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SYMMEDIANS AND THE SYMMEDIAN CENTER OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Abstract: The concepts of symmedians and the symmedian center of the triangle in isotropic plane are defined. Some interesting relationships between the introduced concepts and other elements of the triangle in an isotropic plane are also studied. A certain number of these statements results from euclidean geometry, but some of them are new.

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In [4] it is shown that each triangle ABC in an isotropic plane can be set, by a suitable choice of the coordinate system, into the so-called standard position, in which its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ and its circumscribed circle $\mathcal{K}$ has the equation

$$y = x^2, \tag{1}$$

where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ some useful equalities hold, as for example $q = bc - a^2$, $q = -(b^2 + bc + c^2)$, $a^2 + b^2 + c^2 = -2q$, $(c - a)(a - b) = 2q - 3bc$.

Indeed, we have

$$b^2 + bc + c^2 = (b + c)^2 - bc = a^2 - bc = -q,$$

$$(c - a)(a - b) = -a^2 - bc + ca + ab = -(bc - q) - 2bc + q = 2q - 3bc.$$

If the point $T_1 = (x_1, {x_1}^2)$ is any point on the circle $\mathcal{K}$, then the line $\mathcal{T}_1$ with the equation

$$y = 2x_1 x - {x_1}^2 \tag{2}$$

is the tangent of that circle at the point T_1 because the equation $x^2 - 2x_1 x + x_1^2 = 0$ with the double solution $x = x_1$ from (1) and (2) follows. The tangents $\mathcal{T}_1$ and $\mathcal{T}_2$ with the equation $y = 2x_2 x - {x_2}^2$ to the circle $\mathcal{K}$ at the point T_2 have the intersection point

$$T_{12} = \left(\frac{1}{2}(x_1 + x_2), x_1 x_2\right)$$

because, for example

$$2x_1 \cdot \frac{1}{2}(x_1 + x_2) - {x_1}^2 = x_1 x_2.$$

The tangents $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to the circle $\mathcal{K}$ at the points A, B, C determine the triangle $A_t B_t C_t$ with the vertices $A_t = \mathcal{B} \cap \mathcal{C}$, $B_t = \mathcal{C} \cap \mathcal{A}$, $C_t = \mathcal{A} \cap \mathcal{B}$, the so-called tangential triangle of the triangle ABC. Because of $a + b + c = 0$ we get

$$A_t = \left(-\frac{a}{2}, bc\right), \quad B_t = \left(-\frac{b}{2}, ca\right), \quad C_t = \left(-\frac{c}{2}, ab\right). \tag{3}$$

In [4] it is shown that the centroid of the triangle ABC in the standard position is given by

$$G = \left(0, -\frac{2}{3}q\right) \tag{4}$$

and that its related line $\mathcal{H}$ with the equation

$$y = -\frac{q}{3}, \tag{5}$$

is the orthic axis of the triangle ABC.

Theorem 1. *If $A_tB_tC_t$ is the tangential triangle of the triangle ABC, then the lines AA_t, BB_t, CC_t are symmetric with respect to the bisectors of the angles A, B, C to medians AG, BG, CG of the triangle ABC. The lines AA_t, BB_t, CC_t meet at one point K.*

Proof. The line with the equation

$$y = -\frac{2q}{3a}x + bc - \frac{q}{3} \tag{6}$$

passes through the points A and A_t because for them the right side of (6) becomes

$$-\frac{2q}{3} + bc - \frac{q}{3} = bc - q = a^2, \quad \frac{q}{3} + bc - \frac{q}{3} = bc,$$

and (6) is the equation of the line AA_t. The line with the equation

$$y = \frac{3bc - q}{3a}x - \frac{2q}{3} \tag{7}$$

passes through the point A and centroid G from (4) of the triangle ABC because of

$$\frac{3bc - q}{3a} \cdot a - \frac{2q}{3} = bc - q = a^2,$$

and then (7) is the equation of median AG. Adding (6) and (7) we get the equation

$$2y = \frac{bc - q}{a}x + bc - q,$$

i.e. the equation $2y = ax + a^2$, and that same equation will be obtained by the addition of the equations of the lines CA and AB (see [4]), i.e.

$$y = -bx - ca, \quad y = -cx - ab$$

so it is the equation of the bisector of the angle A. The line (6) passes through the point

$$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right) \tag{8}$$

because $\frac{p}{a} = bc$, and that point also lies on the analogous lines BB_t and CC_t. ◊

The lines AA_t, BB_t and CC_t from Th. 1 are called symmedians of the triangle ABC, and the point K is called the symmedian center of that triangle. The isotropic line through the symmedian center will be called Brocard diameter of the considered triangle.

In [5] the symmedians were defined in a different way and it was proved [5, Th. 1] that three symmedians meet at a point.

Corollary 1. *For the triangle ABC in the standard position the symmedian center K is given by the formula* (8), *Brocard diameter is defined by the equation* $x = \frac{3p}{2q}$, *symmedian AK is given by the equation* (6) *and the symmedians BK and CK appear in the same form.*

Corollary 2. *The symmedian center of the triangle lies on its orthic axis.*

Theorem 2. *The corresponding sides of the triangle and its tangential triangle meet at three points which lie on one line $\mathcal{L}$.*

Proof. The point

$$\left(-\frac{q}{3a}, \frac{q}{3} - bc\right) \tag{9}$$

lies on the line BC with the equation $y = -ax - bc$, and also lies on the tangent line of the circumscribed circle of the triangle ABC at the point A which has the equation $y = 2ax - a^2$ owing to

$$-\frac{2}{3}q - a^2 = -\frac{2}{3}q + q - bc = \frac{q}{3} - bc.$$

The point (9) also lies on the line with the equation

$$y = \frac{3p}{q}x + \frac{q}{3} \tag{10}$$

because of

$$-\frac{p}{a} + \frac{q}{3} = \frac{q}{3} - bc.$$

The symmetry of the equation (10) of a, b, c proves the statement of the theorem. ◊

The line $\mathcal{L}$ from Th. 2 will be called Lemoine line of the considered triangle.

Corollary 3. *The Lemoine line of the triangle from* Cor. 1 *is given by the equation* (10).

The meaning of Ths. 1 and 2 is actually that the triangles ABC and $A_tB_tC_t$ are homologous with the center of homology K and the axis of homology $\mathcal{L}$.

The polar line of the point $T = (x_o, y_o)$ with respect to the circumscribed circle of the triangle ABC with the equation $y = x^2$ is given by the equation $y + y_o = 2x_o x$. In the case of the point K from (8) this equation has got the form (10), i.e. it is valid:

Theorem 3. *The Lemoine line of the triangle is the polar line of its symmedian center with respect to its circumscribed circle.*

Theorem 4. *The symmedian center of the triangle lies on the lines which join the midpoints of its sides and the corresponding altitudes.*

Proof. The midpoint A_m of the side BC has the coordinates

$$\frac{1}{2}(b+c) = -\frac{a}{2}, \quad \frac{1}{2}(b^2+c^2) = -\frac{1}{2}(q+bc),$$

so we find out

$$A_m = \left(-\frac{a}{2}, -\frac{1}{2}(q+bc)\right). \tag{11}$$

The feet A_h of the altitude from the vertex A has the abscissa a and the ordinate $-a^2 - bc = q - 2bc$, and then the midpoint A_s of the points $A = (a, a^2)$ and A_h has the ordinate

$$\frac{1}{2}(a^2 + q - 2bc) = -\frac{1}{2}bc,$$

i.e. we have

$$A_s = \left(a, -\frac{bc}{2}\right). \tag{12}$$

The points A_m and A_s lie on the line with the equation

$$y = \frac{q}{3a}x - \frac{q}{3} - \frac{bc}{2}$$

because we get for them

$$y - \frac{q}{3a}x = -\frac{1}{2}(q+bc) + \frac{q}{6} = -\frac{q}{3} - \frac{bc}{2}$$

namely

$$y - \frac{q}{3a}x = -\frac{bc}{2} - \frac{q}{3}.$$

The symmedian center K from (8) lies on that line too, because we get for it

$$y - \frac{q}{3a}x = -\frac{q}{3} - \frac{p}{2a} = -\frac{q}{3} - \frac{bc}{2}.$$

Theorem 5. [8, Th. 6.] *The spans of the symmedian center of the triangle to its sides are proportional to the lengths of its sides.*

Proof. We find that

$$2\triangle = 2\,\text{area}(ABC) = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = (b-c)(c-a)(a-b),$$

$$\begin{aligned} BC^2 + CA^2 + AB^2 &= (b-c)^2 + (c-a)^2 + (a-b)^2 = \\ &= 2(a^2+b^2+c^2) - 2(bc+ca+ab) = -6q, \end{aligned}$$

$$2\,\text{area}(BCK) = \begin{vmatrix} b & b^2 & 1 \\ c & c^2 & 1 \\ \frac{3p}{2q} & -\frac{q}{3} & 1 \end{vmatrix} = \frac{3p}{2q}(b^2-c^2) + \frac{q}{3}(b-c) - bc(b-c) =$$

$$= (b-c)\left(\frac{q}{3} - \frac{3ap}{2q} - bc\right) = \frac{b-c}{6q}(2q^2 - 9ap - 6bcq) =$$

$$= -\frac{b-c}{6q}(b-c)^2(c-a)(a-b) = -\frac{BC^2}{6q}\cdot 2\triangle = \frac{BC^2}{BC^2+CA^2+AB^2}\cdot 2\triangle$$

owing to

$$\begin{aligned} (b-c)^2(c-a)(a-b) &= (a-b)(b-c)\cdot(b-c)(c-a) = \\ &= (2q-3ca)(2q-3ab) = \\ &= 4q^2 - 6a(b+c)q + 9a^2bc = 4q^2 + 6a^2q + 9ap = \\ &= 4q^2 + 6(bc-q)q + 9ap = -(2q^2 - 9ap - 6bcq). \end{aligned}$$

Therefore

$$\text{area}(BCK) = \triangle \cdot \frac{BC^2}{BC^2+CA^2+AB^2}.$$

If d_a is the span of the point K to the line BC, then $2\,\text{area}(BCK) = BC \cdot d_a$, so it implies that

$$d_a = \frac{2\triangle}{BC^2+CA^2+AB^2}\cdot BC.$$

Similar expressions are got for the spans d_b and d_c of the point K to the lines CA and AB, so it follows $d_a : d_b : d_c = BC : CA : AB$.

Corollary 4. *If K is the symmedian center of the triangle ABC, then the triangles BCK, CAK, ABK have the areas BC^2k, respectively, CA^2k, AB^2k, where $k = \dfrac{\triangle}{BC^2+CA^2+AB^2}$.*

Theorem 6. *The lines which are parallel to the lines BC, CA, AB and moved away off it for the distances proportional to the lengths of the sides BC, CA, AB, determine the triangle $A'B'C'$ which is homothetic* *to the triangle ABC, and the center of homothecy is the symmedian center K of the triangle ABC.*

Proof. The considered lines have the equations of the form

$$y = -ax - bc + (b-c)t,$$
$$y = -bx - ca + (c-a)t,$$
$$y = -cx - ab + (a-b)t.$$

The point

$$A' = \left(a - \frac{3at}{b-c}, a^2 + \frac{2qt}{b-c}\right)$$

lies on the last two lines because for example

$$a^2 + \frac{2qt}{b-c} + b\left(a - \frac{3at}{b-c}\right) + ca - (c-a)t =$$
$$= a(a+b+c)+t\left(a-c+\frac{2q-3ab}{b-c}\right) = t\left[a-c+\frac{(b-c)(c-a)}{b-c}\right] = 0.$$

Besides that, the point A' lies on the symmedian AK with the equation (6) owing to

$$a^2 + \frac{2qt}{b-c} + \frac{2q}{3a}\left(a - \frac{3at}{b-c}\right) + \frac{q}{3} - bc = a^2 + q - bc = 0. \quad \diamond$$

The statements of the past theorems and Cor. 4 are valid in the euclidean plane and there they are well-known properties of the symmedian center (see for example Johnson [3]).

Theorem 7. *If A_m and C_m are the midpoints of the sides CA and AB, then the circles ABB_m and ACC_m meet again at the point T on the symmedian AK of the triangle ABC.*

If the point T' is the second intersection (except A) of that symmedian and the circle AB_mC_m then $AT' : AT = 2:3$ is valid.

Proof. The circle with the equation

$$3by = 2(b-c)x^2 + c(a-c)x + ab(a-c) \tag{13}$$

passes through the points $A(a, a^2)$, $B(b, b^2)$, $B_m = \left(-\frac{b}{2}, -\frac{1}{2}(q+ca)\right)$ because for them we get on the right side of (13) the following

$$2a^2(b-c) + ac(a-c) + ab(a-c) =$$
$$= 2a^2(b-c) - a^2(a-c) = a^2(2b-a-c) = 3b \cdot a^2,$$
$$2b^2(b-c) + bc(a-c) + ab(a-c) =$$

$$= 2b^2(b-c) - b^2(a-c) = b^2(2b-a-c) = 3b \cdot b^2,$$

$$\frac{1}{2}b^2(b-c) - \frac{1}{2}bc(a-c) + ab(a-c) = \frac{b}{2}(2a^2+b^2+c^2-bc-3ca) =$$

$$= \frac{b}{2}(a^2-2q-bc-3ca) = \frac{b}{2}(-3q-3ca) = -3b \cdot \frac{1}{2}(q+ca).$$

Analogously, the circle ACC_m has the equation

$$3cy = 2(c-b)x^2 + b(a-b)x + ac(a-b). \tag{14}$$

Adding equations (13) and (14) we find out the equation of the potential axis of these two circles. Because of $3b+3c=-3a$ and

$$c(a-c)+b(a-b) = ca+ab-b^2-c^2 = q-bc-b^2-c^2 = 2q,$$

$$ab(a-c)+ac(a-b) = a^2(b+c)-2p = -a^3-2p =$$

$$= -a(a^2+2bc) = -a(3bc-q) = -3a\left(bc-\frac{q}{3}\right)$$

after dividing by $-3a$ the equation of that potential axis gets the form (6) i.e. it is the equation of the symmedian AK. If we add the equations (13) and (14) previously multiplied by c and $-b$, we get for the abscissa of the point of intersection of these two circles the following equation

$$2(b+c)(b-c)x^2 + [c^2(a-c)-b^2(a-b)]x + abc(b-c) = 0$$

which, because of $b+c=-a$ and

$$c^2(a-c)-b^2(a-b) = b^3-c^3-a(b^2-c^2) =$$

$$= (b-c)(b^2+bc+c^2-ab-ca) = (b-c)(-q+a^2) = (b-c)(2a^2-bc)$$

after dividing by $-2a(b-c)$ can be written in the form

$$x^2 + \left(\frac{bc}{2a}-a\right)x - \frac{bc}{2} = 0.$$

This equation has the solutions $x=a$ and $x=x_T=-\dfrac{bc}{2a}=0$. The second solution is the abscissa of the common point T of these two circles on the line AK.

The circle with the equation

$$y = 2x^2 - 2ax + a^2 \tag{15}$$

passes through the point A and through the points B_m and C_m, too because of, for example for the point B_m on the right side of (15) we get the following

$$\frac{b^2}{2} + ab + a^2 = \frac{1}{2}(ca - q) - ca = -\frac{1}{2}(q + ca).$$

For the abscissas of the points of intersections A and T' of the circle AB_mC_m and the line AK, according to (6) and (15) we get the equation

$$2x^2 + 2\left(\frac{q}{3a} - a\right)x + a^2 - bc + \frac{q}{3} = 0,$$

which, because of

$$a^2 - bc + \frac{q}{3} = -q + \frac{q}{3} = -\frac{2q}{3}$$

and after dividing by 2 it gets the form

$$x^2 + \left(\frac{q}{3a} - a\right)x - \frac{q}{3} = 0$$

and it has the solutions $x = a$ and $x = x_{T'} = -\dfrac{q}{3a}$. Now, we get

$$AT' = x_{T'} - a = -\frac{q}{3a} - a = -\frac{1}{3a}(q + 3a^2),$$

$$AT = x_T - a = -\frac{bc}{2a} - a = -\frac{1}{2a}(bc + 2a^2) = -\frac{1}{2a}(q + 3a^2)$$

and so, finally $AT' : AT = 2 : 3$. ◊

In the euclidean case the first statement of Th. 7 can be found at Bradley [2], and the second statement at Herzig and Kovač [2].

Theorem 8. *If G is the centroid of the triangle ABC, then the point T such that*

(16) $$\angle(AG, AB) = \angle(BA, BT), \quad \angle(AG, AC) = \angle(CA, CT)$$

lies on the symmedian AK of that triangle. (In the euclidean case d'Ocagne [6] has this statement without proof.)

Proof. Let $T = (x, y)$ be the point. With $A = (a, a^2)$, $G = \left(0, -\frac{2}{3}q\right)$ the lines AG and BT have the slopes

$$\frac{a^2 + \frac{2}{3}q}{a} = a + \frac{2q}{3a}, \quad \frac{y - b^2}{x - b},$$

and the line AB has the slope $-c$, so we get

$$\angle(AC, AB) = -c - a - \frac{2q}{3a} = b - \frac{2q}{3a},$$

$$\angle(BA, BT) = \frac{y - b^2}{x - b} + c.$$

After multiplying by $3a(x - b)$ the first equation (16) gets the form

$$3a(y-b^2)+3ca(x-b)=3ab(x-b)-2q(x-b),$$

i.e.

$$3ay=(3ab-3ca-2q)x+3p+2bq. \tag{17}$$

Analogously, the second equality (16) can be written in the form

$$3ay=(3ca-3ab-2q)x+3p+2cq. \tag{18}$$

By the addition of these two equations and dividing the result by $6a$ we get the equation (6) of the symmedian AK. ◇

Subtracting (17) and (18) we get, for the abscissa of the point T, the equation $6a(b-c)x+2(b-c)q=0$ with the solution $x=-\frac{q}{3a}$. Because of that the point T from Th. 8 coincides with the point T' from Th. 7, and this is exactly the vertex A_2 of the second Brocard triangle $A_2B_2C_2$ of the triangle ABC.

Theorem 9. *Let B' and C' be the feet of the perpendicular lines from the point B and C to the bisector of the angle A. The parallel lines to the lines AB and AC through the points B' and C' meet at the point, which lies on the symmedian AK of the triangle ABC* (d'Ocagne [6] has the Euclidean case) *and it also lies on the bisector of the side BC and this is in fact the vertex A_t of the tangential triangle $A_tB_tC_t$ of the triangle ABC.*

Proof. The bisector of the angle A has, according to the proof of Th. 1, the equation

$$y=\frac{a}{2}x+\frac{a^2}{2}.$$

From the previous equation with $x=b$ it follows

$$y=\frac{a}{2}(a+b)=-\frac{ca}{2},$$

so we get

$$B'=\left(b,-\frac{ca}{2}\right),$$

The line with the equation

$$y=-cx+bc-\frac{ca}{2}$$

obviously passes through the point B' and it is parallel to the line AB. It passes through the point A_t from (3) too, which because of the symmetry of b and c lies on the parallel line to the line CA through the point C'. It also lies on the bisector of the side BC, and on the symmedian AK with the equation (6). ◇

The intersection of the symmedian of the triangle from its one vertex and its opposite side will be called the foot of that symmedian on that side.

Theorem 10. *The foot of the symmedian on the side of the triangle divides that side in the ratio of the squares of the lengths of the adjacent sides of that triangle.*

Proof. From the equation (6) and the equation $y = -ax - bc$ of the line BC we get for the abscissa of the foot A_k of the symmedian AK the following equation

$$\left(a - \frac{2q}{3a}\right) x = \frac{q}{3} - 2bc,$$

i.e. the equation $(3a^2 - 2q)x = aq - 6p$ with the solution

$$x = \frac{aq - 6p}{3a^2 - 2q} \tag{19}$$

which represents the abscissa of the point A_k. Now, we get

$$\begin{aligned} BA_k \cdot (3a^2 - 2q) &= (x - b)(3a^2 - 2q) = aq - 6p - 3a^2b + 2bq = \\ &= a(bc - a^2) - 6p - 3a^2b + 2b(ca - b^2) = \\ &= -3p - a^3 - 3a^2b - 2b^3 = \\ &= 3ab(a + b) - a^3 - 3a^2b - 2b^3 = -a^3 + 3ab^2 - 2b^3 = \\ &= -(a - b)^2(a + 2b) = -(a - b)^2(b - c), \end{aligned}$$

and similarly it follows $CA_k(3a^2 - 2q) = -(a - c)^2(c - b)$, wherefrom

$$BA_k : CA_k = -(a - b)^2 : (a - c)^2 = -AB^2 : AC^2. \quad \Diamond$$

Theorem 11. *The intersections of the medians of the triangle and the corresponding sides of its orthic triangle are the points parallel to the feet of the corresponding symmedians of the given triangle.* (In the euclidean case cf. Mineur [6].)

Proof. The line with the equation

$$y = 2ax + 2bc - q \tag{20}$$

passes through the feet $B_h = (b, q - 2ca)$ and $C_h = (c, q - 2ab)$ of the altitudes from the vertices B and C of the triangle ABC, analogous to the foot A_h from the proof of Th. 4, because of, for example for the point B_h we get

$$2ab + 2bc - q = q - 2ca,$$

and (20) is the equation of the side B_hC_h of the orthic triangle $A_hB_hC_h$.

From the equation (20) and the equation (7) of the median AG it follows for the abscissa of their intersection the following equation

$$2ax + 2bc - q = \frac{3bc - q}{3a}x - \frac{2q}{3},$$

i.e. the equation $6a^2x + 6p - 3aq = (3bc - q)x - 2aq$, which, because of,

$$6a^2 - 3bc + q = 6a^2 - 3(q + a^2) + q = 3a^2 - 2q$$

has the solution (19). ◊

Theorem 12. *If m_a and k_a are the lengths of the median AA_m and symmedian AA_k of the triangle ABC, then the following equality*

$$\frac{m_a}{k_a} = -\frac{CA^2 + AB^2}{2 \cdot CA \cdot AB}$$

is valid.

Proof. Firstly we have

$$m_a = AA_m = -\frac{a}{2} - a = -\frac{3}{2}a,$$

and then, because of (19), we get

$$k_a = AA_k = \frac{aq - 6p}{3a^2 - 2q} - a = \frac{3aq - 6p - 3a(bc - q)}{3a^2 - 2q} =$$
$$= \frac{3(2aq - 3p)}{3a^2 - 2q} = \frac{3a(2q - 3bc)}{3a^2 - 2q}.$$

Therefore

$$\frac{m_a}{k_a} = -\frac{3a^2 - 2q}{2(2q - 3bc)}. \tag{21}$$

However, owing to

$$2q - 3bc = (c - a)(a - b) = CA \cdot AB,$$
$$CA^2 + AB^2 = (a - c)^2 + (a - b)^2 =$$
$$= 2a^2 + (b + c)^2 - 2bc - 2ca - 2ab = 3a^2 - 2q,$$

the statement of the theorem follows. ◊

Alasia ([1], p. 296) has the statement in the euclidean plane: If $ACEE'$ and $ABFF'$ are such rhombuses that the points E' and F' lie on the extensions of the segments $\overline{BA}$ and $\overline{CA}$ through the vertex A, then the point $D = BE \cap CF$ lies on the symmedian AK, where $AD : A_kD = -(CA^2 + AB^2) : (CA \cdot AB)$. In the isotropic plane the feet C_h and B_h of the altitudes of the triangle ABC from the vertices

C and B will have the role of the points E' and F'. The corresponding statement is given by the following theorem.

Theorem 13. *If B_s and C_s are the midpoints of the altitudes $\overline{BB_h}$ and $\overline{CC_h}$ of the triangle ABC and if E and F are the points symmetric to the point A with respect to the points C_s and B_s, then the point $D = BE \cap CF$ lies on the symmedian AA_k of the triangle ABC while*

$$\frac{AD}{A_kD} = -\frac{CA^2 + AB^2}{CA \cdot AB} = -\left(\frac{CA}{AB} + \frac{AB}{CA}\right).$$

Proof. Analogously to the formula (12) we have the formula $C_s = \left(c, -\frac{1}{2}ab\right)$. The point $E = (2c - a, ac)$ is symmetric to the point $A = (a, a^2)$ with respect to the point C_s because of $-ab - a^2 = ac$. The line with the equation

$$y = \frac{q}{3c}x + b^2 - \frac{bq}{3c} \tag{22}$$

passes through the point $B = (b, b^2)$ and it also passes through the point E because of

$$\frac{q}{3c}(2c - a) + b^2 - \frac{bq}{3c} = \frac{2}{3}q - \frac{q}{3c}(a + b) + b^2 = \frac{2}{3}q + \frac{q}{3} + ac - q = ac.$$

From the equation (6) of the symmedian AK and the equation (22) of the line BE we get for the abscissa of the intersection of these lines the following equation

$$\left(\frac{q}{3c} + \frac{2q}{3a}\right)x = bc - \frac{q}{3} - b^2 + \frac{bq}{3c},$$

i.e. the equation

$$\frac{q}{a}(a + 2c)x = (3bc - q)(c - b),$$

which, because of $a + 2c = c - b$, has the solution $x = d$ given by

$$d = \frac{3p}{q} - a.$$

The symmetry of b and c proves that the line CF passes through the point D, too. According to the proof of Th. 12 we get the equality

$$AA_k = 3a\frac{2q - 3bc}{3a^2 - 2q},$$

and now

$$AD = d - a = \frac{3p}{q} - 2a = -\frac{a}{q}(2q - 3bc),$$

so it follows

$$A_kD = AD - AA_k = -a(2q - 3bc)\left(\frac{1}{q} + \frac{3}{3a^2 - 2q}\right) =$$

$$= -\frac{a(2q - 3bc)}{q(3a^2 - 2q)}(3a^2 + q)$$

and because of (21) finally

$$\frac{AD}{A_kD} = \frac{3a^2 - 2q}{3a^2 + q} = \frac{3a^2 - 2q}{3bc - 2q} = \frac{2m_a}{k_a} = -\frac{CA^2 + AB^2}{CA \cdot AB}. \ \diamond$$

Theorem 14. *The lines through the point A, which are antiparallel to the symmedian AK of the triangle ABC with respect to the angles B and C of that triangle, meet the line BC at the points B' and C', whose midpoint is the midpoint of the side BC while $\angle(AB', AC') = \angle(AB, AC)$.* (In euclidean geometry d'Ocagne [6] has this statement too.)

Proof. The lines BA and BC have the slopes $-c$ and $-a$ with the sum b. The line AK from (6) has the slope $-\frac{2q}{3a}$, and if k is the slope of the line AB', then

$$k - \frac{2q}{3a} = b, \quad \text{i.e.} \quad k = b + \frac{2q}{3a}.$$

The line with the equation

$$y = \left(b + \frac{2q}{3a}\right)x + a^2 - ab - \frac{2}{3}q \tag{23}$$

has that slope and passes through the point $A = (a, a^2)$, and that is the line AB'. From the equation (23) and the equation $y = -ax - bc$ of the line BC for the abscissa of the point B' we get the equation

$$\left(a + b + \frac{2q}{3a}\right)x = \frac{2}{3}q - a^2 + ab - bc,$$

i.e. it is equivalent to

$$(2q - 3ca)\frac{x}{a} = -2(a^2 + ab + b^2) - 3a^2 + 3ab + 3b(a + b) = -5a^2 + 4ab + b^2,$$

$$(a - b)(b - c)\frac{x}{a} = -(a - b)(5a + b),$$

with the solution $x = b'$ where the abscissa b' of the point B' is given by the first of the two analogous equalities

$$b' = -a\frac{5a+b}{b-c}, \quad c' = -a\frac{5a+c}{c-b}, \tag{24}$$

while c' is the abscissa of the analogous point C'. From (24) easily follows the equality $b' + c' = -a$, i.e. $b' + c' = b + c$ so the segments $\overline{BC}$ and $\overline{B'C'}$ have the same midpoint. The lines AB' and AC' have the slopes

$$b + \frac{2q}{3a} \quad \text{and} \quad c + \frac{2q}{3a},$$

so we get

$$\angle(AB', AC') = c - b = \angle(AB, AC). \quad \diamond$$

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