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A PLANE MODEL OF THE BRONCHIAL TREE

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Abstract: In his book "The fractal Geometry in Nature" B. B. Mandelbrot gives computer pictures of plane models for the bronchial tree. He renounces of an exact mathematical handling. We present here another model, together with a detailed description of the construction and with an exact mathematical investigation. So our paper may be considered as a completion of Mandelbrot's work.

It is important to describe the reality using mathematics. Today people are speaking about models.

For a model it is not necessary to grasp all the details of a procedure or an object. On the contrary! Sometimes we must give up certain properties within a model. Then the model describes the reality only approximately – it is incomplete.

Today also in the field of medicine such models are constructed. So there exist for example models describing arterial trees in the kidneys or in the retina. Other models are used investigating the emergence of cancer or special metabolic diseases. In the following the bronchial tree is investigated.

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1. Demands to a model of the bronchial tree

Fig. 1 shows the X-ray picture of one lobe of the human lung. The bronchial tubes are clearly to recognize.

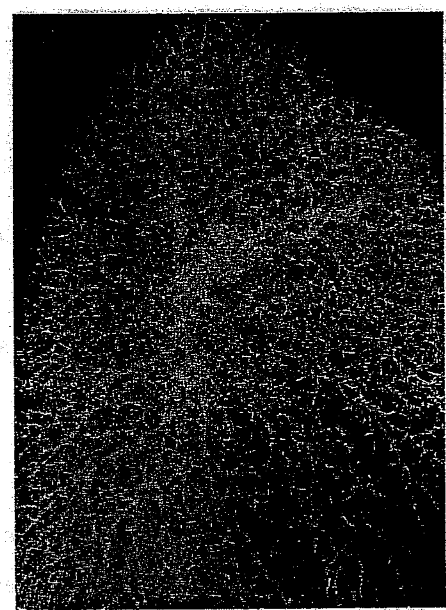

Fig. 1. X-ray picture of one lobe of the human lung.

Anatomists are looking for a suitable geometric model of the bronchial tree. They gave some demands which the model should fulfil. In this paper we restrict ourselves to consider only plane models (X-ray pictures).

A. Our model shall reach a certain limit curve (thorax) – but never leave it. The model then is embedded in the limit curve.
B. The closed limit curve shall be completely filled by the model.
C. The model shall have no overlappings because bronchial tubes do not intersect. (In the X-ray picture this is not fulfilled. In spite of this fact we maintain Demand C – to avoid mathematical difficulties.)
D. The model shall be fractal. (In this paper we denote a point set fractal if the corresponding fractal dimension is not an integer.)

Sometimes objects fulfilling all these four demands are denoted as *physiological fractals*.

2. Construction of models

2.1. A first model

Fig. 2 (left) shows a string tree. In each ramification point two branches emerge. The opening angle 2ϑ is always the same with $0° < < \vartheta \leq 90°$. The intervals are shortened in respect to the preceding ones by the factor r with $0 < r < 1$. The tree shall be symmetric.

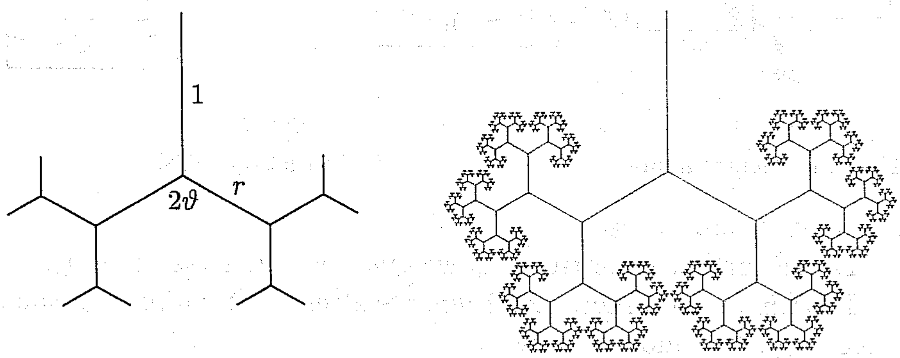

Fig. 2. A string tree and the "golden tree".

Using this procedure in case $\vartheta = 60°$, $r = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}$ (ϕ golden number) the computer yields the "golden tree" (Fig. 2, right). This tree may be a first model of the lung. Demands A, C and D are fulfilled. But the situation is described very roughly. The bronchial tubes are not at all strings.

2.2. Another model

Stimulated by B. B. Mandelbrot [3], we now use rectangles instead of the intervals in Fig. 2.

2.2.1. The starting figure

We start with a rectangle (a, b). Reduction with the factors x, y and $0 < x, y < 1$ gives two rectangles (ax, bx), (ay, by). These two rectangles are added to the starting rectangle in the following way (Fig. 3). The rectangle (ay, by) touches (a, b) and (ax, bx) is moved away. Both rectangles are "translated". The point Z in Fig. 3 is called *ramification point*. The small hatched rectangle $(b - pb, pbx + b(y - x))$ is added. The factor $p < 1$ gives the degree of the respective translation.

With this the fundamental figure is perfect, using the fundamental numbers a, b, x, y, p.

The figures in all the following text are only sketches and not exactly scaled – excepting the computer pictures.

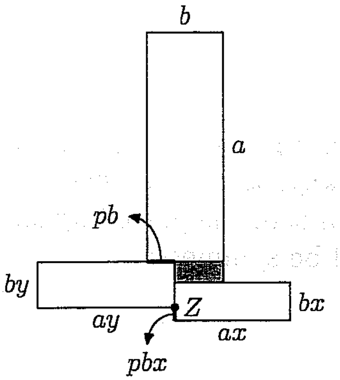

Fig. 3.
The fundamental figure.

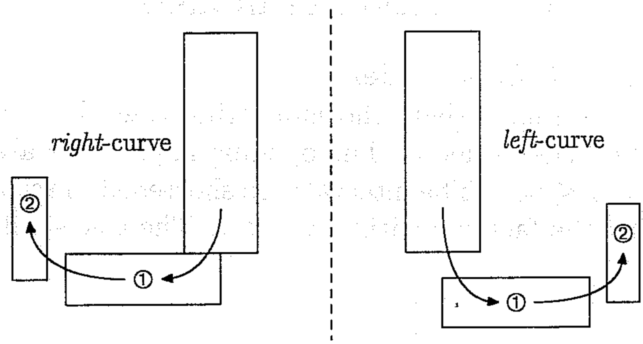

Fig. 4.
Construction rules.

2.2.2. Further procedure

Doing further constructions we give two rules (see Fig. 4).

RULE I: Let a rectangle 1 emerge after a left-curve then the next left-rectangle 2 is moved away.

RULE II: Let a rectangle 1 emerge after a right-curve then the next right-rectangle 2 is moved away.

With our rules I and II we have an algorithm. Using the rules again and again we finally obtain our new (plane) model for the bronchial tubes. Fig. 5 shows some iterations.

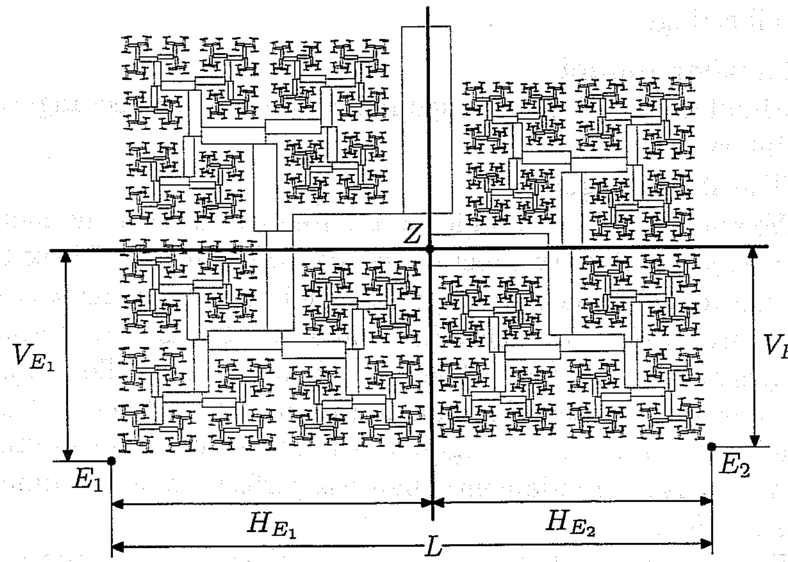

Fig. 5. The new lung model.

2.2.3. What about selfsimilarity?

Looking at Fig. 5 we discover that each rectangle is a trunk of a "subtree". These smaller trees can be developed in different directions – horizontal or vertical. A suitable magnification of subtrees yields again the complete tree.

There is no selfsimilarity in the strict sense [6]. Therefore we speak only about quasi-selfsimilarity. We shall use this notation very often.

3. Calculation of some border segments

Now we calculate the border sides H_{E_1}, H_{E_2}, V_{E_1}, V_{E_2}, L shown in Fig. 5. In the first four cases the calculations are similar, but sometimes different too.

3.1. The segment H_{E_1}

(1)
$$H_{E_1} = y\frac{a + by - pby}{1 - y^2}.$$

The **proof** of this formula is given in several steps.

a) The first step: From Z to F_1.

The horizontal road from Z to F_1 is composed by three sections – as Fig. 6 (left) shows.

$$A_1 = ay + by^2 - pby^2 = y(a + by - pby).$$

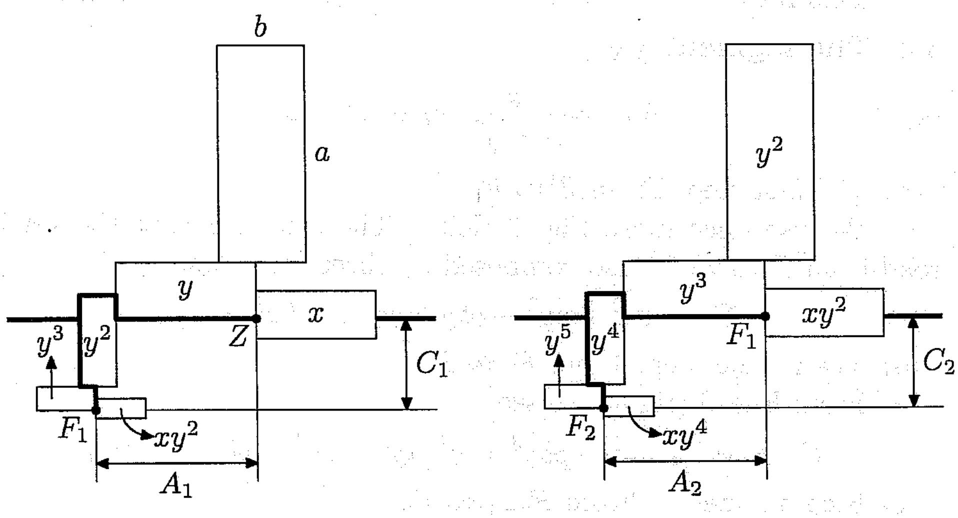

Fig. 6. The segment H_{E_1}.

b) The second step: From F_1 to F_2.
The interval A_2 is composed by three sections too (Fig. 6, right).

$$A_2 = ay^3 + by^4 - pby^4 = y^3(a + by - pby) = y^2 A_1.$$

c) Step number n: From F_{n-1} to F_n.

$$A_n = y^{2n-2}\left(ay + by^2 - pby^2\right) = y^{2n-2} A_1 \quad \text{with} \quad n \in \mathbb{N}.$$

d) Adding up all steps:

$$A_1 + \cdots + A_n = \left(1 + y^2 + \cdots + y^{2n-2}\right) A_1 = A_1 \frac{1 - y^{2n}}{1 - y^2}.$$

e) Limit:
Because of $y < 1$ we have

$$H_{E_1} = \lim_{n\to\infty} \frac{1 - y^{2n}}{1 - y^2} A_1 = \frac{y}{1 - y^2}(a + by - pby). \quad \Diamond$$

3.2. The segment H_{E_2}

(2) $$H_{E_2} = \frac{x}{1 - y^2}(a + by - pby).$$

The proof is done in complete analogy to 3.1. ◊

3.3. The segment L

(3). $$L = \frac{x + y}{1 - y^2}(a + by - pby)$$

This result follows immediately with (1) and (2) by addition. ◊

3.4. The segment V_{E_1}

(4) $$V_{E_1} = \frac{y}{1 - y^2}(ay + by^2 - pb).$$

a) The first step: From Z to F_1.
We use once more Fig. 6 (left). There we see that the vertical road from Z to F_1 is also composed by three intervals.

$$C_1 = ay^2 + by^3 - pby = y(ay + by^2 - pb).$$

b) The second step: From F_1 to F_2.
From Fig. 6 (right) we see

$$C_2 = ay^4 + by^5 - pby^3 = y^2(ay^2 + by^3 - pby) = y^2 C_1.$$

c) Step number n: From F_{n-1} to F_n.
Using complete induction we obtain

$$C_n = y^{2n-2}(ay^2 + by^3 - pby) = y^{2n-2}C_1 \quad \text{with} \quad n \in \mathbb{N}.$$

d) Adding up all steps:

$$C_1 + \cdots + C_n = \left(1 + y^2 + \cdots + y^{2n-2}\right) C_1 = C_1 \frac{1 - y^{2n}}{1 - y^2}.$$

e) Limit:

$$V_{E_1} = \lim_{n \to \infty} \frac{1 - y^{2n}}{1 - y^2} C_1 = \frac{y}{1 - y^2}(ay + by^2 - pb). \quad \Diamond$$

3.5. The segment V_{E_2}

(5) $$V_{E_2} = pbx + \frac{x}{1 - y^2}(ay + by^2 - pb).$$

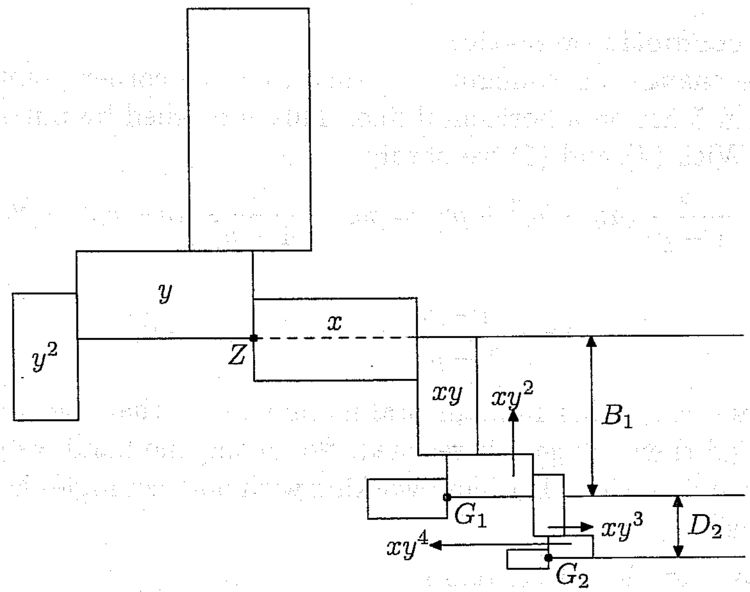

Fig. 7. The segment V_{E_2}.

a) The first step: From Z to G_1.

The vertical road from Z to G_1 is composed in the following way:

$$B_1 = axy + bxy^2 + pbx - pbx = pbx + (axy + bxy^2 - pbx) = pbx + D_1.$$

Now all is running exactly as in the case V_{E_1}.

b) The second step: From G_1 to G_2.

$$B_2 = D_2 = axy^3 + bxy^4 - pbxy^2 = y^2(axy + bxy^2 - pbx) = y^2 D_1.$$

c) Step number n: From G_{n-1} to G_n.

$$B_n = D_n = y^{2n-2} D_1 \quad \text{with} \quad n \in \mathbb{N}.$$

d) Adding up all steps:

$$B_1 + \cdots + B_n = pbx + D_1 \left(1 + y^2 + \cdots + y^{2n-2}\right) = pbx + D_1 \frac{1-y^{2n}}{1-y^2}.$$

e) Limit:

$$V_{E_2} = pbx + \lim_{n\to\infty} \frac{1-y^{2n}}{1-y^2} D_1 = pbx + \frac{x}{1-y^2}(ay + by^2 - pb). \quad \diamond$$

4. Connections between the fundamental numbers a, b, x, y, p

4.1. A cosmetic operation

We change our configuration such that the corner points E_1 and E_2 in Fig. 5 are on a horizontal line. This is reached by putting $V_{E_1} = V_{E_2}$. With (4) and (5) we obtain

$$\frac{y}{1-y^2}(ay + by^2 - pb) = pbx + \frac{x}{1-y^2}(ay + by^2 - pb)$$

or

$$pbx + \frac{x-y}{1-y^2}(ay + by^2 - pb) = 0. \tag{6}$$

If we choose our fundamental numbers such that the equation (6) ist fulfilled then our goal is reached. So on the one hand we get better pictures and on the other hand working with our rectangles is becoming much easier.

4.2. An additional relation

We consider the rectangles $(ay^2,\ by^2)$, $(axy,\ bxy)$ in Fig. 7 as trunks of subtrees. Because of quasi-similarity the limit interval on the line through E_1 and E_2 (Fig. 5) have the lengths y^2L and xyL respectively. Now we require that these two intervals do not overlap. This is reached by the demand

$$L = y^2L + xyL \quad \text{or} \quad x = \frac{1-y^2}{y}. \tag{7}$$

Because of quasi-similarity the property of non-overlapping is inherited to all subtrees. Therefore Demand C given in Sec. 1 is fulfilled.

4.3. And once more a relation

Have a look at Fig. 8, but at the computer picture in Fig. 9 too.

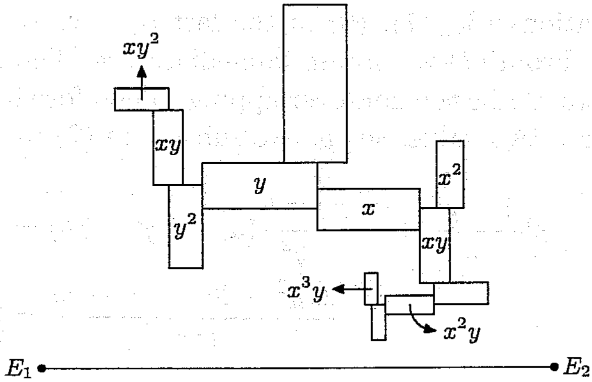

Fig. 8. Concerning the proof in 4.3.

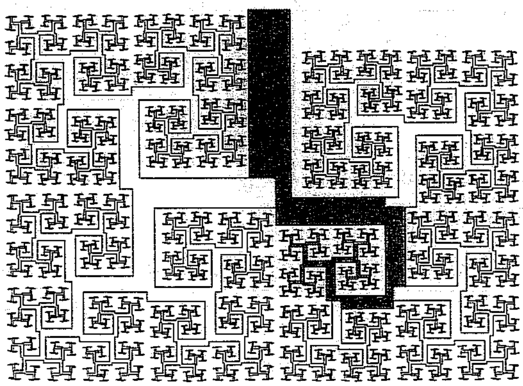

Fig. 9. Once more concerning the proof in 4.3.

We see that above the rectangle $(ax^3y,\ bx^3y)$ a subtree develops. If this tree touches the rectangle $(ax,\ bx)$ and there does not occur any overlapping then we have $x^3yL = ax$. Using 3.3 (3) we obtain

$$x^2y\frac{x+y}{1-y^2}(a+by-pby) = a. \tag{8}$$

With (7) the property of non-overlapping was already secured. Now we have a new formulation of this fact, using a connection between the fundamental numbers.

5. An equation of degree 5 in y

5.1. Some calculations

Let a, b, y be given, then we calculate x and p. To do this we use the equations (6), (7), (8) in the last chapter (3 equations and 2 unknowns!). From (7) we obtain immediately x. The relations (7) and (8) both characterize the non-overlapping. Therefore it is not surprising that we obtain two values for p. We substitute (7) in (6)

$$pb\frac{1-y^2}{y}+\frac{\frac{1-y^2}{y}-y}{1-y^2}(ay+by^2-pb)=0,$$

$$\frac{2ay^2+2by^3-a-by}{by^3}=p. \tag{9}$$

And (7) is substituted in (8)

$$\left(\frac{1-y^2}{y}\right)^2 y\frac{\frac{1-y^2}{y}+y}{1-y^2}(a+by-pby)=a,$$

$$\frac{a+by-2ay^2-by^3}{yb\,(1-y^2)}=p. \tag{10}$$

5.2. Equation of degree 5 in y

We identify (9) and (10)

$$\frac{2ay^2+2by^3-a-by}{by^3}=\frac{a+by-2ay^2-by^3}{yb\,(1-y^2)},$$

$$by^5-2by^3-2ay^2+by+a=0. \tag{11}$$

5.3. Some consequences

5.3.1. $y^2=p$

To prove this equation we transform (11).

$$by^5-by^3=by^3+2ay^2-by-a,$$

$$by^3\left(y^2-1\right)=by^3+2ay^2-by-a,$$

$$y^2=\frac{a+by-2ay^2-by^3}{yb\,(1-y^2)}.$$

Comparison with (10) yields $y^2=p$. ◊

5.3.2. $\frac{1}{2}\sqrt{2}<y<1$

We transform (11) once more, but in another way.

$$b\left(y^5 - 2y^3 + y\right) = 2ay^2 - a,$$
$$yb\left(y^2 - 1\right)^2 = a(2y^2 - 1),$$
$$2y^2 - 1 = \frac{yb(y^2-1)^2}{a}.$$

Because of $a, b > 0$ and $0 < y < 1$ it follows $2y^2 - 1 > 0$ and therefore $y > \frac{1}{2}\sqrt{2}$. ◊

With this result we obtain also $x < y$. Because in case $x \geq$ $\geq y$ we have $\frac{1-y^2}{y} \geq y$ and therefore $y \leq \frac{1}{2}\sqrt{2}$. But this would be a contradiction to $y > \frac{1}{2}\sqrt{2}$.

5.3.3. $x^2 + y^2 < 1$

With (7) we have to prove

$$\left(\frac{1-y^2}{y}\right)^2 + y^2 < 1.$$

Transformation yields

$$1 - 3y^2 + 2y^4 < 0 \quad \text{or} \quad \left(y^2 - \frac{3}{4}\right)^2 < \frac{1}{16} \quad \text{or} \quad \left|y^2 - \frac{3}{4}\right| < \frac{1}{4}.$$

With 5.3.2 it is immediately to see that this inequality is fulfilled.

We distinguish three cases.

$$1) \quad y^2 - \frac{3}{4} > 0 \quad \Longrightarrow \quad \frac{3}{4} < y^2 < 1$$

$$2) \quad y^2 - \frac{3}{4} = 0$$

$$3) \quad y^2 - \frac{3}{4} < 0 \quad \Longrightarrow \quad \frac{1}{2} < y^2 < \frac{3}{4}. \quad \Diamond$$

5.3.4. $L = \dfrac{a}{x^2 y} = \dfrac{b}{y^2 - x^2}$

From $L = \frac{a}{x^2 y}$ with 3.3 (3), (7), 5.3.1 we obtain

$$\frac{x+y}{1-y^2}\left(a + by - by^3\right) = \frac{a}{x^2 y}$$

or

$$\frac{1}{y\left(1-y^2\right)}\left(a + by - by^3\right) = \frac{ay}{\left(1-y^2\right)^2}$$

and finally

$$\left(a + by - by^3\right)\left(1 - y^2\right) = ay^2.$$

We have once more our equation (11). So it remains only to show

$$\frac{a}{x^2y} = \frac{b}{y^2 - x^2}.$$

With $x = \frac{1-y^2}{y}$ it follows

$$ay^2 - a\left(\frac{1-y^2}{y}\right)^2 = by\left(\frac{1-y^2}{y}\right)^2.$$

or

$$by^5 - 2by^3 - 2ay^2 + by + a = 0.$$

That is again equation (11). ◊

6. Limit curve and the enclosed region

6.1. Theorem. *The length of the limit curve (circumference) is:* $U = \frac{2b(1+y)}{y^2-x^2}$.

Proof. We consider the rectangles I and II in Fig. 10 to be trunks of subtrees. These trees have at the top the limit segments $G_1H_1 = Lxy$ and $G_2H_3 = Lx^2$.

In the same way we procede with the rectangles III and IV. To the left respectively to the right the corresponding subtrees have limit segments $E_1G_1 = Ly$ and $E_2G_2 = Lx$. Due to 5.3.2 we have $Lx < Ly$.

It can be proved – the reader should try to do so – that the points G_1, H_1, H_2 are collinear and even that $Lxy + b + Lx^2 = L$.

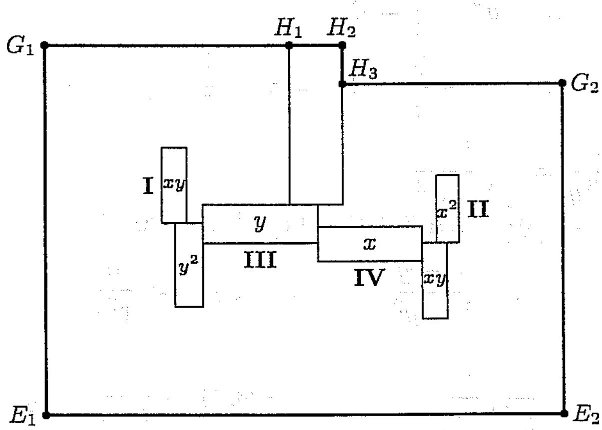

Fig. 10. Limit curve and the enclosed region.

Now we add the lengths of all our limit segments – this yields the total length U of the limit curve (polygon)

$$\begin{aligned} U &= E_1G_1+G_1H_1+H_1H_2+(G_1E_1-G_2E_2)+H_3G_2+G_2E_2+E_2E_1 = \\ &= Ly+Lxy+b+L(y-x)+Lx^2+Lx+L = \\ &= L\left(2y+xy+x^2+1\right)+b. \end{aligned}$$

Now we use $L = \frac{b}{y^2-x^2}$ from 5.3.4 and equation (7). After some calculation we obtain

$$U = \frac{b}{y^2-x^2}\left(2y+y\frac{1-y^2}{y}+\left(\frac{1-y^2}{y}\right)^2+1\right)+b = \frac{2b(y+1)}{y^2-x^2}. \quad \diamond$$

6.2. Theorem. *The region enclosed by the limit curve has the area*

$$F = b^2\frac{y-x^2y+x^3}{\left(y^2-x^2\right)^2}.$$

Proof. The rectangle with side lengths $E_1E_2 = L$ and $E_1G_1 = yL$ from Fig. 10 has area L^2y. To obtain the total area of the enclosed region we must still subtract the area of the small rectangle with side lengths $G_2H_3 = Lx^2$ and $H_2H_3 = E_1G_1 - E_2G_2 = L(y-x)$. Finally we have

$$F = L^2y - L^2(y-x)x^2 = \frac{b^2}{\left(y^2-x^2\right)^2}\left(y-yx^2+x^3\right). \quad \diamond$$

7. Demand *B*

Theorem. *The area of all the bronchial tubes is*

$$F = b^2\frac{y-x^2y+x^3}{\left(y^2-x^2\right)^2}.$$

With Th. 6.2 this means that the bronchial trees completely fill the limit region. Therefore our model fulfils Demand B.

Proof. The proof is a little bit difficult. It is given in several steps.

a) The area in different generations

Generation 0

We consider the rectangle (a, b) and the small rectangle $\big(b(1-p),$ $pbx+b(y-x)\big)$. Both together form the polygon of generation 0. The corresponding area is

$$F_0 = ab + b^2(1-p)(xp+y-x)$$

and with 5.3.1 finally

$$F_0 = ab + b^2\left(1-y^2\right)\left(xy^2+y-x\right) = ab + T.$$

Generation 1

In generation 1 there are two polygons similar to the starting configuration. Fig. 11 shows the situation.

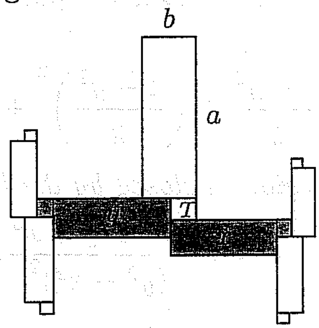

Fig. 11. Generation 11.

We calculate the area.

To the right

$$F_1' = abx^2 + x^2b^2\left(1-y^2\right)\left(xy^2+y-x\right) = x^2(ab+T).$$

To the left

$$F_1'' = aby^2 + y^2b^2\left(1-y^2\right)\left(xy^2+y-x\right) = y^2(ab+T).$$

Together

$$F_1 = F_1' + F_1'' = (ab+T)\left(x^2+y^2\right).$$

Generation 2

Now we obtain 4 polygons similar to the starting configuration from generation 0. Because of this similarity we have

$$F_2 = x^2F_1 + y^2F_1 = (ab+T)\left(x^2+y^2\right)^2.$$

Generation n

Using complete induction it follows

$$F_n = (ab+T)\left(x^2+y^2\right)^n.$$

b) Summing up

$$F_0 + F_1 + \cdots + F_n = (ab+T)\Big(1 + \left(x^2+y^2\right) + \left(x^2+y^2\right)^2 + \\ + \cdots + \left(x^2+y^2\right)^n\Big) = (ab+T)\frac{1-\left(x^2+y^2\right)^{n+1}}{1-\left(x^2+y^2\right)}.$$

Because of 5.3.3 finally we have

$$F = \lim_{n\to\infty} (ab+T)\frac{1-\left(x^2+y^2\right)^{n+1}}{1-(x^2+y^2)} = \frac{ab+T}{1-(x^2+y^2)}.$$

c) Equivalence

Now it remains still the most difficult part of the proof. It must be shown the following equivalence from 6.2:

$$\frac{ab+T}{1-(x^2+y^2)} \overset{!}{=} b^2\frac{y-x^2y+x^3}{\left(y^2-x^2\right)^2}.$$

Doing this needs a lot of effort and many tricky transformations. We transform

$$\frac{ab+T}{1-(x^2+y^2)} = \frac{Z}{N}.$$

With $a = \frac{bx^2y}{y^2-x^2}$ from 5.3.4 and with (7) we have

$$ab = \frac{b^2y\left(1-y^2\right)^2}{2y^2-1}.$$

The numerator is

$$Z = ab+b^2\left(1-y^2\right)\left(xy^2+y-x\right) = \frac{b^2\left(1-y^2\right)}{y\left(2y^2-1\right)}\left[1-4y^2+6y^4-2y^6\right],$$

the denominator

$$N = 1-\left(x^2+y^2\right) = \frac{3y^2-2y^4-1}{y^2} = \frac{\left(2y^2-1\right)\left(1-y^2\right)}{y^2},$$

and together

$$\frac{Z}{N} = \frac{ab+T}{1-(x^2+y^2)} = \frac{b^2y\left[1-4y^2+6y^4-2y^6\right]}{\frac{(2y^2-1)^2}{y^4}y^4}.$$

The trick $\frac{(2y^2-1)^2}{y^4} = \left(y^2-\left(\frac{1-y^2}{y}\right)^2\right)$ yields

$$\frac{Z}{N} = \frac{b^2\frac{\left[1-4y^2+6y^4-2y^6\right]}{y^3}}{\left(y^2-\left(\frac{1-y^2}{y}\right)^2\right)^2}.$$

Now we use a second trick

$$\frac{1}{y^3}\left[1-4y^2+6y^4-2y^6\right] = y-\left(\frac{1-y^2}{y}\right)^2y+\left(\frac{1-y^2}{y}\right)^3.$$

Finally with (7) we obtain

$$\frac{Z}{N} = \frac{b^2}{(y^2 - x^2)^2}\left(y - x^2 y + x^3\right). \quad \Diamond$$

8. What about dimension?

8.1. Definition. Let the rectangle (ax, bx) together with the smaller touching rectangle be trunk of a tree to the right (Fig. 11) – in an analogous way with (ay, by) to the left. The two subtrees magnified with factors $\frac{1}{x}$ and $\frac{1}{y}$ gives the starting tree. We take over the definition of fractal dimension in the case of extended selfsimilarity from [8]. Then the *dimension* d of our model is given by

$$x^d + y^d = 1 \quad \text{or} \quad \left(\frac{1-y^2}{y}\right)^d + y^d = 1.$$

8.2. Are there solutions of the last equation?
Answering the question we define a very special function:

$$f(d) = \left(\frac{1-y^2}{y}\right)^d + y^d = x^d + y^d \quad \text{with} \quad d \geq 0$$

The function has the following properties:

a) $f(d) > 0$ (because $x > 0$ and $y > 0$).
b) $f(d)$ is continuous (because the two functions x^d and y^d are continuous in d).
c) $f(0) = 2$.
d) $\lim_{d\to\infty} f(d) = 0$ (because $x < 1$ and $y < 1$).
e) $f'(d) = x^d \ln x + y^d \ln y < 0$ (because x, $y < 1$ and therefore $\ln x < 0$ and $\ln y < 0$).

We use the "intermediate value theorem". Then with all the properties (a)–(d) it follows that each function value between 2 and 0 occurs at least once. Because of (e) the function $f(d)$ is strictly decreasing over $0 \leq \infty$. Therefore each function value – including number 1 – is reached exactly once. So the proof for the existence of exactly one number d is finished. Fig. 12 explains the situation.

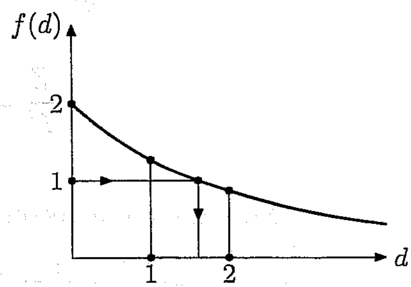

Fig. 12. The graph of $f(d)$.

8.3. Theorem. *If d is the fractal dimension of our model, then we have $1 < d < 2$. This means $d \notin \mathbb{N}$. Due to our definition in Sec. 1 Demand D is therefore fulfilled.*

Proof. Let be $d = 1$. Then it follows

$$f(1) = \frac{1-y^2}{y} + y = \frac{1}{y}.$$

With $\frac{1}{\sqrt{2}} < y < 1$ from 5.3.2 we obtain $\sqrt{2} > f(1) > 1$. Now let be $d = 2$. Then it follows

$$f(2) = x^2 + y^2.$$

With $x^2 + y^2 < 1$ from 5.3.3 and property (a) we obtain $0 < f(2) < 1$. In Fig. 12 the two cases $d = 1$, $d = 2$ are drawn. With this the theorem is proved. ◊

9. Summary – outlook

9.1. What we did

First of all in this paper a very special plane point set was constructed. Then it was proved that this point set fulfils Demands A (embedded, Sec. 3), B (completely filled, Sec. 7), C (no overlapping, Sec. 4.2) and D (fractal, Sec. 8). Due to our definition in Sec. 1 we found a *physiological fractal.*

If we restrict ourselves to the construction of only a finite number of generations (in case of the human lung about 23) then we have a so-called *near-fractal*, an unfinished fractal. Our near-fractal can be considered as an imprecise model for a X-ray picture of the human lung. Fig. 13 shows the result.

9.2. What remains to do?

9.2.1. For anatomists

In an empirical way the fundamental values a, b and y are determined, for instance in [7]. Starting with these parameters the results of our paper allow to calculate not only x and p but a lot of other things: length of bronchial tubes between two ramification points, breath of tubes in each generation, area of the limit region, length of the limit curve, ...

9.2.2. For mathematicians

Our model certainly needs still some improvements. We give an example: The ramification angle in reality is certainly not 180° as in

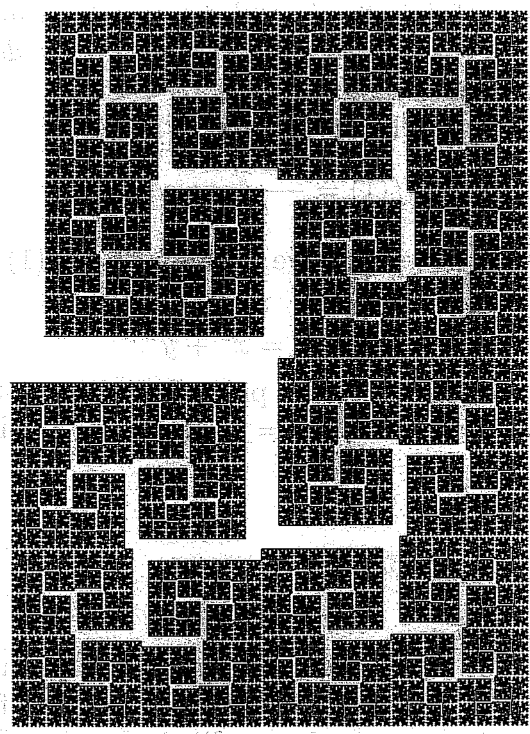

Fig. 13. Our model.

the model. But the main problem for mathematicians is to extend our model in the 3-dimensional space.

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