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# APPROXIMATION OF RANDOM ATTRACTORS AND RANDOM INVARIANT MANIFOLDS WITH SUBDIVISION ALGORITHM

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**Abstract**: We introduce efficient numerical methods for rigorous approximation and visualization of random invariant attractors and manifolds. With these techniques we uncover the dynamics of several well known stochastic differential equations which generate random dynamical systems.

## 1. Introduction

A *global random attractor* of a *random dynamical system* describes the complete long term behavior of the solution of a stochastic system. In fact one can reduce the dynamics of the system to those on the random attractor. It is quite clear that the approximation of such objects is one of the main parts in the analysis of the long term behavior. However, random attractors can be very complicated and it is very hard to gain information about their inner structure. But this is a

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very interesting topic because in general it is possible to decompose a random attractor into *random invariant subsets* such as *random fixed points* and *random manifolds*. At this time there are only a few results about the inner structure of random attractors. Recently the existence of *random stable* and *random unstable manifolds* of hyperbolic fixed points were investigated (see Arnold [1] or Duan, Lu and Schmalfuß [5]). Often random manifolds describe transition and separation between different steady states in the random phase space. They are important for stability investigations inside the attractor.

In this paper we use modern numerical methods (introduced for example by Dellnitz and Hohmann in [3], [4], by Keller in [9], by Keller and Ochs in [10]) to uncover some aspects of the dynamics of random dynamical systems generated by well known stochastic differential equations. As prototypical examples we investigate the *stochastic Duffing van der Pol system* and the *stochastic Lorenz system*. For the stochastic Duffing van der Pol system we investigate the existence of local random manifolds in dependence of the noise intensity and support this with numerical simulations. For the stochastic Lorenz system we prove the existence of the global random attractor and give visualizations for it.

The paper is organized as follows: Sec. 2 gives the necessary theoretical background needed for this paper. Here we state some important concepts from the theory of random dynamical systems. In Sec. 3 we describe the algorithm we use for the numerical approximations. The main results and the application to some *stochastic differential equations* can be found in Sec. 4.

## 2. Theoretical background

In this section we give a brief overview of the *random dynamical system* framework and its theoretical concepts.

A random dynamical system is a dynamical system under random influences. In contrast to the classical, nonrandom theory (see [14]) the well known semigroup property is no more valid. For this reason we need more generalized concepts to understand the behavior of random dynamical systems. A more detailed overview of all aspects discussed here can be found in [1] and also in [2].

Before we can define a random dynamical system we need a model for the underlying random perturbation. Such a model is given by a *metric dynamical system*.

**Definition 1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *metric dynamical system* (MDS) $\theta : \mathbb{R} \times \Omega \mapsto \Omega$, is a $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ measurable flow and fulfills the group property $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$. $(\theta_t)_{t\in\mathbb{R}}$ is supposed to be measure preserving, i.e. $\theta_t \mathbb{P} = \mathbb{P}$, $\forall t \in \mathbb{R}$.

An important example for an MDS is the following: Let $(W_t)_{t\in\mathbb{R}}$ be a $d$-dimensional two-sided standard *Wiener process* over the *canonical Wiener space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, where

$$\tilde{\Omega} = \{\omega \in C(\mathbb{R}, \mathbb{R}^d) : \omega(0) = 0\},$$

$\tilde{\mathcal{F}}$ is the Borel $\sigma$-algebra of $\tilde{\Omega}$ and $\tilde{\mathbb{P}}$ is the Wiener measure. Then

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$$

defines the $d$-dimensional *Wiener shift*. This MDS we use in the investigation of random dynamical systems which arise from stochastic differential equations (SDEs).

From now on let $\theta$ be an MDS over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.** We call $(\phi, \theta)$ a *random dynamical system* (RDS) if

$$\phi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$$

is a $\big(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d)\big)$ measurable mapping and satisfies for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$ the *perfect cocycle property*

$$\phi(0, \omega, x) = x, \quad \phi(t + s, \omega, x) = \phi\big(t, \theta_s \omega, \phi(s, \omega, x)\big).$$

The RDS $(\phi, \theta)$ is *continuous*, if for every $t \in \mathbb{R}^+$ and $\omega \in \Omega$ the mapping $x \mapsto \phi(t, \omega, x)$ is continuous.

An RDS can be generated for example by *random differential equations* (RDE) (i.e. ordinary differential equations (ODEs) with random coefficients), *stochastic differential equations* (i.e. ODEs driven by a white noise process) or *random mappings* (see [1, Ch. 2] or [2, Sec. 1.2]).

**Definition 3.** A mapping $D : \Omega \ni \omega \mapsto D(\omega) \subset \mathbb{R}^d$, where $D(\omega) \neq \emptyset$ is closed, we call *random set* if $\omega \mapsto \inf_{y\in D(\omega)} d(x, y)$ is a random variable for every $x \in \mathbb{R}^d$.

It is necessary to restrict ourselves to a subset of all random sets called *set universe*.

**Definition 4.** A nonempty family $\mathcal{D}$ consisting of random sets is called *set universe*, if it is maximal with respect to inclusions.

In this paper we consider the set universe $\mathcal{D}$ which consists of *tempered sets*.

**Definition 5.** A random closed set $D$ is called *tempered (from above)*, if

$$\mathbb{P}\left(\lim_{t\to\pm\infty}\frac{1}{|t|}\log^{+}\sup\{|x| : x\in D(\theta_t\omega)\}=0\right)=1.$$

**Definition 6.** Let $D$ be a random set. $D$ is *invariant* if

$$\phi\big(t,\omega,D(\omega)\big)=D(\theta_t\omega)\qquad\forall t>0,\quad\forall\omega\in\Omega.$$

Particular invariant sets of interest in the non-random theory are global attractors (see for example [14, Ch. 1]). Such sets encode the long term behavior of the system. It is interesting to look for a random counterpart of an attractor. To be able to define attraction in the random context we need a meaningful concept of *convergence toward a random set*. This concept is the *pullback*-convergence. It is given by the relation

$$\lim_{t\to\infty}\operatorname{dist}\big(\overline{\phi\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big)},A(\omega)\big)=0,$$

where dist denotes the *Hausdorff semi-distance* defined by

$$\operatorname{dist}(A,B)=\sup_{x\in A}\inf_{y\in B}d(x,y).$$

**Definition 7.** A *random attractor* $\{A(\omega)\}_{\omega\in\Omega}$ of an RDS $(\phi,\theta)$ is a random set $A\in\mathcal{D}$ such that the following properties hold:

- $A(\omega)$ is a compact non-empty set for all $\omega\in\Omega$;
- $A$ is invariant: $\phi\big(t,\omega,D(\omega)\big)=D(\theta_t\omega)$ $\forall t>0$, $\forall\omega\in\Omega$;
- $A$ is pullback attracting every closed random set $D\in\mathcal{D}$:

$$\lim_{t\to\infty}\operatorname{dist}\big(\overline{\phi\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big)},A(\omega)\big)=0.$$

To prove the existence of a global random attractor it is important to find an *absorbing random set*.

**Definition 8.** A random set $B$ is called *absorbing* in $\mathcal{D}$, if

$$\phi\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big)\subset B(\omega)\qquad\forall t\geq t_0(\omega,D),\quad\forall\omega\in\Omega$$

holds for any random set $D\in\mathcal{D}$.

In [9, Th. 1.2.19] and in [11, Th. 2.4] the following result is proved.

**Theorem 1.** *We assume that $(\phi,\theta)$ is a continuous RDS. Furthermore we assume the existence of a measurable positive invariant and absorbing random set $B$. Then the RDS $(\phi,\theta)$ has a unique global random attractor given by*

$$A(\omega)=\bigcap_{t\in\mathbb{N}}\phi\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big),\quad\forall\omega\in\Omega.$$

The *Multiplicative Ergodic Theorem* for Euclidian Spaces (MET) is a fundamental tool (see [1, Ch. 3]) to introduce techniques like stability theory for RDS, local random bifurcations and random manifold theory.

**Definition 9.** Let $(\phi, \theta)$ be an RDS which possesses a hyperbolic point in the origin. Suppose that we can apply the MET to the linearized cocycle $\Phi$ in the origin. Then there exists the Lyapunov exponents $\lambda_1 > \ldots > \lambda_r > 0 > \lambda_{r+1} > \ldots > \lambda_p$ with $p \leq d$ and a splitting $\mathbb{R}^d = E_u(\omega) \oplus E_s(\omega)$ into a linear unstable and a linear stable subspace. Where $E_u(\omega) = \oplus_{k=r+1}^{p} E_k(\omega)$ and $E_s(\omega) = \oplus_{k=1}^{r} E_k(\omega)$ respectively. Assume $M_u(\omega)$ to be a random invariant set (i.e. $\phi\big(t, \omega, M_u(\omega)\big) = M(\theta_t\omega)$). If we can represent $M$ by a graph of a measurable Lipschitz mapping $m(\cdot, \omega) : E_u \mapsto E_s$ as the set $M_u(\omega) = \{x + m(x, \omega),\, x \in E_u\}$. Then $M_u(\omega)$ is called a *random unstable Lipschitz continuous manifold.*

In general global random unstable manifolds are not graphs. Often this is only true in a small neighborhood. However, in a lot of cases it is possible to prove at least the existence of a local random manifold

$$M_u^{\mathrm{loc}}(\omega) = \{x + m(x, \omega),\, x \in E_u \cap U(\omega)\} = M_u(\omega) \cap U(\omega)$$

for a random neighborhood $U(\omega) = B_\varepsilon(\omega) := \{x \in \mathbb{R}^d : \|x\|_\omega \leq \xi(\omega)\}$ where $\xi : \Omega \mapsto (0, 1]$ is a random variable. We obtain then the global random unstable manifold by $M_u(\omega) := \bigcup_{t \geq 0} \phi\big(t, \omega, M_u^{\mathrm{loc}}(\omega)\big)$. There exists several techniques to prove local random manifolds (see for example [12] and [1, Ch. 7.5]).

It is very difficult to verify particular dynamical characteristics (such as the existence of a random attractor and a random manifold) for an RDS $(\phi, \theta)$ generated by an SDE. Fortunately sometimes it is possible to find a random coordinate transformation called *stationary conjugation mapping* which lead to another RDS $(\psi, \theta)$ which is easier to handle. This technique is described for example in [8] and [9, Sec. 1.2.2].

**Definition 10.** Assume that $V \subset \Omega \times \mathbb{R}^d$ and $W \subset \Omega \times \mathbb{R}^d$ are $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable. Suppose furthermore that $V(\omega) := \{x \in \mathbb{R}^d : (\omega, x) \in V\}$ and $W(\omega) := \{x \in \mathbb{R}^d : (\omega, x) \in W\}$. A family of mappings $\{T(\omega)\}_{\omega \in \Omega}$ with $T(\omega) : V(\omega) \mapsto W(\omega)$ measurable is called *stationary conjugation mapping* if it is $\forall \omega \in \Omega$ a homeomorphism and its inverse is also measurable.

Two RDSs $(\phi, \theta)$ and $(\psi, \theta)$ are called *conjugated* if $\forall \omega \in \Omega$ the relation

$$\phi(t,\omega) = T(\theta_t\omega) \circ \psi(t,\omega) \circ T^{-1}(\omega)$$

is satisfied.

The main ingredient of the stationary conjugation mappings in this paper is the *Ornstein–Uhlenbeck-process*, which is the stationary solution of the following linear SDE

$$dz = -\mu\, z + dW, \quad z(0) = z_0.$$

Here $\mu > 0$ and $W$ denotes an one dimensional two sided standard Wiener process.

**Lemma 1.** *Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be the canonical Wiener space and let $\{\theta\}_{t\in\mathbb{R}}$ be the Wiener shift. There exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant subset $\hat{\Omega} \in \tilde{\mathcal{F}}$ of full measure with sublinear growth, i.e.*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0,\ \omega \in \hat{\Omega}.$$

*Then $\forall \omega \in \hat{\Omega}$ the Ornstein–Uhlenbeck-process has the following properties:*

(i) *The random variable $z(\omega) = -\mu \int_{-\infty}^{0} e^{\mu\, s}\omega(s)ds$ exists and generates the stationary solution $z(\theta_t\omega) = \omega(t) - \mu e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}\omega(s)ds$. The mapping $t \mapsto z(\theta_t\omega)$ is continuous.*

(ii) *The stochastic process $z$ grows sublinear, i.e.* $\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0$.

(iii) *We have* $\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0$.

The proofs can be found in [5] and also in [9, Ch. 1, p. 29].

## 3. Subdivision algorithm

We are going to use in this paper a set oriented algorithm for the analysis of the long term behavior of dynamical systems. This algorithm is known as the *subdivision algorithm* and was developed by Dellnitz and Hohmann in [3] and [4]. Later on, Keller and Ochs extended this algorithm to RDSs in [10]. We point out that there exists also an earlier approach to use a similar set oriented technique named *cell mapping*, described in Hsu [6].

### Subdivision algorithm for autonomous dynamical systems

Given the solution operator of an autonomous dynamical system (see [14, Ch. 1])

$$S : \mathbb{R}^+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$$

the subdivision algorithm starts with a $d$-dimensional box

$$Q = B(c, r) = \{(x_1, \dots , x_d) \in \mathbb{R}^d : |x_i - c_i| \leq r_i,\ i = 1, \dots , d\}$$

with center $c = (c_1, \dots , c_d) \in \mathbb{R}^d$ and radius $r = (r_1, \dots , r_d) \in \mathbb{R}^d$, $r_i > 0$, $i = \overline{1, d}$. In general it is possible that this box does not cover the complete attractor. Because of this we speak in the following about the $Q$-relative attractor (for more details see [3] and also [4]). The $Q$-relative attractor can be approximated by

$$A_Q := \bigcap_{n \geq 0} S(nT)Q,$$

where $T$ is an appropriate time step which depends on the properties of the dynamics of $S$. In general we can obtain faster convergence of the algorithms for a properly chosen $T$. Moreover, if $S(t)$ possesses a global attractor with $A \subset Q$ then we have $A = A_Q$.

The approximation is done by generating a sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of box collections $\mathcal{B}_k = \{B_j : 1 \leq j \leq n_k\}$ with $Q \supseteq Q_k = \bigcup_{j=1}^{n_k} B_j$ where $B_j \in \mathcal{B}_k$. The first collection $\mathcal{B}_0$ contains only the box $Q$. The next collections were generated successively in two steps.

**1. Subdivision.** Assume we are in the $k$-th step. The collection $\hat{\mathcal{B}}_k$ is obtained by subdividing every box from $\mathcal{B}_k$ and inserting the resulting boxes into $\hat{\mathcal{B}}_k$. For example a given box $B(c, r) \in \mathcal{B}_k$ is separated into $B_-(c^-, \hat{r})$ and $B_+(c^+, \hat{r})$ with

$$\hat{r}_i = \begin{cases} r_i & \text{for } i \neq j \\ r_i/2 & \text{for } i = j \end{cases} \quad \text{and} \quad c_i^{\pm} = \begin{cases} c_i & \text{for } i \neq j \\ c_i \pm r_i/2 & \text{for } i = j \end{cases}.$$

The subdivision is done in every step cyclic in another dimension.

**2. Selection.** Assume we are in the $k$-th step. The selection has the task to decide which boxes $B \in \hat{\mathcal{B}}_k$ are a part of the covering of the attractor. The new collection $\mathcal{B}_{k+1}$ is created by

$$\mathcal{B}_{k+1} := \{B \in \hat{\mathcal{B}}_k : B \cap S(T)Q_k \neq \emptyset\}$$

where $\hat{\mathcal{B}}_k$ is the resulting collection of the preceding subdivision step. To make this decision we map for each box $B \in \hat{\mathcal{B}}_k$ a number of testpoints $x_i \in B$. We have to do this as an approximation of our set valued cocycle. It is not possible to map a complete box in a numerical way. There are several strategies to distribute testpoints over the box (randomly, equidistant, etc.).

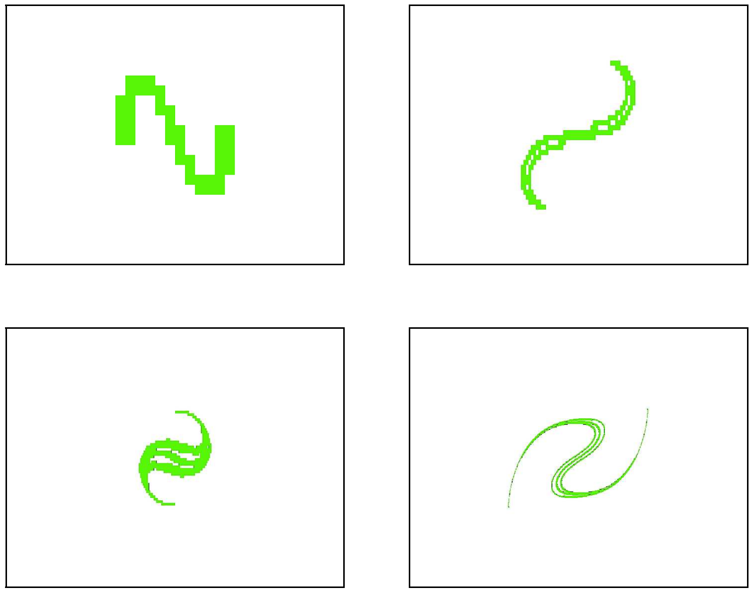

Figure 1. Convergence of the box families towards the random attractor $A(\omega)$ of the stochastic Duffing van der Pol system (see Sec. 4).

**Subdivision algorithm for RDSs**

Some modifications are necessary if we want to apply the algorithm also to RDSs (see [10]). In this case the algorithm can be divided into two parts. Assuming we have given an RDS $(\phi, \theta)$ which possesses a global random attractor $\{A(\omega)\}_{\omega\in\Omega}$.

**Part 1.** We start with a fixed box $Q$ which contains the attractor with high probability. Note, in general we can not be sure that our box covers the random attractor. In fact there exists no comp act set which contains all realizations of $\{A(\omega)\}_{\omega\in\Omega}$ $\mathbb{P}$-almost sure. Then we perform $m$ subdivision and $m$ selection steps as described above. Of course now we use the random map $\phi(T, \tilde{\omega})$ in the selection step, where $T$ is an appropriate time step. We have to fix a $\tilde{\omega}$ for the mapping of all boxes (i.e. testpoints). To take the random character of our RDS into account we have to apply $\theta_T\tilde{\omega}$ after each selection step.

**Part 2.** We perform only selection steps to obtain a better convergence towards the attractor. In fact it is not necessary to apply Part 1. We could instead subdivide $Q$ immediately up to the final depth $m$ and then apply the selection steps to these $2^m$ boxes. Of course a lot of these boxes are unnecessary and we would waste computer resources. The only reason to apply Part 1 is because of efficiency. However, with every selection step the number of boxes in the actual cover decreases until we reach a (sub)optimal covering. We end in the $\omega$-fiber with an approximation of $A(\omega)$.

Fig. 1 shows an application of the algorithm to the stochastic Duffing van der Pol system.

## 4. Results

In this section we will give some analytical and simulation results for SDEs which generate RDS. We will use stationary conjugation mappings to obtain pathwise defined RDEs for which we can make the necessary proofs. From now on we restrict the probability space as in Lemma 1.

**Stochastic Duffing van der Pol system**

The stochastic Duffing van der Pol system (DvP) is a forced nonlinear oscillator given by

$$\ddot{x} = (x^2 - \beta)\dot{x} - U'(x) + \sigma\, x\, \xi_t, \quad U(x) = \frac{x^2}{2} + \frac{x^4}{4}, \quad \beta, \sigma > 0, \tag{1}$$

where $\xi_t$ denotes a white noise process. We rewrite (1) as a second order Stratonovich SDE

$$\begin{aligned} dx_1 &= x_2\, dt \\ dx_2 &= (-x_1 - x_1^3 + (\beta - x_1^2)x_2)\, dt + \sigma x_1 \circ dW \end{aligned} \tag{2}$$

where $W$ denotes an one dimensional two sided standard Wiener process. At first we prove the existence of the global random attractor following [9]. We will give here only a short outline.

**Theorem 2.** *The second order SDE* (2) *generates a perfect cocycle and possesses a global random attractor.*

**Proof.** Using the stationary conjugation mapping $T\big(\theta_t\omega, (y_1, y_2)\big) :=$ $:= \big(x_1, x_2 - \sigma\, z(\theta_t\omega)\big)$ it is possible to show that (2) is stationary conjugated to the RDE

$$\begin{aligned} \dot{y}_1 &= y_2 + \sigma y_1\, z(\theta_t\omega) \\ \dot{y}_2 &= -y_1 - y_1^3 + (\beta - y_1^2)\big(y_2 + \sigma y_1\, z(\theta_t\omega)\big) + \sigma\mu y_1\, z(\theta_t\omega) - \\ &\quad - \sigma\big(y_2 + \sigma y_1\, z(\theta_t\omega)\big)\, z(\theta_t\omega). \end{aligned} \tag{3}$$

This implies that (2) generates a perfect cocycle in the sense of Def. 2.

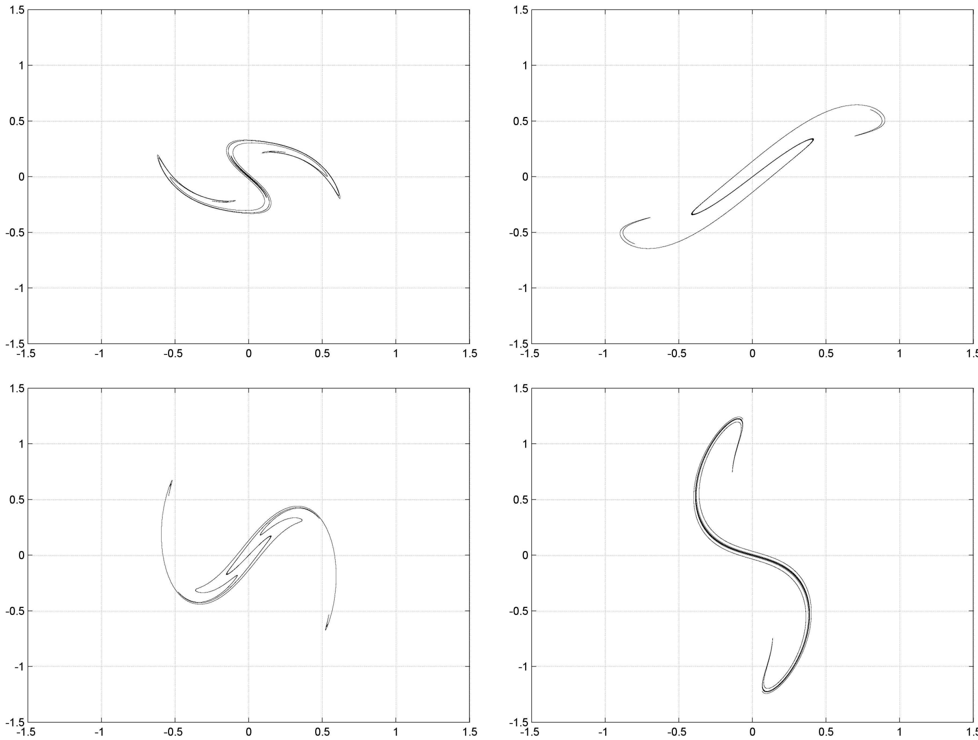


Figure 2. Approximation of the global random attractor $A(\omega)$ of the stochastic DvP system (2) and its evolution in time for $\beta = 0$ (this is the regime where the solution $x = 0$ of the autonomous system undergoes a Hopf bifurcation) and $\sigma = 1$ in a subdivision depth of 30.

We consider the Lyapunov function $V : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $V(y_1, y_2) = U(y_1) + \frac{y_2^2}{2}$ which describes the energy of (3). With $g(y_1) := (\beta - y_1^2)$ the first derivative with respect to $t$ is given by

$$\begin{aligned}\frac{d}{dt}V(y_1,y_2) &= \dot{y_1}U'(y_1) + \dot{y_2}y_2 = \\ &= U'(y_1)\,\sigma y_1\, z(\theta_t\omega) + \Big(g(y_1)\big(y_2 + \sigma y_1\, z(\theta_t\omega)\big) + \\ &\quad + \sigma\mu y_1\, z(\theta_t\omega) - \sigma\big(y_2 + \sigma y_1\, z(\theta_t\omega)\big)\, z(\theta_t\omega)\Big) y_2.\end{aligned}$$

Using particular properties of $U$ and some basic estimation techniques we end up with the random affine differential inequality

$$\frac{d}{dt}V(y_1,y_2) \le -\delta\Big(1 - K\left(|z(\theta_t\omega)| + |z(\theta_t\omega)|^2\right)\Big)V(y_1,y_2) + L.$$

We consider the affine differential equation

$$\frac{d}{dt}v(t,\omega) = -\delta\Big(1 - K\left(|z(\theta_t\omega)| + |z(\theta_t\omega)|^2\right)\Big)v(t,\omega) + L \tag{4}$$

which possesses the stationary solution

$$\tilde{v}(\omega) = \lim_{t\to\infty} v(t,\theta_{-t}\omega) = L\int_{-\infty}^{0} e^{\delta u + \tilde{K}\int_u^0\left(|z(\theta_s\omega)|+|z(\theta_s\omega)|^2\right)ds}\,du. \tag{5}$$

Hence this is a random one point attractor. Because of the comparison principle $V(y_1,y_2) \le v(t,\omega)$ it is clear that there exists an interval $[0,\tilde{v}(\omega)]$ which absorbs every solution of (3). The absorbing set for (3) is then

$$B(\omega) := V^{-1}([0,\tilde{v}(\omega)])$$

where $V^{-1}$ denotes the preimage of $V$. For any random tempered set $D \in \mathcal{D}$ we have

$$V(\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega))) \subset v(t,\theta_{-t}\omega, V(D(\theta_{-t}\omega))).$$

In addition, we know that

$$\lim_{t\to\infty} v(t,\theta_{-t}\omega, V(D(\theta_{-t}\omega))) = [0,\tilde{v}(\omega)] = V(B(\omega)).$$

This implies the inclusion

$$V(\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega))) \subset V(B(\omega))$$

for $t$ sufficiently large. Applying $V^{-1}$ on both sides shows that $B(\omega)$ is an absorbing set. It is easy to see that $B \in \mathcal{D}$. Then Th. 1 implies the existence of a global random attractor. $\Diamond$

In the following we are interested in the situation when system (2) undergoes a Hopf bifurcation. For our investigations we need the linearized system of (2) at the origin $(0,0)$, i.e.

$$\begin{aligned} dv_1 &= v_2\,dt \\ dv_2 &= (-v_1 + \beta x_2)\,dt + \sigma v_1 \circ dW \end{aligned} \tag{6}$$

or equivalently

$$dv = Av\,dt + \sigma Bv \circ dW$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & \beta \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The eigenvalues of $A$ and some additional properties imply that the autonomous version ($\sigma = 0$) of system (2) undergoes a Hopf bifurcation if $\beta$ crosses 0. In the stochastic case (i.e. $\sigma > 0$) it is not possible to give a characterization of the dynamics near the origin with aid of the eigenvalues. Here we need to investigate the Lyapunov exponents. Note that the MET can be applied to (6).

**Theorem 3.** *The second order SDE* (2) *possesses for every* $\sigma > 0$ *large enough a local random unstable manifold with origin* $(0, 0)$.

**Proof.** From [7] we know that there exists an explicit formula for the Lyapunov exponents of (6) independent of $\omega$ given by $\lambda_{1,2}(\beta, \sigma) = = \beta \pm \frac{\sigma^2}{4} C(\beta, \sigma)$ where $C(\beta, \sigma)$ is the fourth moment of the invariant measure of a linear diffusion. We assume $\lambda_1 > \lambda_2$. In our situation (note $\beta = 0$) the trace formula $\lambda_1(0, \sigma) + \lambda_2(0, \sigma) = \mathrm{trace}(A) = 0$ holds. It is enough to consider only the top Lyapunov exponent $\lambda_1$. The explicit formula for $\lambda_1$ forms a real valued monotone increasing function with respect to $\sigma \geq 0$. So this gives us an exponential dichotomy condition in dependence of $\sigma$. This shows that the fixed point in the origin is hyperbolic for some $\sigma > 0$ sufficiently large. It follows that we have also this exponential dichotomy condition for the RDE (3). We can apply Theorem from [1, Th. 7.5.17] to (3). Hence (2) possesses a local random unstable manifold for $\sigma$ large enough. $\diamondsuit$

Fig. 3 illustrates the effect of the exponential dichotomy. If we increase $\sigma$ the limit cycle (upper left) is destroyed and the random unstable manifold (lower right) appears. Obviously the presence of large enough noise leads to a bifurcation of the system.

**Stochastic Lorenz system**

We consider the stochastic Lorenz system given by the following system of Stratonovich SDEs

$$\begin{aligned} dx &= \sigma(y - x)dt + q\, x \circ dW \\ dy &= -(\sigma x + y + xz)dt + q\, y \circ dW \\ dz &= (xy - bz)dt - b(r + \sigma)dt + q\, z \circ dW. \end{aligned} \tag{7}$$

Here $b, r, \sigma, q$ are arbitrary positive constants and $W$ denotes a two sided one dimensional Wiener process. This system was also investigated in [8] and [11]. The dynamics of the autonomous version of the system is studied in [13] and [14, Ch. 1].

We can split the equation into the linear and the nonlinear part and write down the system (7) in the more abstract form

$$du + (Au + F(u))\, dt = f\, dt + q\, u \circ dW, \qquad u(0) = u_0 \in \mathbb{R}^3 \tag{8}$$

with $u = (x, y, z)^T$. The linear part is given by the linear operator

$$A = \begin{pmatrix} \sigma & -\sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & b \end{pmatrix}$$

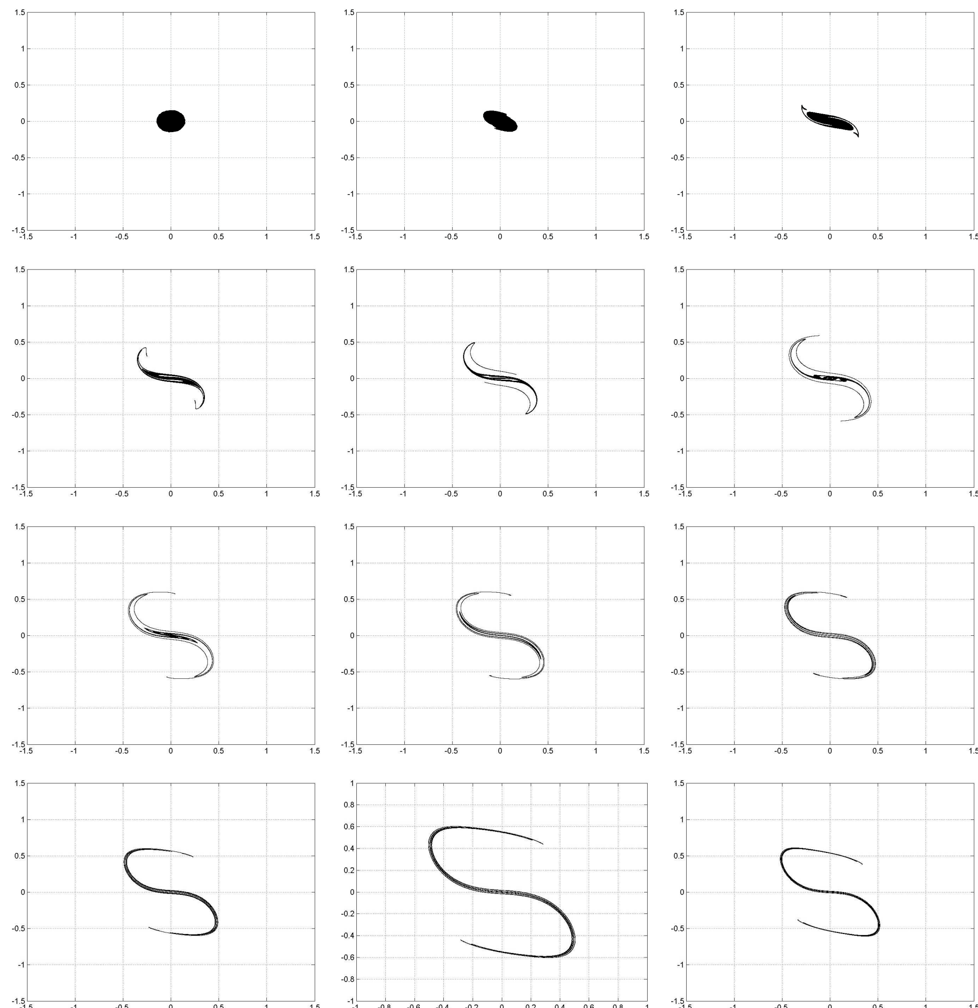

Figure 3. Approximation of the global random attractor $A(\omega)$ of the stochastic DvP system (2) for $\beta = 0$ (this is the regime where the solution $x = 0$ of the autonomous system undergoes a Hopf bifurcation) for increasing $\sigma$.

and the nonlinear part by $F(u) = (0, xz, -xy)^T$. The forcing term $f$ is given by $f := (0, 0, -b(r+\sigma))^T$. The linear operator is positive definite and we can find for $\langle Au, u\rangle$ the following estimate

$$\left\langle Au, u \right\rangle \geq l(x^2 + y^2 + z^2) = l|u|^2 \tag{9}$$

where $l = \min\{1, \sigma, b\}$. The nonlinear part has also some interesting properties, we have

$$\langle F(u), u \rangle = xyz - xyz = 0. \tag{10}$$

Let $\hat{F}(u_1, u_2) = (0, x_1 z_2, -x_1 y_2)^T$, where $u_i = (x_i, y_i, z_i)$. Then $F(u) = \hat{F}(u, u)$. Note that $\hat{F}$ is bilinear and has the properties

$$\begin{aligned} &\left\langle \hat{F}(u_1, u_2), u_2 \right\rangle = 0, \quad \left\langle \hat{F}(u_1, u_2), u_3 \right\rangle = - \left\langle \hat{F}(u_1, u_3), u_2 \right\rangle, \\ &|\hat{F}(u_1, u_2)|^2 \leq |u_1|^2 |u_2|^2. \end{aligned} \tag{11}$$

**Theorem 4.** *The SDE* (8) *generates a perfect cocycle and possesses a global random attractor.*
**Proof.** We consider the following RDE

$$\frac{d\bar{u}}{dt} + (A - q\, z(\omega))\bar{u} + F(\bar{u}) = e^{-q\, z(\omega)} f, \quad \bar{u}(0) = \bar{u}_0 = e^{-q\, z(\omega)} u(0) \tag{12}$$

which generates the cocycle $(\psi, \theta)$. It is easy to show that the stochastic equation (8) is connected to (12) by using the following stationary conjugation mapping $T(\theta_t \omega) = e^{q\, z(\theta_t \omega)}$. This implies that (8) generates a perfect cocycle in the sense of Def. 2.

To prove the existence of the global random attractor we show for all $\delta > 0$ the existence of the random absorbing set

$$B^\delta(\omega) = \{v \in \mathbb{R}^3 : |v| \leq (1 + \delta) 2b(r + \sigma) p_{l,q}(\omega)\} \tag{13}$$

where $p_{l,q}(\omega)$ is a finite random variable. We use the conjugated equation (12) and consider the norm of the cocycle $(\psi, \theta)$ with initial value $\bar{u}_0$. By using (9) and (10) we have

$$\frac{d}{dt} |\psi(t, \omega, \bar{u}_0)| + 2(l - q\, z(\omega))\, |\psi(t, \omega, \bar{u}_0)| \leq e^{-q\, z(\omega)}\, 2b\, (r + \sigma).$$

The variation of constant formula and starting in the $-t$ time fiber gives

$$\begin{aligned} \psi(t, \theta_{-t}\omega, |\bar{u}_0(\theta_{-t}\omega)|) \leq 2b(r + \sigma) \int_{-t}^{0} e^{2l\, s - 2q \int_0^s z(\theta_\tau \omega) d\tau} e^{-q\, z(\theta_s \omega)} ds + \\ + e^{-2lt + 2q \int_0^t z(\theta_{-t}\theta_\tau \omega) d\tau} |\bar{u}_0(\theta_{-t}\omega)|. \end{aligned}$$

From the last inequality it is easy to see that for every $\delta > 0$ and $\bar{u}_0$

$$\lim_{t \to \infty} \psi(t, \theta_{-t}\omega, |\bar{u}_0(\theta_{-t}\omega)|) \leq (1 + \delta) 2b(r + \sigma) p_{l,q}(\omega)$$

holds, where

$$p_{l,q}(\omega) = \int_{-\infty}^{0} e^{2ls - 2q \int_0^s z(\theta_\tau \omega) d\tau} e^{-q\, z(\theta_s \omega)}\, ds$$

defines a finite random variable. Furthermore the finite random bound $2b(r+\sigma)p_{l,q}(\omega)$ is positive invariant. It follows

$$\psi(t, \omega, 2b(r+\sigma)p_{l,q}(\omega)) \leq 2b(r+\sigma)p_{l,q}(\theta_t \omega).$$

We identified $2b(r+\sigma)p_{l,q}(\omega)$ as a positive invariant random variable. Hence, there exists for every $\delta > 0$ an absorbing random set given by (13). The existence of the global attractor follows by using Th. 1. $\Diamond$

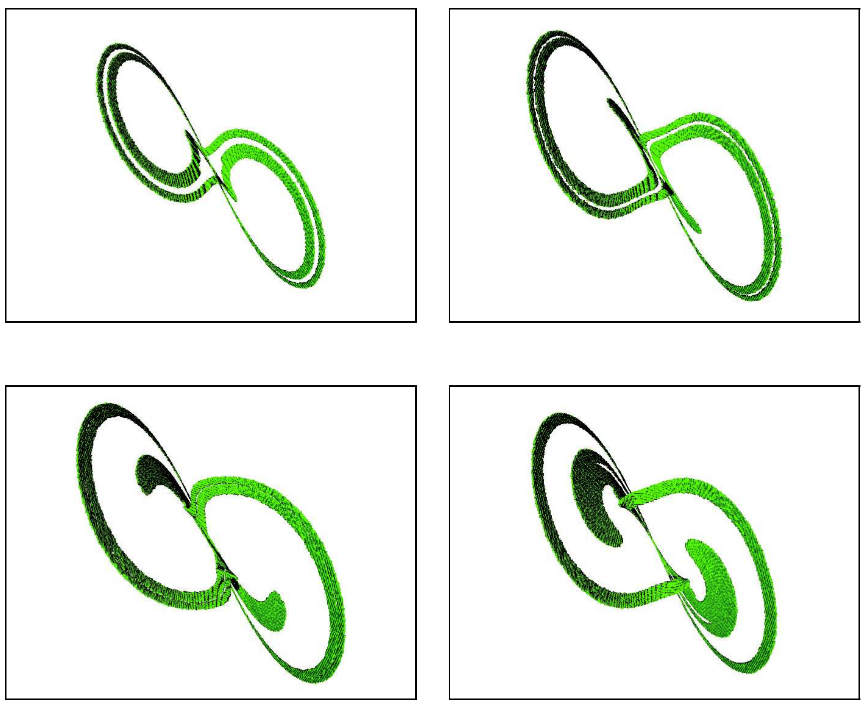

Figure 4. Approximation of the global random attractor $A(\omega)$ of the stochastic Lorenz system (7) and its evolution in time for for $\sigma = 10$, $b = 8/3$, $r = 28$ and $q = 1.0$ in a subdivision depth of 30.

In our simulations we obtain the very complicated global random attractor visualized in Fig. 4. The attractor is folded to itself infinitely often. This makes the attractor a complicated fractal object. We believe that also in this case the presence of noise lead to a bifurcation of the system.

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