

A REMARK ON A PAPER WRITTEN BY J. M. DE KONINCK AND A. IVIĆ

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Abstract: Let \mathcal{F} denote the set of monotonically decreasing functions f defined on $(0, \infty)$ such that $f(x) \rightarrow 0$ ($x \rightarrow \infty$). Let $q_1 < \dots < q_r$ be the prime divisors of n , $\tau(n)$ = number of divisors of n , $d_1 < d_2 < \dots < d_{\tau(n)}$ be the sequence of divisors of n . Let $f \in \mathcal{F}$, and

$$h(n) = \sum_{i=1}^{r-1} f(q_{i+1} - q_i); \quad H(n) := \sum_{i=1}^{\tau(n)-1} f(d_{i+1} - d_i),$$
$$A(x) = \sum_{n \leq x} h(n); \quad B(x) = \sum_{n \leq x} H(n).$$

It is proved:

- $A(x)/x$ tends to a finite limit if and only if $\sum h^{-1} f(2^h) < \infty$. If the series converges then h is almost periodic in Besicovitch sense,
- $B(x)/x$ has a finite limit if and only if $\sum f(2^h) < \infty$. If the series converges then H is almost periodic in Besicovitch sense.

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1. Let $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$, $q_1 < \dots < q_r$ be the prime divisors of n , $\tau(n)$ be the divisor function. Let $d_1 < d_2 < \dots < d_{\tau(n)}$ be the sequence of positive divisors of n . We shall say that a function f belongs to \mathcal{F} , if it is defined for $x \geq 1$ and tends to zero monotonically as $x \rightarrow \infty$, $f(x) > 0$. For a fixed $f \in \mathcal{F}$ let the functions $h : \mathbb{N} \rightarrow \mathbb{R}$, $H : \mathbb{N} \rightarrow \mathbb{R}$ be defined as follows: $h(1) = 0$, $H(1) = 0$, $h(p^\alpha) = 0$ if p^α is a prime-power, otherwise

$$(1.1) \quad h(n) = \sum_{i=1}^{r-1} f(q_{i+1} - q_i); \quad H(n) := \sum_{i=1}^{\tau(n)-1} f(d_{i+1} - d_i).$$

Let

$$(1.2) \quad A(x) = \sum_{n \leq x} h(n); \quad B(x) = \sum_{n \leq x} H(n).$$

In their paper [1] J. M. De Koninck and A. Ivić proved that for $f(x) = 1/x$ the relations

$$A(x) = Ax + O\left(\frac{x \log \log x}{\log x}\right), \quad B(x) = Bx + O\left(\frac{x}{(\log x)^{1/3}}\right)$$

hold with suitable explicitly given constants A, B .

Our aim is to generalize these results for large classes of functions $f \in \mathcal{F}$. (We will not take care about the remainder terms.)

Since $f(q_{i+1} - q_i) \geq f(q_{i+1})$ if $f \in \mathcal{F}$, therefore

$$A(x) \geq \sum_{2 < p < x^{1/4}} f(p) \sum_{2mp \leq x} 1 \geq \frac{x}{2} \sum_{2 < p \leq x^{1/4}} \frac{f(p)}{p},$$

where in the sum p runs over the set of primes.

So for the existence of the mean value of h the condition

$$(1.3) \quad \sum f(p)p^{-1} < \infty,$$

i.e. the equivalent condition

$$(1.4) \quad \sum_{h=1}^{\infty} f(2^h)h^{-1} < \infty$$

is necessary.

Since $f(d_{i+1} - d_i) \geq f(d_{i+1})$, we get similarly that $B(x) \geq \sum f(d) \left[\frac{x}{d} \right]$. If $B(x) = O(x)$, then

$$(1.5) \quad \sum_{d=1}^{\infty} d^{-1} f(d) < \infty,$$

i.e.

$$(1.6) \quad \sum_{h=1}^{\infty} f(2^h) < \infty$$

holds.

It is much more interesting that the conditions (1.4), (1.5) are sufficient for the existence of the mean value for h, H , respectively.

2. Let us consider h under (1.4). For an $y \geq 2$ let P_y be the product of the primes not greater than y . For an $n \in \mathbb{N}$ let $A_y(n)$ be the product of the prime divisors of n not greater than y . Then $n = A_y(n)B_y(n)$, $(B_y(n), P_y) = 1$.

Let $s_y(n) := h(A_y(n))$. Since $h(n)$ does not depend on the multiplicity of the prime divisors of n , therefore $s_y(n) = s_y(\ell)$ if $n \equiv \ell \pmod{P_y^2}$, i.e. $s_y(n)$ is periodic mod P_y^2 .

Let $t_y(n) := h(n) - s_y(n)$. Then

$$t_y(n) = h(B_y(n)) + f(v(n) - u(n))$$

where $v(n)$ is the smallest prime divisor of $B_y(n)$ and $u(n)$ is the largest prime divisor of $A_y(n)$. Consider the sum

$$\sum_1 := \sum_{n \leq x} f(v(n) - u(n)).$$

The number of the integers $n \leq x$ satisfying $u(n) > y/2$ is less than

$$\sum_{\frac{y}{2} < p \leq y} \sum x/p \ll \frac{x}{\log y}.$$

Furthermore, if $u(n) \leq y/2$, then $f(v(n) - u(n)) \leq f(y/2)$, consequently

$$\sum_1 \ll x f(y/2) + \frac{cx}{\log y}.$$

Let

$$\sum_2 := \sum_{n \leq x} h(B_y(n)).$$

Then

$$\sum_2 = \sum_{y < q \leq x} \sum_{y < p < q} f(q - p) D_{p,q},$$

where $D_{p,q}$ denotes the number of those $n \leq x$ for which p and q are consecutive prime divisors. Let $\sum_2 = \sum_{2,1} + \sum_{2,2}$, where in $\sum_{2,1}$ $\frac{q}{2} \leq p < q$, while in $\sum_{2,2}$ $p < q/2$. To estimate $\sum_{2,1}$ we shall use the crude estimation $D_{p,q} \leq x/pq$, and we get

$$\sum_{2,1} \ll x \sum_{y < q < x} \frac{1}{q} \left(\sum_{q/2 < p < q} 1/p \right) \ll x \sum_{y < q} \frac{1}{q \log q},$$

from the convergence of $\sum (q \log q)^{-1}$ we get that $\sum_{2,1} = o_y(1)x$ ($x \rightarrow \infty$). To estimate $\sum_{2,2}$ we observe that $f(p - q) \leq f(q/2)$ and that $\sum_p D_{p,q} \leq x/q$. So by (1.4) we have

$$\sum_{2,2} \leq \sum_{y < q \leq x} f(q/2)[x/q] = o_y(1)x.$$

Collecting our results we get

$$(2.1) \quad \lim_{y \rightarrow \infty} \left(\limsup \frac{1}{x} \sum_{n \leq x} |h(n) - s_y(n)| \right) = 0.$$

Since $s_y(n)$ is periodic mod P_y^2 , therefore (2.1) implies that h is almost periodic, consequently the mean value of h exists. For the proof of this conclusion see [2], Ch. 4, Th. 4. As a consequence we get that h has a limit distribution, i.e.

$$(2.2) \quad \lim \frac{1}{x} \# \{h(n) < \lambda, n \leq x\} = V(\lambda)$$

exists for almost all real number λ .

So we proved

Theorem 1. *Let $f \in \mathcal{F}$. Then $A(x)x^{-1}$ tends to a finite limit if and only if (1.4) holds. If (1.4) holds, then h is almost periodic in Besicovitch sense.*

3. Let us consider now H . Assume that (1.5) holds, and that $f \in \mathcal{F}$. It is obvious that

$$(3.1) \quad \sum_{n=2}^{\infty} n^{-1} f(n/(\log n)^3) < \infty.$$

Indeed, let us sum $1/n$ over those n for which $2^h < \frac{n}{(\log n)^3} < 2^{h+1}$. The smallest n is $A_h \geq c_1 2^h h^3$, the largest n is $B_h \leq c_2 2^{h+1} \cdot h^3$, where c_1, c_2 are positive constants. Then

$$\sum_{A_h \leq n \leq B_h} 1/n \leq \log \frac{c_2}{c_1}.$$

Hence (3.1) immediately follows.

Let $\rho(z) = z(\log z)^{-1}$,

$$t_y(n) := \sum_{d_i > \rho(y)} f(d_{i+1} - d_i), \quad T_y(x) = \sum_{n \leq x} t_y(n).$$

Now we estimate $T_y(x)$. The contribution of the terms in $T_y(x)$ satisfying $d_{i+1} - d_i > \rho(d_i)$ is less than

$$x \sum_{d > \rho(y)} f(\rho(d)) 1/d = o_y(1)x.$$

Now we consider the terms for which $d_{i+1} - d_i < \rho(d_i)$. If d_i, d_{i+1} are consecutive divisors of n such that $\sqrt{n} < d_i < d_{i+1}$, then $n/d_{i+1}, n/d_i$ are consecutive divisors of n as well, furthermore $0 < \frac{n}{d_i} - \frac{n}{d_{i+1}} < d_{i+1} - d_i$. Consequently in the estimation of $T_y(x)$ the terms $d_i > \sqrt{n}$ can be cancelled.

Let us consider the contribution of those pairs d_i, d_{i+1} for which $d_i < \sqrt{n} < d_{i+1}$. Then $d_{i+1} = \frac{n}{d_i}$.

Since $0 < d_{i+1} - d_i < \rho(d_i)$, for a fixed d_i the number of distinct d_{i+1} is less than $O(\rho(d_i))$, consequently the contribution of them is less than

$$\sum_{d < \sqrt{x}} \frac{d}{\log d} = o(x) \quad (x \rightarrow \infty).$$

It has remained to estimate the sum

$$\sum_1 := \sum_m \sum_{\ell} f(\ell) \sum_n 1,$$

where m and ℓ run over the intervals $1 \leq \ell \leq \rho(m)$, $\rho(y) \leq m \leq \sqrt{x}$, $m + \ell \leq \sqrt{x}$, while $n \leq x$ and $m|n$, $m + \ell|n$. The innermost

sum is $\leq \frac{x}{[m, m + \ell]}$. Let $\rho = (m, m, +\ell) = (m, \ell)$, $m = \delta m^*$, $\ell = \delta m^*$.

Then we get

$$\sum_1 \leq x \sum_m \sum_{\delta|m} \sum_{s \leq \frac{\rho(m)}{\delta}} \frac{f(\delta s)}{\delta m^*(m^* + s)} \leq x \sum_m \frac{1}{m} \sum_{\delta, s} \frac{f(\delta s)}{m^*},$$

where in the inner sum $\delta|m$, $s\delta \leq \rho(m)$.

Summing for m in a subinterval of type $[M, 2M]$, we get

$$\sum_{M \leq m \leq 2M} \frac{1}{m} \sum_{s, \delta} \frac{f(\delta s)}{m^*} \leq \frac{1}{m} \sum_{k \leq \rho(2M)} f(k) \sum_{\delta/k} \left(\sum 1/s \right),$$

where in the innermost sum in the right-hand side s runs over the interval $\left[\frac{M}{\delta}, \frac{2M}{\delta} \right]$. So the right-hand side is less than

$$\frac{1}{M} \sum_{k \leq \rho(2M)} f(k)\tau(k).$$

Let now $M_1 = 2$, $M_{\nu+1} = 2M_\nu$ and we sum over those M_t for which $M_{t+1} \geq \rho(y)$. Then we get

$$(3.2) \quad \sum_1 \ll x \sum_{M_t} \frac{1}{M_t} \sum_{k \leq \rho(M_{t+1})} f(k)\tau(k).$$

Let us change the summation in the right-hand side. The least value of M_t for which k occurs in the inner sum is of order $k \log k$. Since $\sum_{t \geq t_0} M_t^{-1} \ll M_{t_0}^{-1}$, we get that

$$(3.3) \quad \sum_1 \ll x \sum_k f(k)\tau(k) \min \left(\frac{1}{\rho(y)}, \frac{1}{k \log k} \right).$$

Now we prove that

$$(3.4) \quad \sum_k \frac{f(k)\tau(k)}{k \log k} < \infty$$

holds.

Let us sum (3.4) for $k \in [2^h, 2^{h+1}]$. The sum is less than

$$\frac{f(2^h)}{2^h \log 2^h} \sum_{k < 2^{h+1}} \tau(k) \ll \frac{f(2^h)}{h},$$

and (3.4) can be overestimated by the convergent sum $\sum h^{-1} f(2^h)$.

Collecting our results we get that $T_y(x) = o_y(1)x$ as $x \rightarrow \infty$.

Now we deduce that H is an almost periodic function. For a prime $p \leq y$ let $\partial_p(y)$ be the least integer for which $p^{\partial_p(y)} \geq y$. Let $Q_y = \prod_{p \leq y} p^{\partial_p(y)}$, furthermore

$$q_y(n) = \sum_{d_{i+1} \leq y} f(d_{i+1} - d_i).$$

It is obvious that $n \equiv \ell \pmod{Q_y}$ implies that $q_y(n) = q_y(\ell)$ so q_y is periodic mod Q_y . Furthermore $0 \leq H(n) \leq t_y(n) + R(n)$, where $R(n) = 0$ except when the largest divisor d of n which is not greater than y is $\leq y/\log y$. In this second case $R(n) = f(k - d)$ where k is the smallest divisor of n that is greater than y . So we have $R(n) \leq f(y - y/\log y) \leq f(y/2)$. Consequently $R(n) = o_y(1)$ uniformly as $n \leq x$. Taking into account that $T_y(x) = o_y(1)x$ ($x \rightarrow \infty$), we get immediately that

$$(3.5) \quad \lim_{y \rightarrow \infty} \left(\limsup \frac{1}{x} \sum_{n \leq x} |H(n) - q_y(n)| \right) = 0.$$

(3.5) means that H is almost periodic. Consequently the following assertion holds true.

Theorem 2. *Let $f \in \mathcal{F}$. Then $B(x)x^{-1}$ has a finite limit if and only if (1.6) holds. If (1.6) holds then $H(n)$ is almost periodic in Besicovitch sense.*

References

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