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ON A PROBLEM OF A. IVIĆ

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Dedicated to the memory of Professor N. M. Timofeev

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Abstract: It is proved that

$$\sum_{n\le x}\tau(n+\tau(n)) = Dx\log x + O\left(\frac{x(logx)}{\log\log x}\right)$$

with positive constants $D>0$, $\delta>0$, where $\tau(m)$ is the number of divisors of m.

1. As usual, $\tau(n)$, $\omega(n)$, $\Omega(n)$ denote the number of divisors, the number of distinct prime factors, the number of prime factors with multiplicity of n.

Let furthermore $\varphi(n)$ be Euler's totient function. For the sake of simplicity we shall write $x_1 := \log x$; $x_2 := \log x_1$; $x_3 := \log x_2$.

A. Ivić [1] formulated the conjecture that

$$D(x) := \sum_{n\le x}\tau(n+\tau(n)) = Dxx_1 + O(x). \tag{1.1}$$

We can deduce the somewhat weaker assertion, namely that

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$$D(x) = Dxx_1 + O\left(\frac{xx_1}{x_2}\right),$$

by using two theorems of N. M. Timofeev and M. B. Khripunova [2], which are analogons of the Vinogradov–Bombieri theorem and the Brun–Titchmarsh inequality. We shall formulate them as Lemmas 1, 2.

In [2] they proved the asymptotic of $\sum_{n<x}\tau(n+a)$ where n runs over the integers with $\Omega(n)=k$, uniformly as $k \le (2-\varepsilon)x_2$, $a=1$. In [4] they proved the asymptotic of

$$\sum_{\substack{n<N \\ \Omega(n)=k}} \tau(N-n)$$

uniformly as $k \le (2-\varepsilon)\log\log N$, or $(2+\varepsilon)\log\log N < k < b\log\log N$.

2. Let $t \ge 2$, $P(t) = \prod_{p<t} p$, p runs over the set of primes. In what follows, $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ are arbitrarily small positive numbers.

Denote

$\mu(x,k,t,a,d) = \#\{n \mid n \le x,\ \Omega(n)=k,\ (n,P(t))=1,\ n \equiv a (\operatorname{mod} d)\}$.

Lemma 1. *Let $2 \le t \le \sqrt{x}$, $k \le x_2^2$, and let*

$$\Delta_k(t) = \sum_{d \le Q} \max_{y \le x} \max_{(a,d)=1} \Bigg| \mu(y,k,t,a,d) - \\ - \frac{1}{\varphi(d)} \#\{n \mid n \le y,\ \Omega(n)=k,\ (n,dP(t))=1\} \Bigg|.$$

Then

$$\Delta_k(t) \ll Q\sqrt{x}\exp\left(x_2^{2+\varepsilon}\right) + \frac{x}{x_1^B},$$

where $\varepsilon > 0$ and B is an arbitrary positive constant.

Lemma 2. *Suppose $k \le (2-\varepsilon)x_2$, $0<\varepsilon<1$, $d \le x^{\frac12+\alpha(k)}$, $2 \le t \le$ $\le x^{\beta(k)}$, $\alpha(k) = 1/3k$, and $\beta(k) = \frac{1}{10}\exp(-k/2)$. Then there exists a constant $c(\varepsilon,\varepsilon_1)$ such that*

$$\mu(x,k,t,a,d) \le c(\varepsilon,\varepsilon_1)\frac{x}{\varphi(d)x_1}(1+\varepsilon_1)^k \frac{\left(\log\frac{x_1}{\log t}\right)^{k-1}}{(k-1)!}$$

where $0<\varepsilon_1<1$.

Lemma 3. *Let $z > 0$, $1 < \beta$, $Q(y) := y \log \frac{y}{e} + 1$. Then*

$$\sum_{k \geq \beta_z} \frac{e^{-z} \cdot z^k}{k!} < \frac{\sqrt{\beta} e^{-Q(\beta) z}}{(\beta - 1)\sqrt{2\pi}}.$$

The proof can be found in [3]. Here the authors proved also that

$$\#\{n \leq x \mid \omega(n) = k\} \leq \frac{c_0 x}{x_1} \frac{(x_2 + c)^{k-1}}{(k-1)!} \quad (x \geq 3,\ k \geq 1)$$

which is called as Hardy–Ramanujan inequality.

3. Let $(a, d) = 1$,

$$S(y; l, K, a, d) := \sum_{\substack{n \leq y \\ \omega(n) = l \\ (n,K)=1 \\ n \equiv a (\mathrm{mod}\, d)}} |\mu(n)|. \tag{3.1}$$

Let χ be a Dirichlet-character $\mathrm{mod}\, d$. Since

$$\prod_{(p,K)=1} \left(1 + \frac{z\chi(p)}{p^s}\right) = \prod_{(p,K)=1} \frac{1}{1 - \frac{z\chi(p)}{p^s}} \cdot \prod_{(p,K)=1} \left(1 - \frac{z^2 \chi^2(p)}{p^{2s}}\right)$$

holds for $z \in \mathbb{C}$, $|z| \leq 1$, therefore

$$\sum_{\substack{n \leq x \\ \omega(n) = l \\ (n,K)=1}} \chi(n) |\mu(n)| = \sum_{\substack{mr^2 \leq x \\ (mr,K)=1 \\ \Omega(m) + 2\omega(r) = l}} (-1)^{\omega(r)} |\mu(r)| \chi(m) \chi^2(r). \tag*{$(3.2)_\chi$}$$

Thus, counting $\frac{1}{\varphi(d)} \cdot \sum_{\chi} (3.2)_\chi$, we obtain

(3.3)

$$S[y; l, K, a, d] = \sum_{\substack{r^2 \leq y \\ (r,Kd)=1}} (-1)^{\omega(r)} |\mu(r)| \cdot \mu\left(\frac{y}{r^2}, l - 2\omega(r),\ 2, b_a, d\right),$$

where b_a $(\mathrm{mod}\, d)$ is defined by $r^2 b_a \equiv a (\mathrm{mod}\, d)$.

Let

$$T(y;l,K,d) := \sum_{\substack{n\le y\\ \omega(n)=l\\ (n,Kd)=1}} |\mu(n)| = \sum\nolimits_{\substack{a\\ (a,d)=1}} S[y;l,K,a,d]. \tag{3.4}$$

It is clear that

$$\begin{aligned}&\left|S[y;l,K,a,d]-\frac{1}{\varphi(d)}T(y;l,K,d)\right|\le\\ &\le \sum_{\substack{r^2\le y\\ (r,Kd)=1}} |\mu(r)|\ \left|\mu\left(\frac{y}{r^2},\ \ l-2\omega(r),\ 2,b_a,d\right)-\right.\\ &\quad\left.-\frac{1}{\varphi(d)}\#\left\{n\le\frac{y}{r^2},\ \Omega(n)=l-2\omega(r)\ (n,d)=1\right\}\right|.\end{aligned} \tag{3.5}$$

Furthermore, from (3.4) and (3.3) we deduce that

$$\begin{aligned}&T(y;l,K,d)=\\ &=\sum_{\substack{r^2\le y\\ (r,Kd)=1}} (-1)^{\omega(r)}|\mu(r)|\#\left\{n\le\frac{y}{r^2}\ \middle|\ \Omega(n)=l-2\omega(r),\ (n,Kd)=1\right\}.\end{aligned} \tag{3.6}$$

4. Theorem. *We have*

$$D(x)=Dxx_1+O\left(\frac{xx_1}{x_2}\right). \tag{4.1}$$

Proof. Let us write every n in the form $n=Km$, where K is square-full, m is square-free and $(K,m)=1$. We say that K is the square-full and m is the square-free part of n.

If $n=Km$, $\omega(m)=t$, then $\tau(n+\tau(n))=\tau(Km+\tau(K)\cdot 2^t)$.

Let

$$E_{K,t}(x)=\sum_{\substack{m\le x/K\\ \omega(m)=t}} \tau(Km+\tau(K)\cdot 2^t), \tag{4.2}$$

where in the summation we assume furthermore that $(m,K)=1$, m is square-free.

It is clear that

$$D(x)=\sum_{K\le x}\sum_{t=0}^{\infty}E_{K,t}(x), \tag{4.3}$$

where K runs over the square-full integers.

We shall prove that

$$\sum_{K\ge x_1^4}\sum_{t}E_{K,t}(x)+\sum_{K\le x_1^4}\sum_{t\ge\beta x_2}E_{K,t}(x)\ll x\cdot x_1^{1-\delta}, \tag{4.4}$$

where β is an arbitrary constant larger than $1/\log 2$.

Since $\tau(n)=O(n^{\varepsilon})$, therefore

$$\sum_{K\ge x^{1/4}}\sum_{t}E_{K,t}(x)=O(x^{1/4}).$$

Let $K\le x^{1/4}$. By using the Hölder inequality

$$\begin{aligned}E_{K,t}(x)&\le\Bigg(\sum_{\substack{m\le x/K\\ \omega(m)=t}}1\Bigg)^{1/2}\left\{\sum_{m\le x/K}\tau^2(Km+\tau(K)\cdot 2^t)\right\}^{1/2}\ll\\ &\ll\left(\frac{x}{Kx_1}\right)^{1/2}\left(\frac{(x_2+c)^{t-1}}{(t-1)!}\right)^{1/2}\cdot\left(\frac{x}{K}x_1^3\right)^{1/2}=\\ &=\frac{x}{K}x_1\cdot\frac{(x_2+c)^{(t-1)/2}}{(t-1)!^{1/2}}.\end{aligned}\tag{4.5}$$

Here we used the Hardy–Ramanujan inequality and the known inequality

$$\max_{t\le x_1}\sum_{m\le x/K}\tau^2(Km+\tau(K)\cdot 2^t)\le\frac{cx}{K}x_1^3,$$

where c does not depend on K and t.

Since

$$\sum_{t\ge 0}\frac{(x_2+c)^{(t-1)/2}}{(t-1)!^{1/2}}\ll x_1^{1/2}\cdot x_2^{1/4}\ll x_1,$$

we obtain that

$$\sum_{K\ge x_1^4}\sum_{t}E_{K,t}(x)=O(x). \tag{4.6}$$

Let $K\ge x_1^4$. From (4.5) we obtain that

(4.7) $$\sum_{t\geq\alpha x_2} E_{K,t}(x) \leq \frac{cxx_1}{K}\sum_{t\geq\alpha x_2}\frac{(x_2+c)^{\frac{t-1}{2}}}{(t-1)!^{1/2}} \ll \frac{x}{K}\cdot\frac{1}{x_1^2},$$

if α is large enough.

Let $\beta > 1/\log 2$. We shall estimate $\sum\limits_{\beta x_2\leq t\leq\alpha x_2} E_{K,t}(x)$.

If $\omega(m) = t \geq \beta x_2$, then $\tau(m) = 2^t \geq 2^{\beta x_2}$, consequently

(4.8) $$\sum_{\beta x_2\leq t\leq\alpha x_2} E_{K,t}(x) \leq 2^{-\beta x_2}\sum_{\beta x_2\leq t\leq\alpha x_2}\sum_{m\leq x/K}\tau(m)\tau(Km+2^t\tau(K)).$$

One can prove elementarily that the inner sum on the right-hand side of (4.8) is less than $\ll \frac{x}{K}x_1^2$, consequently the left-hand side of (4.8) is less than $\ll \frac{x}{K}x_1^{2-\beta\log 2}\cdot x_2$.

We proved (4.4). Thus

(4.9) $$D(x) = \sum_{K\leq x_1^4}\sum_{t<\beta x_2} E_{K,t}(x) + O\left(x\cdot x_1^{1-\delta}\right)$$

with a suitable $\delta > 0$, if $\beta > \frac{1}{\log 2}$.

Observing that

$$\sum_{t\leq\beta x_2}\sum_{m\leq\frac{x}{K}}\tau(Km+\tau(K)\cdot 2^t) \ll \frac{x}{K}\cdot x_1\cdot x_2,$$

if $K\leq x_1^4$, and that $\sum\limits_{K>y} 1/K \ll \frac{1}{\sqrt{y}}$, therefore

$$D(x) = \sum_{K\leq x_2^4}\sum_{t\leq\beta x_2} E_{K,t}(x) + O\left(\frac{xx_1}{x_2}\right).$$

Finally it remained to give the asymptotic of $E_{K,t}(x)$ under the condition $K\leq x_2^4$, $t\leq\beta x_2$. This is the number of solutions of

$$Km+\tau(K)\cdot 2^t = uv \leq x+\tau(K)\cdot 2^t,$$

where u, v run over the positive integers, m over the square-free integers coprime to K, and with $\omega(m) = t$. If we multiply the number of solutions by 2, we can assume that $u < v$. The contribution of the solutions with $u = v$ can be ignored.

Let $Q := \sqrt{x}\cdot e^{-x_1^{1-2\varepsilon}}$, where $\varepsilon > 0$ is the same as in Lemma 1.

Let us overestimate first

$$(4.10)_u \qquad \sum_{t\le\beta x_2} \#\left\{m \le \frac{x}{K} \;\Big|\; Km + \tau(K)\cdot 2^t \equiv O(\operatorname{mod} u),\ \Omega(m) = t\right\}.$$

By using Lemma 2 (substituting $t = 2$ defined there), we deduce that $(4.10)_u$ is less than

$$\frac{cx}{K\varphi(u)x_1} \sum_{t\le\beta x_2} \frac{(1+\varepsilon_1)^{t-1}\cdot x_2^{t-1}}{(t-1)!} \le \frac{cx}{K\varphi(u)x_1}\exp((1+\varepsilon_1)x_2) = $$

$$= \frac{cx}{K\varphi(u)} x_1^{\varepsilon_1}.$$

Furthermore

$$\sum_{Q\le u\le\sqrt{2x_2}} \frac{1}{\varphi(u)} \ll \log\frac{\sqrt{2x_2}}{Q} \ll x_1^{1-2\varepsilon}.$$

Since $\varepsilon = \varepsilon_1$ can be chosen, we obtain that

(4.11)

$$D(x) = \sum_{K\le x_2^4}\sum_{t\le\beta x_2}\sum_{u\le Q} \#\left\{m \le \frac{x}{K} \Big| Km + \tau(K)\cdot 2^t \equiv 0(\operatorname{mod} u),\right.$$

$$\left.\omega(m) = t, (K,m) = 1,\ \mu(m)\ne 0\right\} -$$

$$-\sum_{K\le x_2^4}\sum_{t\le\beta x_2} \#\left\{\frac{m \le u^2 - \tau(K)\cdot 2^t}{K} \Big| Km + \tau(K)\cdot 2^t \equiv 0(\operatorname{mod} u),\right.$$

$$\left.\omega(m) = t, (K,m) = 1,\ \mu(m)\ne 0\right\} + O\left(\frac{xx_1}{x_2}\right).$$

Now we can apply (3.5), (3.6) and Lemma 1 in the usual way. We obtain our theorem quite directly. We omit the details. ◊

5. The following assertion can be proved similarly:

$$\sum_{n\le x} \tau(n + f(n)) = D_f x\log x + O\left(\frac{x(\log x)}{\log\log x}\right)$$

with some constants $D_f > 0$, $\delta > 0$ if $f(n) = \omega(n)$, $\Omega(n)$, $\tau(\tau(n))$, $2^{\omega(n)}$, $\tau_k(n)$, where $\tau_k(n)$ is the number of solutions of $n = u_1 \dots u_k$ in positive integers $u_1, \dots, u_k$. Similar theorems can be proved if we substitute $\tau(m)$ by $2^{\omega(m)}$.

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