion $\mathfrak{g}_I \subseteq \mathfrak{p}(J)$.

(ii) By definition, $P(J) = N_G(\mathfrak{p}(J))$, so the equality follows from (i) and from assertion b) of Prop. 1.9.

(iii) If $J = \emptyset$ then $I = \{1, \dots, n-1\}$, and $\mathfrak{g}(J) = \{0\} = \mathrm{diag}_{\mathfrak{g}}(I)$. If $J = \{1, \dots, n-1\}$ then $I = \emptyset$ and $\Phi(J) = \Phi$. Since $\sum_{\alpha\in\Phi}[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathfrak{a}$, it follows from the formula in 2.5 that $\mathfrak{g}(J) = \mathfrak{g}$, thus $\mathfrak{g}(J) = \mathrm{diag}_{\mathfrak{g}}(I)$. Suppose now that J is a nonempty and proper subset of $\{1, \dots, n-1\}$, that $J = \cup_{\ell=1}^p J_\ell$ is the partition of J into maximal sets of consecutive numbers, and let $I = \{i_1, \dots, i_k\}$ with $1 \leq i_1 < \dots < i_k \leq n-1$. Put $i_0 = 0$ and $i_{k+1} = n$. For every $m \in \{1, ..., k+1\}$ denote by $I^{(m)} := \{i_{m-1}+1, \dots, i_m\}$. Pick now an arbitrary index $j \in J$. Then we find a natural number $m \in \{1, \dots, k+1\}$ such that $j \in I^{(m)}$. Since $j \notin \{i_1, \dots, i_{k+1}\}$, the natural number $j+1$ lies also in the set $I^{(m)}$, hence

$$\mathfrak{g}^{\alpha_j} = \mathbb{R}E_{j,j+1} \subseteq \mathrm{diag}_{\mathfrak{g}}(I).$$

It follows that $\mathfrak{g}(J) \subseteq \mathrm{diag}_{\mathfrak{g}}(I)$, because $\mathrm{diag}_{\mathfrak{g}}(I)$ is a Lie subalgebra of $\mathfrak{g}$. We prove next that the (real) vector spaces $\mathfrak{g}(J)$ and $\mathrm{diag}_{\mathfrak{g}}(I)$ have the same dimension. It follows right from the definition of $\mathrm{diag}_{\mathfrak{g}}(I)$ that

$$\dim(\mathrm{diag}_{\mathfrak{g}}(I)) = \sum_{m=1}^{k+1}((i_m - i_{m-1})^2 - 1).$$

Note that

$$\dim\Bigg(\sum_{\alpha\in\Phi(J)}[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]\Bigg) \geq \dim\Bigg(\sum_{\alpha\in J}[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]\Bigg) = \mathrm{card}(J),$$

hence, according to the formula in 2.5 and to the fact that the root spaces $\mathfrak{g}^\alpha$, $\alpha \in \Phi$, are one-dimensional, the inequality

$$\dim(\mathfrak{g}(J)) \geq \mathrm{card}(\Phi(J)) + \mathrm{card}(J)$$

holds. In view of Prop. 3.8, we obtain further that

$$\begin{aligned}\mathrm{card}(\Phi(J)) &= \sum_{\ell=1}^p (\max(J_k) - \min(J_k) + 2)\big(\max(J_k) - \min(J_k) + 1\big) = \\ &= \sum_{\ell=1}^p \big((\max(J_k) - \min(J_k) + 2)^2 - 1\big) - \mathrm{card}(J),\end{aligned}$$

thus

$$\dim(\mathfrak{g}(J)) \geq \sum_{\ell=1}^{p}\big((\max(J_k)-\min(J_k)+2)^2-1\big).$$

Using assertion (iii) of Lemma 3.10, we conclude that $\dim(\mathfrak{g}(J)) \geq \geq \dim_{\mathfrak{g}}(I)$, hence $\mathfrak{g}(J)=\mathrm{diag}_{\mathfrak{g}}(I)$.

(iv) This equality follows from (iii) and from the fact that the analytic subgroup corresponding to $\mathrm{diag}_{\mathfrak{g}}(I)$ is $\mathrm{diag}_G(I)$. $\Diamond$

3.13. Notation. Let $I=\{i_1,\dots,i_k\}\subseteq\{1,\dots,n-1\}$ with $1\leq i_1 < < \cdots < i_k \leq n-1$. Put $i_0=0$ and $i_{k+1}=n$. We consider the semigroup $\mathbb{R}^{k+1}$ endowed with componentwise multiplication. Define

$$\delta_I\colon \prod_{j=1}^{k+1}\mathrm{Gl}(i_j-i_{j-1},\mathbb{R}) \to \mathbb{R}^{k+1} \text{ by } \delta_I(X_1,\dots,X_{k+1}) = \\ =(\det X_1,\dots,\det X_{k+1}).$$

An easy computation yields that δ_I is a homomorphism. Let

$$D_k := \left\{(x_1,\dots,x_{k+1})\in\mathbb{R}^{k+1} \mid \prod_{i=1}^{k+1} x_i = 1\right\}.$$

Obviously D_k is a submonoid of $\mathbb{R}^{k+1}$ and $(\delta_I\circ\mathrm{pr})^{-1}(D_k)\subseteq G$.

3.14. Theorem. *Let $I=\{i_1,\dots,i_k\}$ be a nonempty subset of $\{1,\dots,n-1\}$ and $M\subseteq G_I$. Then M is a subsemigroup which contains N and* $\mathrm{diag}_G(I)$ *if and only if there exists a submonoid $\mathcal{S}$ of D_k such that $M=(\delta_I\circ\mathrm{pr})^{-1}(\mathcal{S})$, i.e.,*

$$M=\{(\mathrm{pr})^{-1}(X_1,\dots,X_{k+1}) \mid (X_1,\dots,X_{k+1})\in\delta_I^{-1}(\mathcal{S})\},$$

where pr *is the map defined in remark* 3) *of* 1.8.

Proof. ($\Leftarrow$) Denote by 1_k the identity of D_k. Since $\prod_{j=1}^{k+1}\mathrm{Sl}(i_j - - i_{j-1},\mathbb{R})=\delta_I^{-1}(1_k)$ it follows that $\mathrm{diag}_G(I)\subseteq M$. It is also clear that $N\subseteq M$ and that M is a subsemigroup of G_I.

($\Rightarrow$) Let $\mathcal{M}=\mathrm{pr}(M)$. This set is a subsemigroup of $\prod_{j=1}^{k+1}\mathrm{Gl}(i_j - i_{j-1},\mathbb{R})$, hence $\mathcal{S}=\delta_I(\mathcal{M})$ is a subsemigroup of D_k. We show first that $\mathcal{M}=\delta_I^{-1}(\mathcal{S})$. Consider $(Z_1,\dots,Z_{k+1})\in\delta_I^{-1}(\mathcal{S})$ and $(X_1,\dots,X_{k+1})\in \in\mathcal{M}$ such that $\det X_j=\det Z_j$ for every $j\in\{1,\dots,k+1\}$. Thus, for every $j\in\{1,\dots,k+1\}$, the matrix $X_j^{-1}Z_j$ belongs to $\mathrm{Sl}(i_j-i_{j-1},\mathbb{R})$. Since

$$\prod_{j=1}^{k+1}\mathrm{Sl}(i_j-i_{j-1},\mathbb{R})=\mathrm{pr}(\mathrm{diag}_G(I))\subseteq\mathcal{M},$$

we conclude that

$$(Z_1, \ldots, Z_{k+1}) = (X_1, \ldots, X_{k+1})(X_1^{-1} Z_1, \ldots, X_{k+1}^{-1} Z_{k+1}) \in \mathcal{M}.$$

This proves the equality $\mathcal{M} = \delta_I^{-1}(\mathcal{S})$. Finally we have to prove that

$$M = \{(\mathrm{pr})^{-1}(X_1, \ldots, X_{k+1}) \mid (X_1, \ldots, X_{k+1}) \in \mathcal{M}\}.$$

For this it suffices to show that if $(X_1, \ldots, X_{k+1}) \in \delta_I^{-1}(\mathcal{S})$ and if $Y \in$ $\in G_I$ is so that $\mathrm{pr}(Y) = (X_1, \ldots, X_{k+1})$, then $Y \in M$. Let $X \in M$ be so that $\mathrm{pr}(X) = (X_1, \ldots, X_{k+1})$. It follows that $X^{-1}Y \in N \subseteq M$, hence $Y = X(X^{-1}Y) \in M$. $\Diamond$

3.15. Remark. The proof of the above theorem shows in particular that a subset S of $\mathrm{Gl}(n, \mathbb{R})$ is a subsemigroup containing $\mathrm{Sl}(n, \mathbb{R})$ if and only if there exists a submonoid D of the multiplicative group of nonzero reals such that $S = \det^{-1}(D)$.

3.16. Examples. We keep the notations of Th. 3.14 and show how one can construct subsemigroups M of G_I with k 'independent factors' on the main diagonal: For every $j \in \{1, \ldots, k\}$ consider a subsemigroup S_j of $\mathrm{Gl}(i_j - i_{j-1}, \mathbb{R})$ containing $\mathrm{Sl}(i_j - i_{j-1}, \mathbb{R})$ (by the above remark we know how one obtains such subsemigroups). Consider now M to be the set of all matrices of the following type

$$\begin{pmatrix} X_1 & * & * & \ldots & * \\ 0 & X_2 & * & \ldots & * \\ 0 & 0 & X_3 & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & X_{k+1} \end{pmatrix},$$

where $X_j \in S_j$, $j = \overline{1, k}$, $X_{k+1} \in \det^{-1}(\frac{1}{\det X_1 \ldots \det X_k})$, and the stars $*$ stay for arbitrary real matrices of suitable dimension.

Acknowledgment. The research for this article was supported by the Central European University (CEU) Special and Extension Programs.

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