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# $\widetilde{g}$-SEMI-CLOSED SETS IN TOPOLOGICAL SPACES

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**Abstract**: In this paper, we introduce a new class of sets namely, $\widetilde{g}$-semi-closed sets for topological spaces. This class lies between the class of semi-closed sets and the class of $sg$-closed sets. This class also lies between the class of $g^{\#}$-closed sets and the class of ${}^{\#}g$-semi-closed sets.

## 1. Introduction

Levine [13], Mashhour et al. [16], Abd El-Monsef et al. [1] and Njåstad [17] introduced semi-open sets, pre open sets, $\beta$-open sets and

$\alpha$-open sets, respectively. Levine [12] introduced $g$-closed sets and studied their most fundamental properties. Bhattacharya and Lahiri [5], Arya and Nour [3] Maki et al. [14, 15] introduced $sg$-closed sets, $gs$-closed sets, $\alpha g$-closed sets and $g\alpha$-closed sets, respectively.

Recently Sundaram and Sheik John [20] introduced and studied $\omega$-closed sets. Veera Kumar introduced and studied $^*g$-closed sets [22] and $^{\#}g$-semi-closed sets [23] in topological spaces.

In this paper, we introduce a new class of sets, namely $\widetilde{g}$-semi-closed sets for topological spaces. This class lies between the class of semi-closed sets and the class of sg-closed sets.

## 2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively.

We recall the following definitions, which are useful in the sequel.

**Definition 2.1.** A subset $A$ of a space $(X, \tau)$ is called:

(1) semi-open set [13] if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$;

(2) pre-open set [16] if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$;

(3) $\alpha$-open set [17] if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;

(4) $\beta$-open set [1] ( = semi-pre-open [2]) if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$;

(5) regular open [11] if $A = \mathrm{int}(\mathrm{cl}(A))$.

The complements of the above mentioned sets are called semi-closed, pre-closed, $\alpha$-closed, $\beta$-closed and regular closed, respectively.

The semi-closure [6, 7] (resp. $\alpha$-closure [17], semi-pre-closure [2]) of a subset $A$ of $X$, denoted by $scl_X(A)$ (resp. $\alpha\,\mathrm{cl}_X(A)$, $sp\,\mathrm{cl}_X(A)$) briefly $s\,\mathrm{cl}(A)$, (resp. $\alpha\,\mathrm{cl}(A)$, $sp\,\mathrm{cl}(A)$) is defined to be the intersection of all semi-closed (resp. $\alpha$-closed, semi-preclosed) sets of $(X, \tau)$ containing $A$. It is known that $s\,\mathrm{cl}(A)$ (resp. $\alpha\,\mathrm{cl}(A)$, $sp\,\mathrm{cl}(A)$) is a semi-closed (resp. $\alpha$-closed, semi-preclosed) set. For any subset $A$ of an arbitrarily chosen topological space, the semi-interior [6] of $A$, denoted by $s\,\mathrm{int}(A)$ is defined by the union of all semi-open sets of $(X, \tau)$ contained in $A$. If $A \subseteq B \subseteq X$, then $s\,\mathrm{cl}_B(A)$ and $s\,\mathrm{int}_B(A)$ denote the semi-closure of $A$ relative to $B$ and semi-interior of $A$ relative to $B$. The family of all semi-closed subsets of $(X, \tau)$ is denoted by $SC(X, \tau)$.

**Definition 2.2.** A subset $A$ of a space $(X, \tau)$ is called:

(1) a generalized closed (briefly $g$-closed) set [12] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$;

(2) a semi-generalized closed (briefly $sg$-closed) set [5] if $s\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$;

(3) a generalized semi-closed (briefly $gs$-closed) set [3] if $s\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$;

(4) an $\alpha$-generalized closed (briefly $\alpha g$-closed) set [14] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of $\alpha g$-closed set is called $\alpha g$-open set;

(5) a generalized $\alpha$-closed (briefly $g\alpha$-closed) set [15] if $\alpha\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$;

(6) an $\omega$-closed [20] ($= \widehat{g}$-closed [21]) set if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq$ $\subseteq U$ and $U$ is semi-open in $(X, \tau)$. The complement of $\widehat{g}$-closed set is called $\widehat{g}$-open set;

(7) a generalized semi-pre-closed set (briefly $gsp$-closed) [10] if $sp\,\mathrm{cl}(A)$ $\subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of $gsp$-closed set is called $gsp$-open set;

(8) a $^*$g-closed set [22] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\widehat{g}$-open in $(X, \tau)$. The complement of $^*g$-closed set is called $^*g$-open set;

(9) a $^\#g$-semi-closed (briefly $^\#g$s-closed) set [23] if $s\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^*g$-open in $(X, \tau)$. The complement of $^\#g$s-closed set is called $^\#g$s-open set. The set of all $^\#g$-semi-open sets of $(X, \tau)$ is denoted by $^\#GSO(X, \tau)$;

(10) a $\widetilde{g}$-closed set [11] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^\#g$s-open in $(X, \tau)$.

## 3. Basic properties of $\widetilde{g}$-semi-closed sets

We introduce the following notion

**Definition 3.1.** A subset $A$ of $X$ is called a $\widetilde{g}$*-semi-closed* (briefly $\widetilde{g}s$-*closed*) set if $s\,\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^\#gs$-open in $(X, \tau)$. The class of all $\widetilde{g}s$-closed subsets of $X$ is denoted by $\widetilde{G}SC(X, \tau)$.

**Proposition 3.2.** *Every closed set in* $(X, \tau)$ *is* $\widetilde{g}s$*-closed in* $(X, \tau)$.

**Proof.** If $A$ is any closed set in $(X, \tau)$ and $G$ be any $^\#gs$-open set containing $A$, then $G \supseteq A = \mathrm{cl}(A) \supseteq s\,\mathrm{cl}(A)$. Hence $A$ is $\widetilde{g}s$-closed in $(X, \tau)$. $\Diamond$

**Proposition 3.3.** *Every semi-closed set in* $(X, \tau)$ *is* $\widetilde{g}s$*-closed in* $(X, \tau)$.

**Proof.** If $A$ is any semi-closed set in $(X, \tau)$ and $G$ be any $^{\#}gs$-open set containing $A$, then $G \supseteq A = s\,\mathrm{cl}(A)$. Hence $A$ is $\widetilde{g}s$-closed in $(X, \tau)$. $\lozenge$

**Proposition 3.4.** *Every $\alpha$-closed set in $(X, \tau)$ is a $\widetilde{g}s$-closed set in $(X, \tau)$.*

**Proof.** Since every $\alpha$-closed set is semi-closed and by Prop. 3.3, the proof follows immediately. $\lozenge$

The following example shows that the converse of Props. 3.2, 3.3 and 3.4 need not be true.

**Example 3.5.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a, b\}, X\}$. Then the set $\{a, c\}$ is $\widetilde{g}s$-closed but neither closed, semi-closed and $\alpha$-closed in $(X, \tau)$, because $\widehat{G}SC(X, \tau) = \{\varnothing, \{c\}, \{b, c\}, \{a, c\}, X\}$.

**Proposition 3.6.** *Every $\widetilde{g}$-closed set in $(X, \tau)$ is $\widetilde{g}s$-closed in $(X, \tau)$.*

**Proof.** If $A$ is a $\widetilde{g}$-closed subset of $(X, \tau)$ and $G$ be any $^{\#}gs$-open set containing $A$, then $G \supseteq \mathrm{cl}(A) \supseteq s\,\mathrm{cl}(A)$. Hence $A$ is $\widetilde{g}s$-closed in $(X, \tau)$. $\lozenge$

The converse need not be true as seen from the following example.

**Example 3.7.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$. Then the set $\{b\}$ is $\widetilde{g}s$-closed but not $\widetilde{g}$-closed in $(X, \tau)$, because $\widetilde{G}SC(X, \tau) =$ $= \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.

**Proposition 3.8.** *Every $\widetilde{g}s$-closed set in $(X, \tau)$ is $sg$-closed in $(X, \tau)$.*

**Proof.** Suppose that $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$. Since every semi-open set is $^{\#}gs$-open and $A$ is $\widetilde{g}s$-closed, therefore $s\,\mathrm{cl}(A) \subseteq U$. Hence $A$ is $sg$-closed in $(X, \tau)$. $\lozenge$

The converse need not be true as seen from the following example.

**Example 3.9.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$. Then the set $\{b\}$ is sg-closed but not $\widetilde{g}s$-closed in $(X, \tau)$, because $\widetilde{G}SC(X, \tau) =$ $= \{\varnothing, \{a\}, \{b, c\}, X\}$.

In [10], it has been proved that every $sg$-closed set is semi-preclosed ($= \beta$-closed). Hence every $\widetilde{g}s$-closed set is $\beta$-closed.

**Proposition 3.10.** *Every $\widetilde{g}s$-closed set is gs-closed and gsp-closed.*

**Proof.** Let $A$ be a $\widetilde{g}s$-closed set in $(X, \tau)$. By Prop. 3.8, $A$ is $sg$-closed. From the investigation of Devi et al. [9] every $sg$-closed set is $gs$-closed, therefore $A$ is $gs$-closed. Also from the investigation of Dontchev [10] every gs-closed set is $gsp$-closed. Thus, again by Prop. 3.8, $A$ is $gs$-closed and $gsp$-closed in $(X, \tau)$. $\lozenge$

**Example 3.11.** Let $X$ and $\tau$ as in Ex. 3.9, the set $\{b\}$ is $gs$-closed but not $\widetilde{g}s$-closed in $(X, \tau)$.

Thus, the class of $\widetilde{g}s$-closed sets properly contains the class of closed sets, the class of $\alpha$-closed sets, the class of semi-closed sets and the class of $\widetilde{g}$-closed sets, the class of $\widetilde{g}s$-closed sets is properly contained in the class of $sg$-closed sets, the class of $\beta$-closed sets, the class of $gs$-closed sets and the class of $gsp$-closed sets.

**Theorem 3.12.** *$\widetilde{g}s$-closedness is independent from pre-closedness, $g$-closedness, $\alpha g$-closedness, $g\alpha$-closedness and $\omega$-closedness.*

**Proof.** It follows from the following examples. $\Diamond$

**Example 3.13.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$. Then the set $\{b\}$ is preclosed, $\alpha g$-closed, $g\alpha$-closed and $\omega$-closed but not $\widetilde{g}s$-closed in $(X, \tau)$.

**Example 3.14.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Then the set $\{a\}$ is $\widetilde{g}s$-closed but not pre-closed, $g$-closed, $\alpha g$-closed, $g\alpha$-closed and $\omega$-closed in $(X, \tau)$, because $\widetilde{GSC}(X, \tau) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$.

**Remark 3.15.** The following diagrams show the relationships established between $\widetilde{g}s$-closed sets and some other sets. $A \to B$ (resp. $A \not\leftrightarrow \not\leftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent of each other).

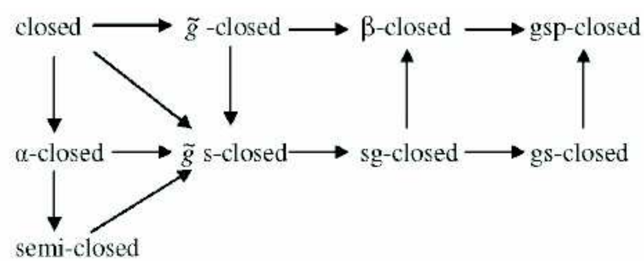


Diagram 1

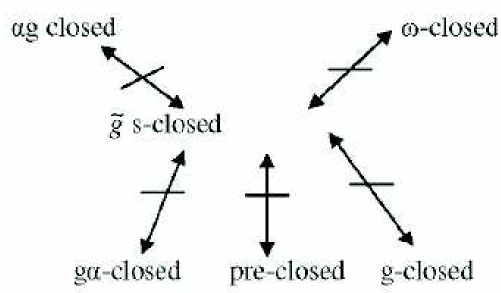


Diagram 2

**Theorem 3.16.** *If $A$ and $B$ are $\widetilde{g}s$-closed sets, then $A \cap B$ is also $\widetilde{g}s$-closed in $(X, \tau)$.*

**Proof.** Let $A$ and $B$ be any two $\widetilde{g}$s-closed sets in $X$ such that $A \subseteq G$ and $B \subseteq G$, where $G$ is $^{\#}gs$-open in $X$ and so $A \cap B \subseteq G$. Since $A$ and $B$ are $\widetilde{g}s$-closed, $G \supseteq s\,\mathrm{cl}(A)$ and $G \supseteq s\,\mathrm{cl}(B)$ and hence $G \supseteq s\,\mathrm{cl}(A) \cap \cap\, s\,\mathrm{cl}(B) \supseteq s\,\mathrm{cl}(A \cap B)$ ([6]; Theorem 1.7(5)). Thus, $A \cap B$ is $\widetilde{g}s$-closed in $(X, \tau)$. $\Diamond$

**Remark 3.17.** The following example shows that the union of two $\widetilde{g}s$-closed sets in $(X, \tau)$ is not, in general, $\widetilde{g}s$-closed in $(X, \tau)$.

**Example 3.18.** Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{c\}, \{a, c\}, X\}$. $A = \{a\}$ and $B = \{c\}$ are $\widetilde{g}s$-closed sets in $(X, \tau)$. But $A \cup B$ is not $\widetilde{g}s$-closed in $(X, \tau)$, because $\widetilde{G}SC(X, \tau) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$.

**Theorem 3.19.** *A set $A$ is $\widetilde{g}s$-closed if and only if $s\,\mathrm{cl}(A)\backslash A$ contains no nonempty $^{\#}gs$-closed subset.*

**Proof. Necessity.** Suppose that $A$ is $\widetilde{g}s$-closed in $(X, \tau)$. Let S be a $^{\#}gs$-closed subset of $s\,\mathrm{cl}(A)\backslash A$. Since $S^c$ is $^{\#}gs$-open and $A \subseteq S^c$, from the definition of $\widetilde{g}s$-closed set it follows that $s\,\mathrm{cl}(A) \subseteq S^c$ ie., $S \subseteq \subseteq (s\,\mathrm{cl}(A))^c$. This implies that $S \subseteq s\,\mathrm{cl}(A) \cap (s\,\mathrm{cl}(A))^c = \varnothing$. Therefore, $S$ is empty.

**Sufficiency.** Suppose that $s\,\mathrm{cl}(A)\backslash A$ contains no non-empty $^{\#}gs$-closed subset. Let $A \subseteq G$ and let $G$ be $^{\#}gs$-open. If $s\,\mathrm{cl}(A) \nsubseteq G$, then $s\,\mathrm{cl}(A) \cap G^c$ is a non-empty $^{\#}gs$-closed subset of $s\,\mathrm{cl}(A)\backslash A$. A contradiction. Therefore, $A$ is $\widetilde{g}s$-closed in $(X, \tau)$. $\Diamond$

**Theorem 3.20.** *If $A$ is $^{\#}gs$-open and $\widetilde{g}s$-closed set in $(X, \tau)$, then $A$ is semi-closed set in $(X, \tau)$.*

**Proof.** Since $A$ is $^{\#}gs$-open and $\widetilde{g}s$-closed in $(X, \tau)$, we get $s\,\mathrm{cl}(A) = A$ and hence $A$ is semi-closed in $(X, \tau)$. $\Diamond$

**Theorem 3.21.** *If $A$ is open and $B$ is $^{\#}gs$-open in $(X, \tau)$, then $A \cap B$ is $^{\#}gs$-open in $(X, \tau)$.*

**Proof.** Similar to Th. 1.9 [6]. $\Diamond$

**Theorem 3.22.** *Let $A \subseteq Y \subseteq X$, where $X$ is a topological space and $Y$ is a subspace. Let $A \in {}^{\#}GSO(X, \tau)$. Then $A \in {}^{\#}GSO(Y, \tau)$.*

**Proof.** Similar to Th. 6 [13]. $\Diamond$

**Theorem 3.23.** *Suppose that $B \subseteq A \subseteq X$, $B$ is a $\widetilde{g}s$-closed set relative to $A$ and that $A$ is open and $\widetilde{g}s$-closed in $(X, \tau)$. Then $B$ is $\widetilde{g}s$-closed in $(X, \tau)$.*

**Proof.** Let $B \subseteq G$, where $G$ is any $^{\#}gs$-open set in $(X, \tau)$. Then $B \subseteq A \cap G$. Since $A$ is open and $G$ is $^{\#}gs$-open, by Th. 3.21, $A \cap G$ is $^{\#}gs$-open in $(X, \tau)$. Since $A \cap G \subseteq A \subseteq X$ and $A \cap G$ is $^{\#}gs$-open in $(X, \tau)$ by Th. 3.22, $A \cap G$ is $^{\#}gs$-open in $A$. So $A \cap G$ is a $^{\#}gs$-open in $A$ such that $B \subseteq A \cap G$. By hypothesis, $B$ is a $\widetilde{g}s$-closed set relative to $A$. Thus, $s\,\mathrm{cl}_A(B) \subseteq A \cap G$. Since $s\,\mathrm{cl}_A(B) = A \cap s\,\mathrm{cl}(B)$, we have $A \cap s\,\mathrm{cl}(B) \subseteq A \cap G$, from which we obtain $A \subseteq G \cup (s\,\mathrm{cl}(B))^c$. Since every semi-open set is $^{\#}gs$-open and union of any two $^{\#}gs$-open sets is also $^{\#}gs$-open set [23], $G \cup (s\,\mathrm{cl}(B))^c$ is a $^{\#}gs$-open set in $(X,\tau)$. By hypothesis, $A$ is $\widetilde{g}s$-closed in $(X,\tau)$, and therefore $s\,\mathrm{cl}(A) \subseteq G \cup$ $\cup (s\,\mathrm{cl}(B))^c$. Since $s\,\mathrm{cl}(B) \subseteq s\,\mathrm{cl}(A)$, $s\,\mathrm{cl}(B) \subseteq G \cup (s\,\mathrm{cl}(B))^c$ and hence $s\,\mathrm{cl}(B) \subseteq G$. Therefore, $B$ is $\widetilde{g}s$-closed relative to $(X,\tau)$. $\Diamond$

**Corollary 3.24.** If $A$ is $\widetilde{g}s$-closed set and $F$ is a semi-closed set in $(X,\tau)$, then $A \cap F$ is a $\widetilde{g}s$-closed set in $(X,\tau)$.

**Proof.** $A \cap F$ is semi-closed set in $A$. Therefore $s\,\mathrm{cl}_A(A \cap F) = A \cap F$ in $A$. Let $A \cap F \subseteq G$, where $G$ is $^{\#}gs$-open in $A$. Then $s\,\mathrm{cl}(A \cap F) \subseteq G$ and hence $A \cap F$ is $\widetilde{g}s$-closed in $A$. By Th. 3.23, $A \cap F$ is $\widetilde{g}s$-closed in $(X,\tau)$. $\Diamond$

**Theorem 3.25.** *If $A$ is $\widetilde{g}s$-closed and $A \subseteq B \subseteq s\,\mathrm{cl}(A)$, then $B$ is $\widetilde{g}s$-closed in $(X,\tau)$.*

**Proof.** Since $B \subseteq s\,\mathrm{cl}(A)$, we have $s\,\mathrm{cl}(B) \subseteq s\,\mathrm{cl}(A)$ and $s\,\mathrm{cl}(B)\backslash B \subseteq$ $\subseteq s\,\mathrm{cl}(A)\backslash A$. But $A$ is $\widetilde{g}s$-closed. Hence $s\,\mathrm{cl}(A)\backslash A$ has no nonempty $^{\#}gs$-closed subsets, neither does $s\,\mathrm{cl}(B)\backslash B$. By Th. 3.19, $B$ is $\widetilde{g}s$-closed in $(X,\tau)$. $\Diamond$

**Theorem 3.26.** *In a topological space $(X,\tau)$, $^{\#}GSO(X,\tau) = SC(X,\tau)$ if and only if every subset of $X$ is a $\widetilde{g}s$-closed set.*

**Proof. Necessity.** Suppose that $^{\#}GSO(X,\tau) = SC(X,\tau)$. Let $A$ be a subset of $(X,\tau)$ such that $A \subseteq G$, where $G \in {}^{\#}GSO(X,\tau)$. Then $s\,\mathrm{cl}(G) = G$. Also $s\,\mathrm{cl}(A) \subseteq s\,\mathrm{cl}(G) = G$. Hence, $A$ is $\widetilde{g}s$-closed in $(X,\tau)$.

**Sufficiency.** Suppose that every subset of $X$ is $\widetilde{g}s$-closed. Let $G \in {}^{\#}GSO(X,\tau)$. Since $G \subseteq G$ and $G$ is $\widetilde{g}s$-closed, we have $s\,\mathrm{cl}(G) \subseteq$ $\subseteq G$. Then $s\,\mathrm{cl}(G) = G$ and $G \in SC(X,\tau)$. Therefore, $^{\#}GSO(X,\tau) \subseteq$ $\subseteq SC(X,\tau)$. If $S \in SC(X,\tau)$, then $S^c$ is semi-open and hence it is $^{\#}gs$-open subset of $(X,\tau)$ [23]. Therefore, $S^c \in {}^{\#}GSO(X,\tau)$. Thus, $^{\#}GSO(X,\tau) = SC(X,\tau)$. $\Diamond$

**Theorem 3.27.** *For each $x \in X$, either $\{x\}$ is the $^{\#}gs$-closed or $\{x\}^c$ is $\widetilde{g}s$-closed in $(X,\tau)$.*

**Proof.** Suppose that $\{x\}$ is not $^{\#}gs$-closed in $(X,\tau)$. Then $\{x\}^c$ is not $^{\#}gs$-open and the only $^{\#}gs$-open set containing $\{x\}^c$ is the space $X$ itself. Therefore, $s\,\mathrm{cl}(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is $\widetilde{g}s$-closed in $(X,\tau)$. $\Diamond$

**Theorem 3.28.** *Let $A$ be a $\widetilde{g}s$-closed set of a topological space $(X,\tau)$. Then,*

(i) *$s\,\mathrm{int}(A)$ is $\widetilde{g}s$-closed in $(X,\tau)$.*

(ii) *If $A$ is regular open, then $p\,\mathrm{int}(A)$ and $s\,\mathrm{cl}(A)$ are also $\widetilde{g}s$-closed sets in $(X,\tau)$.*

(iii) *If $A$ is regular closed, then $p\,\mathrm{cl}(A)$ is also $\widetilde{g}s$-closed set in $(X,\tau)$.*

**Proof.** (i) Since $\mathrm{cl}(\mathrm{int}(A))$ is a closed set in $(X,\tau)$, $A$ and $\mathrm{cl}(\mathrm{int}(A))$ are $\widetilde{g}s$-closed sets in $(X,\tau)$. By ([2]; Th. 1.5(b)) and Th. 3.17, $s\,\mathrm{int}(A) = A \cap \mathrm{cl}(\mathrm{int}(A))$ is $\widetilde{g}s$-closed in $(X,\tau)$.

(ii) Since $A$ is regular open in $(X,\tau)$, by ([2]; Th. 1.5(a), (f)) $s\,\mathrm{cl}(A) = A \cup \mathrm{int}(\mathrm{cl}(A)) = A \cap \mathrm{int}(\mathrm{cl}(A)) = p\,\mathrm{int}(A) = A$. Thus, $s\,\mathrm{cl}(A)$ and $p\,\mathrm{int}(A)$ are $\widetilde{g}s$-closed in $(X,\tau)$.

(iii) Since $A$ is regular closed in $(X,\tau)$, by ([2]; Th. 1.5(e)) $p\,\mathrm{cl}(A) = A \cup \mathrm{cl}(\mathrm{int}(A)) = A$. Thus, $p\,\mathrm{cl}(A)$ is $\widetilde{g}s$-closed in $(X,\tau)$. $\Diamond$

The converses of the statements in the above theorem are not true in generally as the following examples show.

**Example 3.29.** Let $X = \{a,b,c\}$ and $\tau = \{\varnothing,\{a\},\{b,c\},X\}$. $A = \{b\}$ is not a $\widetilde{g}s$-closed set. However, $s\,\mathrm{int}(A) = \varnothing$ is a $\widetilde{g}s$-closed because $\widetilde{GSC}(X,\tau) = \{\varnothing,\ \{a\},\ \{b,c\},\ X\}$.

**Example 3.30.** Let $X = \{a,b,c\}$ and $\tau = \{\varnothing,\{a\},X\}$. Then the set $A = \{b\}$ is not regular open. However, $A$ is $\widetilde{g}s$-closed and $s\,\mathrm{cl}(A) = \{b\}$ is $\widetilde{g}s$-closed, and $p\,\mathrm{int}(A) = \varnothing$ is also $\widetilde{g}s$-closed because $\widetilde{GSC} = \{\varnothing,\ \{b\},\ \{c\},\ \{b,c\},\ X\}$.

**Example 3.31.** Let $X = \{a,b,c\}$ and $\tau = \{\varnothing,\{a\},X\}$. Then the set $A = \{b\}$ is not regular closed. However, $A$ is $\widetilde{g}s$-closed and $p\,\mathrm{cl}(A) = \{b\}$ is $\widetilde{g}s$-closed.

## 4. $\widetilde{g}$-semi-open sets

**Definition 4.1.** A set $A$ in $X$ is called *$\widetilde{g}$-semi-open* (briefly *$\widetilde{g}s$-open*) in $(X,\tau)$ if and only if $A^c$ is $\widetilde{g}s$-closed in $(X,\tau)$.

**Proposition 4.2.**

(i) *Every open set is $\widetilde{g}s$-open, but not conversely.*

(ii) *Every $\alpha$-open set is $\widetilde{g}s$-open, but not conversely.*

(iii) *Every semi-open set is $\widetilde{g}s$-open, but not conversely.*

(iv) *Every $\widetilde{g}s$-open set is $^{\#}gs$-open, but not conversely.*

(v) *Every $\widetilde{g}s$-open set is sg-open, $\beta$-open, gs-open and hence gsp-open, but the converses are not true.*

By means of simple examples, it can be shown that the converses of (i), (ii), (iii), (iv) and (v) of Prop. 4.2 need not hold.

**Theorem 4.3.** *A set $A$ is $\widetilde{g}s$-open if and only if $F \subseteq s\operatorname{int}(A)$ whenever $F$ is $^{\#}gs$-closed and $F \subseteq A$.*

**Proof. Necessity.** Let $A$ be $\widetilde{g}s$-open in $(X,\tau)$ and suppose $F \subseteq A$ where $F$ is $^{\#}gs$-closed. By definition $X \backslash A$ is $\widetilde{g}s$-closed. Also $X \backslash A$ is contained in the $^{\#}gs$-open set $X \backslash F$. This implies $s\operatorname{cl}(X \backslash A) \subseteq X \backslash F$. Now $s\operatorname{cl}(X \backslash A) = X \backslash s\operatorname{int}(A)$ ([7]; Th. 1.6(2)). Hence $X \backslash s\operatorname{int}(A) \subseteq$ $\subseteq X \backslash F$. i.e., $F \subseteq s\operatorname{int}(A)$.

**Sufficiency.** If $F$ is $^{\#}gs$-closed set with $F \subseteq s\operatorname{int}(A)$ whenever $F \subseteq A$, it follows that $X \backslash A \subseteq X \backslash F$ and $X \backslash s\operatorname{int}(A) \subseteq X \backslash F$, ie. $s\operatorname{cl}(X \backslash A) \subseteq X \backslash F$. Hence $X \backslash A$ is $\widetilde{g}s$-closed and $A$ becomes $\widetilde{g}s$-open in $(X,\tau)$. $\lozenge$

**Theorem 4.4.** *If $A$ and $B$ are $\widetilde{g}s$-open sets in $(X,\tau)$, then $A \cup B$ is $\widetilde{g}s$-open in $(X,\tau)$.*

**Proof.** Let $A$ and $B$ are any two $\widetilde{g}s$-open sets in $(X,\tau)$. Then $A^c$ and $B^c$ are $\widetilde{g}s$-closed in $(X,\tau)$. By Th. 3.17, a set $A^c \cap B^c$ is $\widetilde{g}s$-closed and consequently $A \cup B$ is $\widetilde{g}s$-open in $(X,\tau)$. $\lozenge$

**Remark 4.5.** The following example shows that the intersection of two $\widetilde{g}s$-open sets in $X$ is not, in general, $\widetilde{g}s$-open in $(X,\tau)$.

**Example 4.6.** Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, \{a\}, \{c\}, \{a,c\}, X\}$. $A =$ $= \{b,c\}$ and $B = \{a,b\}$ are $\widetilde{g}s$-open sets in $(X,\tau)$. But $A \cap B = \{b\}$ is not $\widetilde{g}s$-open in $(X,\tau)$.

**Theorem 4.7.** *A set $A$ is $\widetilde{g}s$-open in $(X,\tau)$ if and only if $G = X$ whenever $G$ is $^{\#}gs$-open and $s\operatorname{int}(A) \cup A^c \subseteq G$.*

**Proof. Necessity.** Let A be $\widetilde{g}s$-open, G be $^{\#}gs$-open and $s\operatorname{int}(A) \cup$ $\cup A^c \subseteq G$. This gives $G^c \subseteq (s\operatorname{int}(A) \cup A^c)^c = (s\operatorname{int}(A))^c \cap A =$ $= (s\operatorname{int}(A))^c \backslash A^c = s\operatorname{cl}(A^c) \backslash A^c$. Since $A^c$ is $\widetilde{g}s$-closed and $G^c$ is $^{\#}gs$-closed, by Th. 3.19, it follows that $G^c = \varnothing$. Therefore $G = X$.

**Sufficiency.** Suppose that $F$ is $^{\#}gs$-closed and $F \subseteq A$. Then $s\operatorname{int}(A) \cup A^c \subseteq s\operatorname{int}(A) \cup F^c$. It follows by hypothesis that $s\operatorname{int}(A) \cup$ $\cup F^c = X$ and hence $F \subseteq s\operatorname{int}(A)$. Therefore, by Th. 4.3 A is $\widetilde{g}s$-open in $(X,\tau)$. $\lozenge$

**Theorem 4.8.** *If $A \subseteq B \subseteq X$ where $A$ is $\widetilde{g}s$-open relative to $B$ and $B$ is $\widetilde{g}s$-open in $(X,\tau)$, then $A$ is $\widetilde{g}s$-open in $(X,\tau)$.*

**Proof.** Let $F$ be a $^{\#}gs$-closed set in $(X,\tau)$ and suppose that $F \subseteq A$. Then $F = F \cap B$ is $^{\#}gs$-closed in $B$. But $A$ is $\widetilde{g}s$-open relative to $B$. Therefore $F \subseteq s\operatorname{int}_B(A)$. Since $s\operatorname{int}_B(A)$ is a semi-open set relative to $B$, we have $F \subseteq G \cap B \subseteq A$, for some semi-open set $G$ in $(X,\tau)$ ([18]; Th. 3.2). Since $B$ is $\widetilde{g}s$-open in $(X,\tau)$, we have $F \subseteq s\operatorname{int}(B) \subseteq$ $\subseteq B$. Therefore $F \subseteq s\operatorname{int}(B) \cap G \subseteq B \cap G \subseteq A$. It follows then that

$F \subseteq s\,\mathrm{int}(A)$. Therefore, by Th. 4.3 $A$ is $\widetilde{g}s$-open in $(X,\tau)$. ◊

**Theorem 4.9.** *If $s\,\mathrm{int}(A) \subseteq B \subseteq A$ and if $A$ is $\widetilde{g}s$-open in $X$, then $B$ is $\widetilde{g}s$-open in $(X,\tau)$.*

**Proof.** Suppose that $s\,\mathrm{int}(A) \subseteq B \subseteq A$ and $A$ is $\widetilde{g}s$-open in $(X,\tau)$. Then $A^c \subseteq B^c \subseteq s\,\mathrm{cl}(A^c)$ and since $A^c$ is $\widetilde{g}s$-closed in $X$, we have by Th. 3.25, $B^c$ is $\widetilde{g}s$-closed in $(X,\tau)$. Hence $B$ is $\widetilde{g}s$-open in $(X,\tau)$. ◊

**Lemma 4.10** ([7]; Cor. 1.1)**.** *For a subset $A$ of $X$, $s\,\mathrm{int}(s\,\mathrm{cl}(A) - A) = \varnothing$.*

**Proof.** We shall give another proof (shorter) of this result. Let $A \subseteq X$. By ([2]; Th. 1.5(a)) and ([8]; Th. 1.6(4)) we have, $s\,\mathrm{int}(s\,\mathrm{cl}(A)\backslash A) = s\,\mathrm{int}((A \cup \mathrm{int}(\mathrm{cl}(A))) \cap A^c) = s\,\mathrm{int}(\mathrm{int}(\mathrm{cl}(A)) \cap A^c) \subset \mathrm{int}(\mathrm{cl}(A)) \cap s\,\mathrm{int}(A^c)$. Using ([2]; Th. 1.5(6)) we obtain $s\,\mathrm{int}(s\,\mathrm{cl}(A)\backslash A) \subset A^c \cap (\mathrm{int}(cl(A)) \cap \mathrm{cl}(\mathrm{int}(A^c))) \subset A^c \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{int}(A^c) \subset A^c \cap \mathrm{cl}(A \cap A^c)) = \varnothing$. ◊

**Theorem 4.11.** *A set $A$ is $\widetilde{g}s$-closed in $(X,\tau)$ if and only if $s\,\mathrm{cl}(A)\backslash A$ is $\widetilde{g}s$-open in $(X,\tau)$.*

**Proof. Necessity.** Suppose that $A$ is $\widetilde{g}s$-closed in $(X,\tau)$. Let $F \subseteq s\,\mathrm{cl}(A)\backslash A$ where $F$ is ${}^{\#}gs$-closed. By Th. 3.19, $F = \varnothing$. Therefore $F \subseteq s\,\mathrm{int}(s\,\mathrm{cl}(A)\backslash A)$ and by Th. 4.3, $s\,\mathrm{cl}(A)\backslash A$ is $\widetilde{g}s$-open in $(X,\tau)$.

**Sufficiency.** Suppose $s\,\mathrm{cl}(A)\backslash A$ is $\widetilde{g}s$-open in $(X,\tau)$. Let $A \subseteq G$ where $G$ is ${}^{\#}gs$-open in $(X,\tau)$. Then $s\,\mathrm{cl}(A) \cap G^c \subseteq s\,\mathrm{cl}(A) \cap A^c = s\,\mathrm{cl}(A)\backslash A$. Since every semi-closed set is ${}^{\#}gs$-closed and intersection of any two ${}^{\#}gs$-closed sets is also ${}^{\#}gs$-closed set [23], $s\,\mathrm{cl}(A) \cap G^c$ is ${}^{\#}gs$-closed in $(X,\tau)$. Then $s\,\mathrm{cl}(A) \cap G^c \subseteq s\,\mathrm{int}(s\,\mathrm{cl}(A)\backslash A) = \varnothing$, by Th. 4.3 and Lemma 4.10. Hence $s\,\mathrm{cl}(A) \subseteq G$ and $A$ is $\widetilde{g}s$-closed in $(X,\tau)$. ◊

**Theorem 4.12.** *For a subset $A$ of $X$ the following are equivalent:*

(i) *$A$ is $\widetilde{g}s$-closed.*
(ii) *$s\,\mathrm{cl}(A)\backslash A$ contains no non-empty ${}^{\#}gs$-closed set.*
(iii) *$s\,\mathrm{cl}(A)\backslash A$ is $\widetilde{g}s$-open.*

**Proof.** Follows from Th. 3.19 and Th. 4.11. ◊

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