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THE SEMIGROUP OF RIGHT IDEALS OF A RING

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Abstract: The structure of the multiplicative semigroup of all right ideals of a ring is investigated. Of particular interest is the case when such a semigroup is von Neumann regular. Examples are given which illustrate and delimit the theory developed.

1. Introduction

This paper investigates connections between the structure of a ring and that of its multiplicative group of right ideals. Here R is a ring with unity, $1 \neq 0$, and $\mathbb{R}(R)$ is the semigroup, under right ideal multiplication, of all right ideals of R. Observe that R is a right identity for $\mathbb{R}(R)$ and the zero ideal (0) of R is the zero for the semigroup $\mathbb{R}(R)$.

In Sec. 2 the Green's relations for $\mathbb{R}(R)$ are characterized and some radical-like ideals of $\mathbb{R}(R)$ are discussed. In Sec. 3 von Neumann regularity and some related conditions are considered. Recall that a semigroup S is regular (von Neumann regular) if for each $s \in S$ there

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exists $s' \in S$ such that $s = ss's$. Such a regular semigroup is inverse if s' is unique. It is well known that S is inverse if and only if it is regular and the idempotents of S commute with each other; see [2, Th. 1.17]. If every element of a semigroup S is idempotent, then S is called a band. A commutative band is called a semilattice.

In Sec. 4 the concept of an "almost" property is introduced and $\mathbb{R}(R)$ is characterized when it is almost a rectangular band, which in turn is equivalent to the ring R being simple. Examples are given throughout which illustrate and delimit the theory developed.

2. General results

A semigroup S is ordered if S is a partially ordered set and the multiplication in S is consistent with the partial order; i.e., if $s, t, a \in S$ and $s \leq t$, then $as \leq at$ and $sa \leq ta$. A semigroup S is left ordered if $st \leq s$ for each $s, t \in S$.

Of course, $\mathbb{R}(R)$ is a partially ordered set under set inclusion, but it is not an ordered semigroup. If H, K are right ideals of R, we do not have, in general, that $HK \subseteq K$. However, we still have the following result.

Theorem 2.1. (a) *The semigroup $\mathbb{R}(R)$ is a left ordered semigroup under set inclusion of right ideals.*

(b) *The element R is the unique maximum element of $\mathbb{R}(R)$, and the zero ideal of R is the unique minimum element of $\mathbb{R}(R)$.*

(c) *If $\mathbb{R}(R)$ has DCC on chains of elements, then it has ACC on chains of elements.*

Proof. The proofs of (a) and (b) are immediate from the definitions.

(c) This is the Hopkins–Levitzki Theorem; see [5, Th. 18.13]. $\Diamond$

We next describe the Green's relations for $\mathbb{R}(R)$. Here we use the standard notation for the Green's relations in a semigroup as found in [2, Sec. 2.1].

Theorem 2.2. *In* $\mathbb{R}(R)$,

(a) $\mathcal{J} = \mathcal{D} = \mathcal{L}$;

(b) $\mathcal{R} = \mathcal{H} = \epsilon$, *the equality relation.*

Proof. (a) Let $H, K \in \mathbb{R}(R)$ such that $H \mathcal{J} K$. Then there exist X, Y, U, W such that $H = XKY$ and $K = UHW$. We show that $\mathbb{R}(R)H = \mathbb{R}(R)K$. For any $A \in \mathbb{R}(R)$ we have $AH = AXKY \subseteq$ $\subseteq AKY \subseteq AK$. Dually, $AK = AUHW \subseteq AHW \subseteq AH$. This implies that $AH = AK$. Hence $\mathbb{R}(R)H \subseteq \mathbb{R}(R)K$. The same argument shows that $\mathbb{R}(R)K \subseteq \mathbb{R}(R)H$. Therefore $H\mathcal{L}K$. Since $\mathcal{D} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{D} = \mathcal{L}$.

(b) Let $H, K \in \mathbb{R}(R)$ such that $H\mathcal{R}K$. Then there exist $X, Y \in \mathbb{R}(R)$ such that $H = KX$ and $K = HY$. Then $H \subseteq K$ and $K \subseteq H$, so that $H = K$. Since $\mathcal{H} \subseteq \mathcal{R}$, we have $\mathcal{R} = \mathcal{H} = \epsilon$. $\Diamond$

We next consider some "radical-like" ideals of $\mathbb{R}(R)$. First some terminology is needed.

Definition 2.3. Let S be a semigroup with zero. An element $s \in S$ is a *left (right) zero-divisor* in S if there exists a nonzero $t \in S$ such that $st = 0$ (respectively, $ts = 0$). An ideal I of S is *completely prime* if S/I has no zero-divisors; i.e., if $s, t \in S$ such that $st = 0$, then $s = 0$ or $t = 0$. We denote by $D_l(S)$ and $D_r(S)$ the sets of left and right zero-divisors in S, respectively.

Proposition 2.4. (a) $D_l(\mathbb{R}(R))$ *is a completely prime ideal of* $\mathbb{R}(R)$.

(b) $D_l(\mathbb{R}(R)/D_l(\mathbb{R}(R))) = 0$.

(c) $D_r(\mathbb{R}(R))$ *is a completely prime ideal of* $\mathbb{R}(R)$.

(d) $D_r(\mathbb{R}(R)/D_r(\mathbb{R}(R))) = 0$.

Proof. Let $H \in D_l(\mathbb{R}(R))$. Then there exists $0 \neq K \in \mathbb{R}(R)$ such that $HK = 0$. For each $X \in \mathbb{R}(R)$ we have $(XH)K = X(HK) = 0$ and $(HX)K \subseteq H = 0$, so $D_l(R)$ is an ideal of $\mathbb{R}(R)$. If $HK \in D_l(\mathbb{R}(R))$ for $0 \neq H, K \in \mathbb{R}(R)$, then there exists $0 \neq Y \in \mathbb{R}(R)$ such that $HKY = 0$. Either $KY = 0$, in which case $K \in D_l(\mathbb{R}(R))$, or $KY \neq 0$, in which case $H \in D_l(\mathbb{R}(R))$; thus $D_l(\mathbb{R}(R))$ is completely prime.

Part (b) follows immediately from part (a). The proofs of (c) and (d) are strictly analogous to those of parts (a) and (b), respectively. $\Diamond$

The beginning of insight into the structure of $\mathbb{R}(R)$ in connection with the ideal structure of R is given by the next result, whose proof follows from routine considerations.

Proposition 2.5. *Each of the following is a right ideal in the semigroup* $\mathbb{R}(R)$*:*

(a) $\mathbb{I}(R)$*, the set of all ideals of* R *(which is a monoid);*

(b) *the set of all nilpotent (right, two-sided) ideals of* R*;*

(c) *the set of all nil (right, two-sided) ideals of* R*;*

(d) *the set of all minimal (right, two-sided) right ideals of* R *and* (0).

Furthermore, the set of all nilpotent right ideals of R *is an ideal of* $\mathbb{R}(R)$.

For a semigroup S with zero we use $N(S)$ for the set of all nilpotent elements of S. Observe that $N(\mathbb{R}(R)/N(\mathbb{R}(R))) = 0$.

In light of Prop. 2.4, it is natural to ask if $N(\mathbb{R}(R))$ is completely prime. The next example shows that this is not the case.

Example 2.6. The ideal $N(\mathbb{R}(R))$ is not completely prime.

Let $R = \begin{pmatrix} Z & Z_4 \\ 0 & Z_4 \end{pmatrix}$, let $H = \begin{pmatrix} 0 & Z_4 \\ 0 & Z_4 \end{pmatrix}$, and let $K = \begin{pmatrix} (2) & (\bar{2}) \\ 0 & (\bar{2}) \end{pmatrix}$. Then $HK = \begin{pmatrix} 0 & (\bar{2}) \\ 0 & (\bar{2}) \end{pmatrix}$, which is nilpotent. Hence, $HK \in N(\mathbb{R}(R))$ but neither H nor K is nilpotent, so $H, K \notin N(\mathbb{R}(R))$.

Corollary 2.7. *If $\mathbb{R}(R)$ is 0-simple, then R is semiprime.*

Proof. If $N(R) = \mathbb{R}(R)$, then R would be nilpotent, contrary to R having unity. Otherwise, the only nilpotent right ideal of R is the zero ideal; i.e., R is semiprime. $\Diamond$

The converse of Cor. 2.7 is false, as the case where R is the sum of two fields illustrates.

Observe that if $\mathbb{R}(R)$ is 0-simple and $(\mathbb{R}(R), \supseteq)$ satisfies the DCC, then R is right Artinian and semisimple. So, by the Artin–Wedderburn Theorem, R is regular and $\mathbb{R}(R)$ is a band.

Finally, we note that if R_1 and R_2 are isomorphic rings, then $\mathbb{R}(R_1) \cong \mathbb{R}(R_2)$. The converse fails badly; e.g., if R_1 and R_2 are non-isomorphic skew fields then $\mathbb{R}(R_1) \cong \mathbb{R}(R_2)$.

3. Von Neumann regular

In this section we consider the consequences of $\mathbb{R}(R)$ being regular (von Neumann regular) together with some related conditions. Recall that if the ring R is regular, then every right ideal of R is idempotent, [6, p. 2]. Hence R regular implies $\mathbb{R}(R)$ regular. However, the converse does not hold. If R is a simple ring, then every one-sided ideal of R is idempotent; see Prop. 4.3 below, or [9, Cor. 18]. For example, the n-th Weyl algebra, W_n, over a field of characteristic zero is a simple ring without divisors of zero and is not regular; see [7, p. 19]. We use $J(R)$ for the Jacobson radical of R.

Theorem 3.1. *If $\mathbb{R}(R)$ is regular, then $\mathbb{R}(R)$ is a band and $J(R) = 0$.*

Proof. Let $H, K \in \mathbb{R}(R)$ such that $H = HKH$. Then $H \subseteq HK \subseteq H$, or $H = HK$, which is an idempotent. So $\mathbb{R}(R)$ regular implies $\mathbb{R}(R)$ is a band.

Now let $r \in J(R)$. Then $rR = (rR)^2$ and hence $r = rxry$ for some $x, y \in R$. But $xry \in J(R)$, so $1 - xry$ is invertible in R, and hence $r = 0$. $\Diamond$

Rings in which every right ideal is idempotent have been investigated by Courter [3] and Ramamurthi [9], albeit not in the context of studying the semigroup $\mathbb{R}(R)$. Ramamurthi called such rings "right weakly regular", and he showed that such a ring has zero Jacobson radical.

Proposition 3.2. *Let* $\mathbb{R}(R)$ *be regular.*

(a) *If* R/P *is regular for each prime ideal* P *of* R*, then* R *is regular.*

(b) *If* R *has bounded index of nilpotence, then* R *has no nil one-sided ideals.*

Proof. (a) By Th. 3.1, $\mathbb{R}(R)$ is a band, so every right ideal of R is idempotent. This and the given condition on prime ideals imply that R is regular by [6, Th. 1.17].

(b) A ring of bounded index with a non-zero nil right ideal has a non-zero nilpotent ideal. This would be contrary to $\mathbb{R}(R)$ a band. $\Diamond$

Recall that S is *π-regular* if for each $s \in S$ there exist $n \in \mathbb{N}$ ($\mathbb{N}$ being the set of natural numbers), $t \in S$ such that $s^n = s^n t s^n$. Recall also that the class of π-regular rings includes all regular rings and all nil rings, and is closed under direct sums and homomorphic images.

Proposition 3.3. *If* $\mathbb{R}(R)$ *is* π*-regular, then:*

(a) *for each* $H \in \mathbb{R}(R)$ *there exists* $n \in \mathbb{N}$ *such that* H^n *is idempotent;*

(b) *every cyclic subsemigroup of* $\mathbb{R}(R)$ *is finite.*

Proof. Let $H \in \mathbb{R}(R)$. Then $H^n = H^n K H^n$ for some $n \in \mathbb{N}$ and some $K \in \mathbb{R}(R)$. As in the proof of Th. 3.1, we have that H^n is an idempotent. Hence the subsemigroup of $\mathbb{R}(R)$ generated by H is finite. $\Diamond$

Lemma 3.4. *Let* $\mathbb{R}(R)$ *be commutative. If* $r, s \in R$*, then there exists* $x \in R$ *such that* $rs = sx$*.*

Proof. From $(rR)(sR) = (sR)(rR)$ we get $rs \in sRrR$ and hence $rs = surv$ for some $u, v \in R$. Use $x = urv$. $\Diamond$

Theorem 3.5. *The following are equivalent:*

(a) $\mathbb{R}(R)$ *is inverse;*

(b) $\mathbb{R}(R)$ *is a semilattice;*

(c) R *is strongly regular.*

Proof. From Th. 3.1 we get (a) implies (b). Assume (b). Then (a) follows immediately. Also, for each $r \in R$ we use Lemma 3.4 to get $r = r^2 xy$ for some $x, y \in R$. Thus R is strongly regular.

Finally, assume R is strongly regular. So R is regular and hence every right ideal is idempotent. (See [6, Th. 3.5 and Cor. 1.2].) Thus

$\mathbb{R}(R)$ is a band. Andrunakevic [1] has shown that if R is strongly regular, then $HK = H \cap K$ for all right ideals H, K of R. Thus R is commutative. $\Diamond$

Corollary 3.6. *If $\mathbb{R}(R)$ is a band, then $\mathbb{R}(Z(R))$ is a semilattice and $Z(R)$ is a strongly regular ring, where $Z(R)$ is the center of the ring R.*

Proof. Ramamurthi [9, Prop. 12] showed that if $\mathbb{R}(R)$ is a band, then so is $\mathbb{R}(Z(R))$. Thus $\mathbb{R}(Z(R))$ is a commutative band. By Th. 3.5, $Z(R)$ is strongly regular. $\Diamond$

Corollary 3.7. *The following are equivalent:*

(a) $\mathbb{R}(R)$ *is a finite inverse semigroup;*

(b) $\mathbb{R}(R)$ *is a semilattice and* $(\mathbb{R}(R), \subseteq)$ *has DCC;*

(c) R *is a finite direct sum of skewfields.*

Proof. The implication (a) $\Rightarrow$ (b) follows immediately from Th. 3.5. Assume (b). Then R is semiprime and right Artinian, so the Artin–Wedderburn Theorem and Th. 3.5 give (c). Finally, assume (c). Since this gives $\mathbb{R}(R)$ finite and R strongly regular, we immediately get (a). $\Diamond$

Corollary 3.8. *The following are equivalent:*

(a) $\mathbb{R}(R)$ *is inverse and* $(\mathbb{I}(R), \subseteq)$ *has ACC;*

(b) R *is strongly regular and every one-sided ideal of* R *is principal.*

Proof. Assume (a). Then R is strongly regular and $\mathbb{R}(R) = \mathbb{I}(R)$. Thus every right ideal of R is finitely generated, and since R is also regular, every right ideal of R is principal. Thus (a) implies (b). The converse is immediate. $\Diamond$

Example 3.9. Here we give an example of a ring R in which every right ideal of R is idempotent but not every left ideal is idempotent.

Let F be a field and let V be a vector space over F with basis $\{x, y\}$. On this basis define a multiplication via $x^2 = x,\ yx = y,\ xy = = 0 = y^2$. Extend this multiplication via linearity to all of V. This yields a ring on V. The non-zero, proper right ideals of the ring V will all be one-dimensional as subspaces, so each of them is idempotent.

Now take F to have characteristic 0. Let S be any non-zero subgroup of $(F, +)$. Then Sy will be a left ideal of the ring V. All of these left ideals are nilpotent.

This ring V has no identity. However, we can embed V in its Dorroh extension using $F \times V = V^1$. Then V is an ideal in V^1 and $V^1/V \cong F$. Since the right ideals of V and F are idempotent, the same is true of V^1 by [9, Prop. 5]. However, the left ideals of V are left ideals of V^1 also, so V^1 has some non-idempotent left ideals.

By Th. 3.1 we have that $J(R) = 0$. However, the Brown–McCoy radical of the ring V^1 in Ex. 3.9 is V, which is non-zero.

Note that this example, using $F = \mathbb{R}$, is given as an example of a ring that has DCC on right ideals, but not DCC on left ideals; see [4, pp. 37–38].

4. Almost rectangular bands

We now consider the case where $\mathbb{R}(R)$ is a band. By Th. 3.1, this condition is equivalent to $\mathbb{R}(R)$ being regular. It is well known that any band S is a semilattice of rectangular bands, by [8, Cor. II.1.7], where a rectangular band is a direct product $A \times B$ with A being a left zero semigroup ($st = s$ for all $s, t \in S$) and B being a right zero semigroup ($st = t$ for all $s, t \in S$). However, we cannot assume that $\mathbb{R}(R)$ is right zero, left zero, or a rectangular band, because $\mathbb{R}(R)$ has a right identity, namely R, and a zero element, namely the zero ideal of R. Therefore we make the following definitions.

Definition 4.1. Let $\mathbf{P}$ be any semigroup property. A semigroup S is a $\mathbf{P}$-*semigroup* if S satisfies property $\mathbf{P}$. A semigroup S is *almost* $\mathbf{P}$ if S contains a right identity e and a zero element 0, and $S \setminus \{0, e\}$ forms a $\mathbf{P}$-semigroup.

Lemma 4.2. *The semigroup $\mathbb{R}(R)$ is almost a rectangular band if and only if $\mathbb{R}(R)$ is almost left zero.*

Proof. If $\mathbb{R}(R)$ is almost left zero, then it is almost a rectangular band. Conversely, suppose that $\mathbb{R}(R)$ is almost a rectangular band. By Th. 2.2, $\mathcal{L} = \mathcal{J}$. Since a rectangular band is simple, then every pair of elements in $\mathbb{R}(R)$ is $\mathcal{L}$-related by Th. 2.2.

As mentioned above, any rectangular band can be represented as a direct product $A \times B$, where A is a left zero semigroup and B is a right zero semigroup. Then $(a, b)\mathcal{L}(c, d)$, which implies that $b = d$. Hence the set B is a singleton, which implies that $A \times B$ is left zero. Therefore $\mathbb{R}(R)$ is almost left zero. $\Diamond$

Proposition 4.3. $\mathbb{R}(R)$ *is almost left zero if and only if R is simple.*

Proof. Let R be simple, and let $H, K \in \mathbb{R}(R)$. Then RK is a non-zero ideal of R and hence $RK = R$, so $HK = (HR)K = H(RK) = HR = H$.

Conversely, take $\mathbb{R}(R)$ to be almost left zero. Let H be a non-zero right ideal of R and let I be a non-zero proper ideal of R. Then $H = HI \subseteq I$, so every proper right ideal of R is contained in I and

I is the only proper non-zero ideal of R. Consequently I is a unique maximal ideal of R and $I = J(R)$. However, by Th. 3.1, $J(R) = 0$. Thus R has no proper non-zero ideals. $\Diamond$

We summarize the previous results.

Theorem 4.4. *The following are equivalent.*

(a) *The semigroup* $\mathbb{R}(R)$ *is almost a rectangular band.*

(b) *The semigroup* $\mathbb{R}(R)$ *is almost left zero.*

(c) *The ring* R *is simple.*

We now consider the case in which $\mathbb{R}(R)$ is almost right zero. Note that such a semigroup is an almost rectangular band by definition.

Theorem 4.5. *The following are equivalent.*

(a) $\mathbb{R}(R)$ *is almost right zero.*

(b) $\mathbb{R}(R)$ *is a two-element semilattice.*

(c) R *is a division ring.*

Proof. (a) $\Rightarrow$ (c). Assume that $\mathbb{R}(R)$ is almost right zero. Let H and K be proper right ideals of R; that is, $0 \neq H, K \neq R$. Then $H = KH \subseteq K$ and $K = HK \subseteq H$ so that $H = K$. Therefore R has only one proper right ideal H, which must therefore be minimal. Since every element of $\mathbb{R}(R)$ is idempotent, then R is semiprime. Hence $H = eR$ for some idempotent R. But then $(1-e)R$ is a right ideal of R distinct from H, a contradiction. Therefore R has no proper right ideals, and hence is a division ring.

The implications (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) are immediate. $\Diamond$

For similar results, see [9, Lemma 16, Th. 17, Cors. 17 and 18].

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