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ON A SPECIAL FAMILY OF COMPACT CONVEX SETS IN THE EUCLIDEAN PLANE $\mathbb{R}^2$

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Abstract: We are proving the existence of a supporting circle for a family of compact convex sets K_1, K_2, K_3 which are verifying the conditions: $K_1 \cap \cap K_2 \neq \emptyset$, $K_2 \cap K_3 \neq \emptyset$, $K_3 \cap K_1 \neq \emptyset$, and $K_1 \cap K_2 \cap K_3 = \emptyset$.

The reader unfamiliar with the theory of convex sets is referred to the books [4], [12] or [13]. Let M be a set in the n-dimensional Euclidean space $\mathbb{R}^n$. In the following we shall denote by $\operatorname{int} M$, $\operatorname{cl} M$, ϑM, $\operatorname{conv} M$ the interior, the closure, the boundary and respectively the convex hull of the set M. With $\mathrm{d}(x,y)$ we denote the Euclidean distance of the points x and y and with $L(x,y)$ the line determined by the points x and y. The distance $\mathrm{d}(x,C)$ of a point x in $\mathbb{R}^2$ to the set C is defined by $\mathrm{d}(x,C) = \inf\{\mathrm{d}(x,y) : y \in C\}$. A set C is said to be a Chebyshev set in $\mathbb{R}^2$ if there exists for every $x \in \mathbb{R}^2$ an unique point $n(x)$ in C such that $\mathrm{d}(x, n(x)) = \mathrm{d}(x,C)$. The resulting map $n : \mathbb{R}^2 \to \to C$ is the nearest point map of C. The following theorem was proved by Motzkin in 1935 in [9], [10] (see also Mani-Levitska [8]):

Theorem 1. *A compact subset of the Euclidean plane $\mathbb{R}^2$ is a Chebyshev set if and only if it is convex.*

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We need also the following 2 definitions:

Definition 1. The family $\mathcal{K}$ of sets in $\mathbb{R}^n$ will said to be *independent*, if for any $n+1$ pairwise distinct members $K_1, \ldots, K_{n+1}$ of $\mathcal{K}$, any set of points $p_1, \ldots, p_{n+1}$, where $p_i \in K_i, i = 1, \ldots, n+1$ determines a simplex of dimension n, or equivalently the vectors $p_2 - p_1, \ldots, p_{n+1} - p_1$ are linearly independent.

Definition 2. There will be said that a family $\mathcal{K}$ of sets in the Euclidean space $\mathbb{R}^n$ has a *supporting sphere*, if there exists a sphere S in $\mathbb{R}^n$ having common points with each member of the family $\mathcal{K}$ and the interior of S contains no point of any member of $\mathcal{K}$. If $n = 2$ we shall use instead of supporting sphere the notion of *supporting circle*.

We need the following theorem of C. Berge [2] (see also Th. 3.7.5 in [11]):

Theorem 2. *If M is a convex set and $K_i, i = 1, \ldots, m$, $m \geq 1$ are closed convex sets in $\mathbb{R}^n$ satisfying*

$$\text{(i)} \qquad M \cap \bigcap_{\substack{i=1 \\ i \neq j}}^{m} K_i \neq \emptyset \quad \textit{for} \quad j = 1, \ldots m,$$

$$\text{(ii)} \qquad M \cap \bigcap_{i=1}^{m} K_i = \emptyset,$$

then M *is not contained in the union of the K_i: $M \not\subseteq \bigcup_{i=1}^{m} K_i$.*

An immediate corollary (see also Cor. to Th. 3.7.5 in [11]) is:

Corollary. *Let K_i, $i = 1, \ldots, m$, $m \geq 2$, be closed convex sets in $\mathbb{R}^n$. If their intersection is empty and if the intersection of any $m-1$ of the sets K_i is not empty, then their union is not convex.*

Let us now consider a special family of 3 convex sets in $\mathbb{R}^2$ which is not an independent family.

Lemma 1. *Let K_1, K_2, K_3 be compact convex sets in $\mathbb{R}^2$ such that:*

(1) $K_1 \cap K_2 \neq \emptyset$, $K_2 \cap K_3 \neq \emptyset$, $K_3 \cap K_1 \neq \emptyset$,

(2) $K_1 \cap K_2 \cap K_3 = \emptyset$.

The family $\mathcal{F} = \{K_1 \cap K_2, K_2 \cap K_3, K_3 \cap K_1\}$ is then an independent family of three compact convex sets.

Proof. Indeed if we consider three arbitrary points a, b, c such that $a \in K_2 \cap K_3, b \in K_3 \cap K_1$ and $c \in K_1 \cap K_2$ we have to show that the three points can not be collinear. Let us assume the contrary, then we can suppose without loss of generality that $b \in \operatorname{conv}\{a, c\}$. Because $a \in K_2, c \in K_2$ and K_2 is a convex set it follows that $b \in K_2$. This with $b \in K_3 \cap K_1$ means that $b \in K_1 \cap K_2 \cap K_3$ in contradiction to the property (2) of the sets K_1, K_2, K_3. $\Diamond$

Lemma 2. *Let d, e be two arbitrary distinct points interior to the triangle $T = \triangle abc$. There is then verified one of the conditions:*

(3) *There is a point $m \in \{a, b, c\}$ such that d, e and m are collinear.*

(4) *There are two distinct points p, q in the set $\{a, b, c\}$ and a point n in the interior of the triangle T such that:*

$$d \in \operatorname{int}\{\operatorname{conv}\{n, p\}\} \text{ and } e \in \operatorname{int}\{\operatorname{conv}\{n, q\}\}.$$

Proof. Let us denote with a_1, b_1, c_1 the following points:

$$a_1 = L(b,c) \cap L(a,d), \quad b_1 = L(c,a) \cap L(b,d), \quad c_1 = L(a,b) \cap L(c,d).$$

Because the point $e \in \operatorname{int} T$ we have to distinguish the two cases:

(i) $e \in \operatorname{conv}\{a, a_1\} \cup \operatorname{conv}\{b, b_1\} \cup \operatorname{conv}\{c, c_1\}$, or

(ii) $e \in \operatorname{int}\{\operatorname{conv}\{a, b, c\}\} \setminus \{\operatorname{conv}\{a, a_1\} \cup \operatorname{conv}\{b, b_1\} \cup \operatorname{conv}\{c, c_1\}\}$.

In the case (i) let us first suppose that $e \in \operatorname{conv}\{a, a_1\}$. As we have also $d \in \operatorname{conv}\{a, a_1\}$ by the definition of the point a_1, we have to distinguish then the two subcases: (α) $e \in \operatorname{conv}\{a, d\}$ or (β) $e \in \operatorname{conv}\{d, a_1\}$. In both subcases ($\alpha$) and ($\beta$) we can put $m = a$. If $e \in \operatorname{conv}\{b, b_1\}$ we can put $m = b$ and if $e \in \operatorname{conv}\{c, c_1\}$ we can put $m = c$ and it follows that the points m, d, e are collinear, i.e. the condition (3) of Lemma 2 is verified.

In the case (ii) we have to distinguish 6 subcases:

$$(\gamma)\ \ e \in \operatorname{int}\{\operatorname{conv}\{a, d, c_1\}\}, \quad (\delta)\ \ e \in \operatorname{int}\{\operatorname{conv}\{a, d, b_1\}\},$$

$$(\epsilon)\ \ e \in \operatorname{int}\{\operatorname{conv}\{c, d, b_1\}\}, \quad (\varepsilon)\ \ e \in \operatorname{int}\{\operatorname{conv}\{c, d, a_1\}\},$$

$$(\zeta)\ \ e \in \operatorname{int}\{\operatorname{conv}\{b, d, a_1\}\}, \quad (\eta)\ \ e \in \operatorname{int}\{\operatorname{conv}\{b, d, c_1\}\}.$$

In the cases: (γ) let be $n = L(a,e) \cap L(c,d)$ and we can put $p = c$ and $q = a$, (δ) let be $n = L(a,e) \cap L(b,d)$, $p = b$ and $q = a$, (ϵ) let be $n = L(c,e) \cap L(b,d)$, $p = b$ and $q = c$, (ε) let be $n = L(c,e) \cap L(a,d)$, $p = a$ and $q = c$, (ζ) let be $n = L(b,e) \cap L(a,d)$, $p = a$ and $q = b$, (η) let be $n = L(b,e) \cap L(c,d)$, $p = c$ and $q = b$. With that we have shown that in these 6 cases condition (4) is verified. $\Diamond$

Theorem 3. *Let K_1, K_2, K_3 be three compact convex sets in $\mathbb{R}^2$ which are verifying the conditions* (1) *and* (2) *of Lemma 1. The set $\mathbb{R}^2 \setminus \{K_1 \cup K_2 \cup K_3\}$ consists then of two connected components: an open bounded set S_b and an open unbounded set S_u, i.e. we have*

$$\mathbb{R}^2 \setminus \{K_1 \cup K_2 \cup K_3\} = S_b \cup S_u.$$

Proof. The family $\mathcal{F} = \{K_1 \cap K_2, K_2 \cap K_3, K_3 \cap K_1\}$ is by Lemma 1 an independent family of three compact convex sets. Let us consider three arbitrary points a, b, c such that $a \in K_1 \cap K_2$, $b \in K_2 \cap K_3$ and $c \in K_3 \cap K_1$. The points a, b, c determine then a non-degenerated triangle $\triangle abc$. Let us now consider the set $T = \text{conv}\{a, b, c\}$ and the three compact convex sets: $C_1 = T \cap K_1$, $C_2 = T \cap K_2$ and $C_3 = T \cap K_3$. We have of course: $a \in C_1 \cap C_2$, $b \in C_2 \cap C_3$, and $c \in C_3 \cap C_1$. The sets C_1, C_2, C_3 verify thereby the conditions:

(1′) $C_1 \cap C_2 \neq \emptyset$, $C_2 \cap C_3 \neq \emptyset$, $C_3 \cap C_1 \neq \emptyset$

(2′) $C_1 \cap C_2 \cap C_3 = \emptyset$.

By the Cor. to Th. 2 it follows then that the union $C_1 \cup C_2 \cup C_3$ is not a convex set. As $C_1 \cup C_2 \cup C_3 \subset T$ results hereby the existence of a point $x \in T$ such that $x \notin C_1 \cup C_2 \cup C_3$, i.e. we have proved the existence of at least one bounded connected component of $\mathbb{R}^2 \setminus \{C_1 \cup C_2 \cup C_3\}$.

Let us suppose there are at least two bounded connected components B_1 and B_2 of the set $\mathbb{R}^2 \setminus \{C_1 \cup C_2 \cup C_3\}$. Consider now two arbitrary points d and e such that $d \in B_1$ and $e \in B_2$. The points d and e are of course in the interior of the triangle $\triangle abc$. By Lemma 2 there is verified one of the conditions (3) or (4).

Let us suppose there is verified condition (3), i.e. there is a point $m \in \{a, b, c\}$ such that d, e and m are collinear. Without loss of generality we can suppose that we have $m = a$ and $d \in \text{int}\{\text{conv}\{a, e\}\}$. Because B_1 and B_2 are two distinct connected components of the set $\mathbb{R}^2 \setminus \{C_1 \cup C_2 \cup C_3\}$ there must exist on the segment de a point f such that $f \in \{C_1 \cup C_2 \cup C_3\}$. We have of course also $a \in C_1 \cap C_2$. If we have $f \in C_1$ follows immediately from the convexity of the set C_1 that we must have $d \in C_1$ in contradiction to the definition of the set B_1. If we have $f \in C_2$ follows immediately from the convexity of the set C_2 that we must have $d \in C_2$ in contradiction to the definition of the set B_1. If we have $f \in C_3$ it follows from $b \in C_3$, $c \in C_3$ and from $e \in \text{conv}\{b, c, f\}$ that $e \in C_3$ in contradiction to the definition of the connected component B_2. In conclusion the condition (3) can not be verified. Let us now suppose that condition (4) is verified. We can then select the points p and q from the set $\{a, b, c\}$ and a point $n \in \text{int}\, T$ such that:

$$d \in \text{int}\{\text{conv}\{n, p\}\} \text{ and } e \in \text{int}\{\text{conv}\{n, q\}\}.$$

Without loss of generality we can suppose $p = a$ and $q = b$ and that:

$$d \in \text{int}\{\text{conv}\{n, a\}\} \text{ and } e \in \text{int}\{\text{conv}\{n, b\}\}.$$

On the broken line $\text{conv}\{d, n\} \cup \text{conv}\{n, e\}$ there must be at least one point p_0 such that $p_0 \in C_1 \cup C_2 \cup C_3$, because B_1 and B_2 are two distinct connected components of the set $T \setminus \{C_1 \cup C_2 \cup C_3\}$. Let us first suppose that we have $p_0 \in \text{conv}\{d, n\}$. If $p_0 \in C_1$ it follows immediately from $a \in C_1$ and the convexity of the set C_1 that $d \in C_1$ contrary to the definition of the component B_1. If $p_0 \in C_2$ it follows immediately from $a \in C_2$ and the convexity of the set C_2 that $d \in C_2$, again in contradiction to the definition of the component B_1. If $p_0 \in$ $\in C_3$ it is easy to see that the point e is interior to the triangle Δbcp_0. But because $b \in C_3$ and $c \in C_3$ that means $e \in \text{conv}\{b, c, p_0\} \subset C_3$, in contradiction to the definition of the component B_2. In a similar way we get a contradiction if we suppose that $p_0 \in \text{conv}\{n, e\}$. In conclusion there is exactly one bounded connected component of the set $T \setminus \{C_1 \cup C_2 \cup C_3\}$.

The existence and the unicity of the open unbounded connected component S_u is obvious. ◊

Definition 3. Let K_1, K_2, K_3 be smooth compact convex sets in $\mathbb{R}^2$, which are verifying the conditions (1) and (2) of Lemma 1. The boundary T_c (i.e. $T_c = \vartheta S_b$) of the bounded component S_b of the set $\mathbb{R}^2 \setminus \{K_1 \cup K_2 \cup K_3\}$ will be called a *curved triangle* with the vertices a, b and c, where a is the unique point of $T_c \cap K_1 \cap K_2$, b is the unique point of $T_c \cap K_2 \cap K_3$ and c is the unique point of $T_c \cap K_3 \cap K_1$. The sides of the curved triangle T_c are the arcs α, β, γ, where:

$$\alpha = \vartheta K_3 \cap \vartheta S_b,\ \beta = \vartheta K_1 \cap \vartheta S_b \text{ and } \gamma = \vartheta K_2 \cap \vartheta S_b.$$

If the unique supporting line L_a^1 for the smooth convex set K_1 going through the point a and the unique supporting line L_a^2 for the smooth convex set K_2 going through the point a are verifying the condition $L_a^1 \cap L_a^2 = \{a\}$ we shall say that a is a *crossing point.*

In the following we need also the notion of the ϵ-neighbourhood of a convex body B ([3] p. 2), which is also known in the German literature as the "Parallelkörper" ([1] p. 48, [4] p. 30, [12] p. 160) i.e. the parallel body of a convex body: $N_\epsilon(B) = \{x \in \mathbb{R}_n : \text{d}(x, B) \leq \epsilon\}$, which is also a convex set.

The following simple lemma is perhaps known; we supply its proof here for the sake of completeness.

Lemma 3. *The ϵ-neighbourhood $N_\epsilon(C)$ of a compact convex set C in the Euclidean space $\mathbb{R}^n$ is smooth.*

Proof. Let us suppose the contrary, i.e. there is a point $a \in \vartheta N_\epsilon(C)$ such that through the point a there are going two supporting hyperplanes H_1 and H_2 for the convex body $N_\epsilon(C)$. Let us the denote with H_1' and H_2' the closed half-spaces determined by the hyperplanes H_1

and respectively H_2 which contain the set $N_\epsilon(C)$. As $a \in \vartheta N_\epsilon(C)$, there must exist a point b in the convex set C such that $\mathrm{d}(b,a) \leq \epsilon$. The point b belongs of course to the intersection $H'_1 \cap H'_2$. Consider now the closed ball of radius ϵ with the center b, $B = \{x : x \in \mathbb{R}^n, \mathrm{d}(x,b) \leq \epsilon\}$. The set B is a subset of $N_\epsilon(C)$ and we must have $B \subset N_\epsilon(C) \subset H'_1 \cap \cap H'_2$. This is impossible because the point a belongs to the boundary of $H'_1 \cap H'_2$ and $\mathrm{d}(b,a) \leq \epsilon$. ◊

In the papers [6], [7] we have used Brouwer's fixed point theorem for the proof of some geometric properties of families of compact convex sets such as the existence of an equally spaced point (respectively the existence of a supporting sphere) for a family of compact convex sets in $\mathbb{R}^d$.

In [6] we have proved the following theorem (see also [5]):

Theorem 4. *Let $K_1, \ldots, K_{n+1}$ be a family of independent convex and compact sets in $\mathbb{R}^n$. Then this family of convex sets admits exactly one supporting sphere.*

The independence of the sets was an essential condition in the proof of Th. 4. We prove now for the case $n = 2$ a theorem similar to the Th. 4 for a family of compact smooth convex sets which are not independent:

Theorem 5. *Let T_c be a curved triangle determined by the three compact smooth convex sets K_1, K_2, K_3 in $\mathbb{R}^2$, which are verifying the conditions (1) and (2) of Lemma 1 and such that the vertices a, b and c are crossing points for the curved triangle T_c. There exists then a supporting circle for the three sets K_1, K_2, K_3.*

Proof. Let us consider the triangle $T = \mathrm{conv}\{a,b,c\}$ and the 3 convex sets: $C_1 = \mathrm{conv}\{\mathrm{conv}\{a,c\},\beta\} = T \cap K_1$, $C_2 = \mathrm{conv}\{\mathrm{conv}\{a,b\},\gamma\} = = T \cap K_2$, and $C_3 = \mathrm{conv}\{\mathrm{conv}\{b,c\},\alpha\} = T \cap K_3$. Consider also the closed set $S = T_c \cup S_b \subset T$, where S_b is the bounded component of $\mathbb{R}^2 \setminus \{K_1 \cup K_2 \cup K_3\}$, and $f : T \to S$ defined by:

(a) for $x \in C_1$ $f(x) = \beta \cap \mathrm{conv}\{x,b\}$,
(b) for $x \in C_2$ $f(x) = \gamma \cap \mathrm{conv}\{x,c\}$,
(c) for $x \in C_3$ $f(x) = \alpha \cap \mathrm{conv}\{x,a\}$,
(d) for $x \in T \setminus \{C_1 \cup C_2 \cup C_3\}$ $f(x) = x$.

The map $f : T \to S$ so defined is a continuous map.

We define then a map $g : S \to S$ in the following way: Let $x \in S$ be an arbitrary point. We consider again the nearest point maps:

$n_i : \mathbb{R}^2 \to K_i$ the nearest point map on the set K_i, for i = 1,2,3.

From the smoothness of the convex sets K_1, K_2, K_3 follows the existence of an unique supporting line L_1 for the set K_1 such that $n_1(x) \in L_1$, an unique supporting line L_2 for the set K_2 such that $n_2(x) \in L_2$ and an unique supporting line L_3 for the set K_3 such that $n_3(x) \in L_3$. Because the points a, b, c are crossing points for the curved triangle T_c it follows that the three lines are pairwise distinct lines even in the special case, when the point x coincides with one of the vertices of the curved triangle T_c i.e. $x \in \{a, b, c\}$. Let us consider the points $a' = L_1 \cap L_2$, $b' = L_2 \cap \cap L_3$ and $c' = L_3 \cap L_1$. The triangle $\Delta a'b'c'$ determined by the tree lines L_1, L_2, L_3 is contained in the set S and the three lines L_1, L_2, L_3 depend continuously on x. Let us now consider the center $g(x)$ of the circle inscribed to the triangle $\Delta a'b'c'$. The map $g : S \to S$ defined in this way is also a continuous map. We get by the composition of the two continuous maps $f : T \to S$ and $g : S \to S$ a continuous gf map $gf : T \to T$ of the convex set T on itself defined by $gf(x) = g(f(x))$. By Brouwer's fixed point theorem follows then the existence of a point $x_0 \in T$ such that $gf(x_0) = x_0$. It follows then immediately that the point x_0 is equally spaced from the three convex smooth sets K_1, K_2 and K_3 i.e. $\mathrm{d}(x_0, K_1) = \mathrm{d}(x_0, K_2) = \mathrm{d}(x_0, K_2)$. With this we have proved the existence of an supporting circle for the sets K_1, K_2 and K_3. $\Diamond$

Lemma 4. *Let C be a compact convex set in $\mathbb{R}^2$ and $\{x_k : k = 1, 2, \ldots\}$ a sequence of points from $\mathbb{R}^2$ such that $\lim_{k\to\infty} x_k = p$, where $p \notin C$. Consider $N_{1/k}(C)$ the $1/k$-neighbourhood of the set C, $n_{1/k} : \mathbb{R}^2 \to \to N_{1/k}(C)$ the nearest point map of $N_{1/k}(C)$ and $n : \mathbb{R}^2 \to C$ the nearest point map of C. We have then:*

$$\lim_{k\to\infty} n_{1/k}(p) = n(p), \tag{6}$$

and

$$\lim_{k\to\infty} n_{1/k}(x_k) = n(p). \tag{7}$$

Proof. There is a k_0 such that we have $n_{1/k}(p) \in \vartheta N_{1/k}(C) \subset \subset N_{1/k_0}(C)$ for $k \geq k_0$. We show first that: $\lim_{k\to\infty} n_{1/k}(p) = n(p)$. If we suppose the contrary, there exists a number $\epsilon > 0$ such that the points of a subsequence $s_u = \{n_{1/k_m}(p) : k_m = 1, 2, \ldots\}$ of the sequence $\{n_{1/k}(p) : k = 1, 2, \ldots\}$ are all in the compact set $S = = N_{1/k_0}(C) \backslash B(n(p), \epsilon)$, where $B(n(p), \epsilon) = \{x : x \in \mathbb{R}^2, \mathrm{d}(x, n(p)) < \epsilon\}$. By the compactness of the set S there is a subsequence $\{n_{1/k_{m_n}}(p) : k_{m_n} = = 1, 2, \ldots\}$ of the sequence s_u and a point $q \in S$ such that:

$$\lim_{n\to\infty} n_{1/k_{m_n}}(p) = q. \tag{8}$$

From $C \subset N_{\epsilon_1}(C) \subset N_{\epsilon_2}(C)$ for $\epsilon_1 < \epsilon_2$ and (8) follows immediately:

$$q \in \bigcap_{n=1}^{\infty} N_{1/k_{m_n}}(C) = C.$$

By the definition of the set S we have of course $q \neq n(p)$.

Because $n_{1/k_{m_n}}(p) \in \vartheta N_{1/k_{m_n}}(C)$ there is a point $q_{k_{m_n}} \in C$ such that: $\mathrm{d}(n_{1/k_{m_n}}(p), q_{k_{m_n}}) = 1/k_{m_n}$ and $\mathrm{d}(p, q_{k_{m_n}}) - \mathrm{d}(q_{k_{m_n}}, n_{1/k_{m_n}}(p)) \leq \mathrm{d}(p, n_{1/k_{m_n}}(p))$ i.e. we have $\mathrm{d}(p, q_{k_{m_n}}) - 1/k_{m_n} \leq \mathrm{d}(p, n_{1/k_{m_n}}(p))$. As $\mathrm{d}(p, n(p)) \leq \mathrm{d}(p, q_{k_{m_n}})$ we have also: $\mathrm{d}(p, n(p)) - 1/k_{m_n} \leq \mathrm{d}(p, n_{1/k_{m_n}}(p))$. Because $n(p) \in C \subset N_{1/k_{m_n}}(C)$ we have also the inequality $\mathrm{d}(p, n_{1/k_{m_n}}(p)) \leq \mathrm{d}(p, n(p))$ i.e. we have

$$\mathrm{d}(p, n(p)) - 1/k_{m_n} \leq \mathrm{d}(p, n_{1/k_{m_n}}(p)) \leq \mathrm{d}(p, n(p))$$

and therefore we have: $\lim_{n\to\infty} \mathrm{d}(p, n_{1/k_{m_n}}(p)) = \mathrm{d}(p, n(p))$. Using the continuity of the distance function and (8) we get:

$$\mathrm{d}(p, n(p)) = \lim_{n\to\infty} \mathrm{d}(p, n_{1/k_{m_n}}(p)) = \mathrm{d}(p, \lim_{n\to\infty} n_{1/k_{m_n}}(p)) = \mathrm{d}(p, q).$$

The convex set C is by Th. 1 a Chebyshev set. As $q \in C$, $n(p) \in C$, $q \neq n(p)$ and $\mathrm{d}(p, q) = \mathrm{d}(p, n(p))$ we have got a contradiction. It follows that:

$$\lim_{k\to\infty} n_{1/k}(p) = n(p).$$

Consider now the sequence $\{x_k : k = 1, 2, \ldots\}$ such that $\lim_{k\to\infty} x_k = = p$. Because $n_{1/k}(x_k)$ and $n_{1/k}(p)$ can be considered as the projections of the point x_k and respectively of the point p on the convex set $N_{1/k}(C)$ we have: $0 \leq \mathrm{d}(n_{1/k}(x_k), n_{1/k}(p)) \leq \mathrm{d}(x_k, p)$. From this inequality and from $\lim_{k\to\infty} n_{1/k}(p) = n(p)$ we can deduce that we have also:

$$\lim_{k\to\infty} n_{1/k}(x_k) = n(p). \quad \diamond$$

We shall prove a generalization of Th. 5 by renouncing to the smoothness hypotheses of the convex sets K_1, K_2, K_3 and the "crossing point property" of the vertices of the curved triangle determined by the 3 convex sets:

Theorem 6. *Let K_1, K_2, K_3 be three compact convex sets in $\mathbb{R}^2$ which are verifying the conditions* (1) *and* (2) *of Lemma 1. There exists then a supporting circle for the three sets K_1, K_2, K_3.*

Proof. Let us consider the curved triangle T_c with the vertices $a, b,$ and c in the sense of Def. 3 and a point p in the interior of the curved triangle T_c. Consider now the distances from the point p to the 3 convex compact sets K_1, K_2, K_3: $\mathrm{d}_i = \mathrm{d}(p, K_i), i = 1,2,3$. Let us select a value of ϵ such that $\epsilon < \min\{\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3\}$. The ϵ-neighbourhoods $N_\epsilon(K_1), N_\epsilon(K_2), N_\epsilon(K_3)$ of the three convex sets K_1, K_2, K_3 are by Lemma 3 smooth compact convex sets and it is easy to show that they verify the conditions:

(1″) $N_\epsilon(K_1) \cap N_\epsilon(K_2) \neq \emptyset$, $N_\epsilon(K_2) \cap N_\epsilon(K_3) \neq \emptyset$, $N_\epsilon(K_3) \cap$ $\cap N_\epsilon(K_1) \neq \emptyset$

(2″) $N_\epsilon(K_1) \cap N_\epsilon(K_2) \cap N_\epsilon(K_3) = \emptyset$.

The three smooth convex compact sets $N_\epsilon(K_1), N_\epsilon(K_2), N_\epsilon(K_3)$ determine then a curved triangle T_ϵ with the vertices $a_\epsilon, b_\epsilon, c_\epsilon$ and the curved sides $\alpha_\epsilon, \beta_\epsilon, \gamma_\epsilon$. Consider now the unique supporting line L^1_ϵ through the point a_ϵ for the smooth convex set $N_\epsilon(K_1)$ and the unique supporting line L^2_ϵ through the point a_ϵ for the smooth convex set $N_\epsilon(K_2)$. We assert now that the two lines L^1_ϵ and L^2_ϵ cannot coincide, respectively that $L^1_\epsilon \cap L^2_\epsilon = \{a_\epsilon\}$. Let us suppose the contrary. The supporting lines L^1_ϵ and L^2_ϵ must then coincide. Consider now the lines L^1 and L^2 parallel to the line L^1_ϵ at distance ϵ to it in the halfplane determined by the line L^1_ϵ which contains the convex set $N_\epsilon(K_1)$ and respectively the convex set $N_\epsilon(K_2)$. The interior of the band bounded by the lines L^1_ϵ and L^1 cannot contain any point of the convex set K_1, because if we would suppose the contrary, i.e. the existence of a point p_1 in that band such that $p_1 \in K_1$ it would follow that the disc $D(p_1, \epsilon)$ with the center p_1 and the radius ϵ has to contain points from the halfplane determined by the line L^1_ϵ, which contains the set K_2, in contradiction to the definition of the supporting line L^1_ϵ. Analogous we can show that he interior of the band bounded by the lines L^1_ϵ and L^2 cannot contain any point of the convex set K_2. That means that the band determined by the two lines L^1 and L^2 is separating the two convex sets K_1 and K_2. But this is contradicting condition (1) of Th. 6. We have so proved that $L^1_\epsilon \cap L^2_\epsilon = \{a_\epsilon\}$, i.e. a_ϵ is a crossing point for the curved triangle T_ϵ. In the same way we can prove that b_ϵ and c_ϵ are crossing points for the curved triangles T_ϵ. From Th. 5 follows then the existence of a point x_ϵ such that:

$$\mathrm{d}(x_\epsilon, N_\epsilon(K_1)) = \mathrm{d}(x_\epsilon, N_\epsilon(K_2)) = \mathrm{d}(x_\epsilon, N_\epsilon(K_3)).$$

For $0 \leq \epsilon < \min\{d_1, d_2, d_3\}$ we have of course $T_\epsilon \subset T_c$. We consider again:

$n_i^\epsilon : \mathbb{R}^2 \to N_\epsilon(K_i)$ the nearest point map on the set $N_\epsilon(K_i)$, $i=1,2,3$. If we select now $\epsilon = 1/m$, then there exists a number m_0 such that for $m \geq m_0$ we have: $1/m \leq \min\{d_1, d_2, d_3\}$. Consider now for the so selected $\epsilon = 1/m$ the corresponding points x_m such that:

$$\mathrm{d}(x_m, n_1^{1/m}(x_m)) = \mathrm{d}(x_m, n_2^{1/m}(x_m)) = \mathrm{d}(x_m, n_3^{1/m}(x_m)).$$

Consider the 4 sequences of points from the compact set T_c:

$$\{x_m : m = 1,2,\ldots\}, \qquad \{n_1^{1/m}(x_m) : m = 1,2,\ldots\},$$
$$\{n_2^{1/m}(x_m) : m = 1,2,\ldots\}, \quad \{n_3^{1/m}(x_m) : m = 1,2,\ldots\}.$$

From the compactness of the set T_c and by 4 repeated selections follows the existence of the four subsequences:

$$\{x_{m_k} : k = 1,2,\ldots\}, \qquad \{n_1^{1/m_k}(x_{m_k}) : k = 1,2,\ldots\},$$
$$\{n_2^{1/m_k}(x_{m_k}) : k = 1,2,\ldots\}, \quad \{n_3^{1/m_k}(x_{m_k}) : k = 1,2,\ldots\}$$

and the existence of the points $k_i \in K_i, i = 1,2,3$ and $x_0 \in T_c$ such that:

$$\lim_{k\to\infty} x_{m_k} = x_0,$$

$$\lim_{k\to\infty} n_1^{1/m_k}(x_{m_k}) = k_1, \lim_{k\to\infty} n_2^{1/m_k}(x_{m_k}) = k_2, \lim_{k\to\infty} n_3^{1/m_k}(x_{m_k}) = k_3,$$

and also:

$$\mathrm{d}(x_{m_k}, n_1^{1/m_k}(x_{m_k})) = \mathrm{d}(x_{m_k}, n_2^{1/m_k}(x_{m_k})) = \mathrm{d}(x_{m_k}, n_3^{1/m_k}(x_{m_k}))$$

Hence it follows that: $\mathrm{d}(x_0, k_1) = \mathrm{d}(x_0, k_2) = \mathrm{d}(x_0, k_1)$. By Lemma 4 we have then: $k_1 = n_1(x_0), k_2 = n_2(x_0), k_3 = n_3(x_0)$, where n_i is the nearest point map on K_i for $i = 1,2,3$ and $\mathrm{d}(x_0, n_1(x_0)) = \mathrm{d}(x_0, n_2(x_0)) = \mathrm{d}(x_0, n_3(x_0))$. ◊

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