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FUNCTIONAL EQUATIONS ON STURM–LIOUVILLE HYPERGROUPS

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Abstract: This paper presents some recent results concerning functional equations on Sturm–Liouville hypergroups. The general form of additive functions, exponentials and moment functions of second order on these types of hypergroups is given.

1. Introduction

The concept of DJS-hypergroup (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) can be introduced using different axiom systems. The way of introducing the concept here is due to R. Lasser (see e.g. [2], [6]). One begins with a locally compact Haussdorff space K, the space $\mathcal{M}(K)$ of all finite complex regular measures on K, the space $\mathcal{M}_c(K)$ of all finitely supported measures in $\mathcal{M}(K)$, the space $\mathcal{M}^1(K)$ of all probability measures in $\mathcal{M}(K)$, and the space $\mathcal{M}_c^1(K)$ of all compactly supported probability measures in $\mathcal{M}(K)$. The point mass concentrated at x is denoted by δ_x. Suppose that we have the following:

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(H^*) There is a continuous mapping $(x,y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $\mathcal{M}_c^1(K)$, the latter being endowed with the weak*-topology with respect to the space of compactly supported complex valued continuous functions on K. This mapping is called convolution.

($H^\vee$) There is an involutive homeomorphism $x \mapsto x^\vee$ from K to K. This mapping is called involution.

(He) There is a fixed element e in K. This element is called identity.

Identifying x by δ_x the mapping in (H^*) has a unique extension to a continuous bilinear mapping from $\mathcal{M}(K) \times \mathcal{M}(K)$ to $\mathcal{M}(K)$. The involution on K extends to an involution on $\mathcal{M}(K)$. Then a DJS-hypergroup, or simply hypergroup is a quadruple $(K, *, \vee, e)$ satisfying the following axioms: for any x, y, z in K we have

($H1$) $\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z$;

($H2$) $(\delta_x * \delta_y)^\vee = \delta_{y^\vee} * \delta_{x^\vee}$;

($H3$) $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

($H4$) e is in the support of $\delta_x * \delta_{y^\vee}$ if and only if $x = y$;

($H5$) the mapping $(x,y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space of nonvoid compact subsets of K is continuous, the latter being endowed with the Michael–topology (see [2]).

If $\delta_x * \delta_y = \delta_y * \delta_x$ holds for all x, y in K, then we call the hypergroup commutative. If $x^\vee = x$ holds for all x in K then we call the hypergroup Hermitian. By ($H2$) any Hermitian hypergroup is commutative. For instance, if $K = G$ is a locally compact Haussdorff–group, $\delta_x * \delta_y = \delta_{xy}$ for all x, y in K, $x^\vee$ is the inverse of x, and e is the identity of G, then we obviously have a hypergroup $(K, *, \vee, e)$, which is commutative if and only if the group G is commutative. However, not every hypergroup originates in this way.

In any hypergroup K we identify x by δ_x and we define the right translation operator T_y by the element y in K according to the formula:

$$T_y f(x) = \int_K f \, d(\delta_x * \delta_y),$$

for any f integrable with respect to $\delta_x * \delta_y$. In particular, T_y is defined for any continuous complex valued function on K. Similarly, we can define left translation operators but at this moment we do not need any extra notation for them.

Sometimes one uses the suggestive notation

$$f(x * y) = \int_K f \, d(\delta_x * \delta_y),$$

for any x, y in K. However, we call the attention to the fact that actually $f(x * y)$ has no meaning in itself, because $x * y$ is in general not an element of K, hence f is not defined at $x * y$.

If K is a discrete topological space, then we call the hypergroup a discrete hypergroup. An important special class of discrete hypergroups are the polynomial hypergroups which are closely related to orthogonal polynomials. For the definition and a detailed study of polynomial hypergroups the reader should refer to [2].

Another important class of hypergroups is the class of Sturm–Liouville hypergroups. The definition and some basic properties of Sturm–Liouville hypergroups will be given in the following section.

In our former paper [9] we presented some recent results concerning functional equations on hypergroups. The aim was to give some idea for the treatment of classical functional equation problems in the hypergroup setting. We described the general form of additive functions, exponentials and moment functions of second order on polynomial hypergroups. Here we consider similar problems concerning classical functional equations on Sturm–Liouville hypergroups.

2. Sturm–Liouville hypergroups

Sturm–Liouville hypergroups represent another important class of hypergroups, which arise from Sturm–Liouville boundary value problems on the nonnegative reals. In order to build up the Sturm–Liouville operator basic to the construction of hypergroups one introduces the Sturm–Liouville functions. For further details see [2]. In what follows $\mathbb{R}_0$ denotes the set of nonnegative real numbers.

The continuous function $A : \mathbb{R}_0 \to \mathbb{R}$ is called a Sturm–Liouville function, if it is positive and continuously differentiable on the positive reals. Different assumptions on A can be found in [2] which lead to the desired Sturm–Liouville problem. For a given Sturm–Liouville function A one defines the Sturm–Liouville operator L_A by

$$L_A f = -f'' - \frac{A'}{A} f',$$

where f is a twice continuously differentiable real function on the positive reals. Using L_A one introduces the differential operator l by

$$l[u](x,y) = (L_A)_x u(x,y) - (L_A)_y u(x,y) =$$
$$= -\partial_1^2 u(x,y) - \frac{A'(x)}{A(x)}\partial_1 u(x,y) + \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\partial_2 u(x,y)\,,$$

where u is twice continuously differentiable for all positive reals x, y. Here $(L_A)_x$ and $(L_A)_y$ indicates that L_A operates on functions depending on x or y, respectively.

A hypergroup on $\mathbb{R}_0$ is called a Sturm–Liouville hypergroup if there exists a Sturm–Liouville function A such that given any real-valued C^∞-function f on $\mathbb{R}_0$ the function u_f defined by

$$u_f(x,y) = f(x*y) = \int_{\mathbb{R}_0} f\, d(\delta_x * \delta_y)$$

for all positive x, y is twice continuously differentiable and satisfies the partial differential equation

$$l[u_f] = 0$$

with $\partial_2 u_f(x,0) = 0$ for all positive x. Hence u_f is a solution of the Cauchy–problem

$$\partial_1^2 u(x,y) + \frac{A'(x)}{A(x)}\,\partial_1 u(x,y) = \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u(x,y),$$
$$\partial_2 u_f(x,0) = 0$$

for all positive x, y. From general properties of one-dimensional hypergroups given in [2] it follows that $u_f(y,0) = u_f(0,y) = f(y)$ and $\partial_1 u_f(0,y) = 0$ holds, whenever y is a positive real number. In other words, u_f is the unique solution of the boundary value problem

$$\begin{gathered}\partial_1^2 u(x,y) + \frac{A'(x)}{A(x)}\,\partial_1 u(x,y) = \partial_2^2 u(x,y) + \frac{A'(y)}{A(y)}\,\partial_2 u(x,y)\,,\\ \partial_1 u_f(0,y) = 0, \qquad \partial_2 u_f(x,0) = 0\,,\\ u_f(x,0) = f(x), \qquad u_f(0,y) = f(y)\end{gathered} \tag{1}$$

for all positive x, y. As this boundary value problem uniquely defines u_f for any f, we may consider it the boundary value problem defining the Sturm–Liouville hypergroup.

If the Strum–Liouville function A satisfies

$$\frac{A'(x)}{A(x)} = \frac{\alpha_0}{x} + \alpha_1(x) \tag{2}$$

for all $x \neq 0$ in a neighborhood of 0 with $\alpha_0 > 0$ such that α_1 is an odd

C^∞-function on $\mathbb{R}$ and the function $\frac{A'}{A}$ is nonnegative and decreasing, further A is increasing with $\lim_{x\to+\infty} A(x) = +\infty$, then A is called a Chébli–Trimèche function and the corresponding Sturm–Liouville hypergroup is called a Chébli–Trimèche hypergroup. Special cases are represented by the Bessel–Kingman hypergroups with $A(0) = 0$ and

$$A(x) = x^\alpha$$

for all positive x and some $\alpha > 0$, and the hyperbolic hypergroups where $A(0) = 0$ and

$$A(x) = \sinh^a x$$

for all positive x and some $a > 0$.

If the Sturm–Liouville function A is twice continuously differentiable on the positive reals and satisfies (2), where $\alpha_0 = 0$ and α_1 is continuously differentiable on the positive reals, then A is called a Levitan function and the corresponding Sturm–Liouville hypergroup is called a Levitan hypergroup. Special cases are represented by the cosh hypergroup, where

$$A(x) = \cosh^2 x$$

for all nonnegative x, and the square hypergroup, where

$$A(x) = (1+x)^2$$

for all nonnegative x (see [8]). For more about these hypergroups and their applications see [2].

3. Exponentials, additive functions and moment functions on hypergroups

Let K be a commutative hypergroup and for any y in K let T_y denote translation operator on the space of all complex valued functions on K which are integrable with respect to $\delta_x * \delta_y$ for any x, y in K. In particular, any continuous complex valued function belongs to this class.

The continuous complex valued function m on K is called an exponential, if it is not identically zero, and

$$T_y m(x) = m(x)m(y)$$

holds for all x, y in K. In other words m satisfies the functional equation

$$m(x * y) = \int_K m(t)\, d(\delta_x * \delta_y)(t) = m(x)m(y)\,.$$

The continuous complex valued function a on K is called additive, if it satisfies

$$T_y a(x) = a(x) + a(y)$$

for all x, y in K. In more details this means that

$$a(x * y) = \int_K a(t)\, d(\delta_x * \delta_y)(t) = a(x) + a(y)$$

holds for any x, y in K. It is obvious that any linear combination of additive functions is additive again. However, in contrast to the case of groups, the product of exponentials is not necessarily an exponential.

The third important class of functions we want to study in this work is the class of moment functions. Moments of probability measures on a hypergroup can be introduced in terms of moment functions. The notion of moment functions has been formalized in [8] (see also [2]). For any nonnegative integer N the complex valued function f on K is called a moment function of order N, if there are complex valued continuous functions f_k on K for $k = 0, 1, \ldots, N$ such that $f_0 = 1$, $f_N = f$, and

$$f_k(x * y) = \sum_{j=0}^{k} \binom{k}{j} f_j(x) f_{k-j}(y) \tag{3}$$

holds for $k = 0, 1, \ldots, N$ and for all x, y in K. In this case we say that the functions f_k $(k = 0, 1, \ldots, N)$ form a moment sequence of order N. Hence moment functions of order 1 are exactly the additive functions. In [3] the general form of moment functions of order $N = 1$ and $N = 2$ have been determined in the case of polynomial hypergroups. We can generalize this concept by omitting the hypothesis $f_0 = 1$, but supposing that f_0 is not identically zero. In this case f_0 is an exponential function and we say that f_0 generates the generalized moment sequence of order N further f_k is a generalized moment function of order k with respect to f_0 $(k = 0, 1, \ldots, N)$. For instance, generalized moment functions of order 1 with respect to the exponential f_0 are solutions of the sine functional equation

$$f_1(x * y) = f_0(x) f_1(y) + f_0(y) f_1(x)$$

for any x, y in K.

The study of moment functions and moment sequences on hypergroups leads to the study of the above system of functional equations. We remark that a similar system of functional equation on groupoids has been investigated and solved in [1].

4. Exponentials on Sturm–Liouville hypergroups

Let K be a Sturm–Liouville hypergroup. Now we describe all exponentials defined on K (see also [2]).

Theorem 4.1. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the continuous function $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on K if and only if it is C^∞ and there exists a complex number λ such that*

$$(4) \qquad m''(x) + \frac{A'(x)}{A(x)} m'(x) = \lambda m(x), \quad m(0) = 1, \quad m'(0) = 0$$

holds for any positive x.

Proof. First suppose that the function $m : \mathbb{R}_0 \to \mathbb{C}$ is C^∞ on $\mathbb{R}_0$ and it satisfies the given boundary value problem. Then the function

$$m(x * y) = \int_0^\infty m(t)\, d(\delta_x * \delta_y)(t)$$

and also the function $(x, y) \to m(x)m(y)$ is a solution of the boundary value problem defining the hypergroup, hence they are equal and m is an exponential.

Conversely, suppose that $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on the hypergroup K. Then the function $u_m(x, y) = m(x)m(y)$ is a solution of the boundary value problem defining the hypergroup, hence we obtain

$$\left(m''(x) + \frac{A'(x)}{A(x)} m'(x)\right) m(y) = \left(m''(y) + \frac{A'(y)}{A(y)} m'(y)\right) m(x)$$

holds for each positive x, y, and there exists a complex λ with

$$m''(x) + \frac{A'(x)}{A(x)} m'(x) = \lambda m(x)$$

for all positive x, consequently m is C^∞ on $\mathbb{R}_0$. The relations $m(0) = 1$ and $m'(0) = 0$ are immediate consequences of the fact that m is an exponential and the neutral element of the hypergroup is zero. $\Diamond$

Hence any exponential function on a Sturm–Liouville hypergroup is an eigenfunction of the Sturm–Liouville operator corresponding to the given hypergroup. Each complex number is an eigenvalue and there is a one-to-one correspondence between complex numbers and exponentials. For any fixed complex λ we shall denote by $x \mapsto \varphi(x, \lambda)$ the unique solution of the boundary value problem (4). Then the function $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ represents a one-parameter family of exponentials of the

Sturm–Liouville hypergroup K, which is called the exponential family of K. We obviously have

$$\partial_1^2\,\varphi(x,\lambda) + \frac{A'(x)}{A(x)}\,\partial_1\,\varphi(x,\lambda) = \lambda\,\varphi(x,\lambda) \tag{5}$$
$$\varphi(0,\lambda) = 1, \quad \partial_1\,\varphi(0,\lambda) = 0$$

holds for each positive x.

For instance, the complex number $\lambda = 0$ corresponds to the eigenvalue problem

$$m''(x) + \frac{A'(x)}{A(x)} m'(x) = 0, \quad m(0) = 1, \quad m'(0) = 0\,,$$

which obviously has the unique solution $m \equiv 1$, hence $\varphi(x,0) = 1$ for each x in $\mathbb{R}_0$.

5. Additive functions on Sturm–Liouville hypergroups

Let K be a Sturm–Liouville hypergroup. Now we describe all additive functions defined on K (see also [2]).

Theorem 5.1. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the continuous function $a : \mathbb{R}_0 \to \mathbb{C}$ is an additive function on K if and only if it is C^∞ and there exists a complex number λ such that*

$$a''(x) + \frac{A'(x)}{A(x)} a'(x) = \lambda, \quad a(0) = 0, \quad a'(0) = 0 \tag{5}$$

holds for any positive x.

Proof. The proof is very similar to that of the previous theorem. First suppose that the function $a : \mathbb{R}_0 \to \mathbb{C}$ is C^∞ and it satisfies the given boundary value problem. Then the function

$$a(x * y) = \int_0^\infty a(t)\, d(\delta_x * \delta_y)(t)$$

and also the function $(x,y) \to a(x) + a(y)$ is a solution of the boundary value problem defining the hypergroup, hence they are equal and a is an additive function.

Conversely, suppose that $a : \mathbb{R}_0 \to \mathbb{C}$ is an additive function on the given hypergroup K. Then the function $u_a(x,y) = a(x) + a(y)$ is a solution of the boundary value problem defining the hypergroup, hence we obtain

$$a''(x) + \frac{A'(x)}{A(x)}\, a'(x) = a''(y) + \frac{A'(y)}{A(y)}\, a'(y)$$

holds for each positive x, y, and there exists a complex λ with

$$a''(x) + \frac{A'(x)}{A(x)}\, a'(x) = \lambda$$

for all positive x, consequently a is C^∞ on $\mathbb{R}_0$. The relations $a(0) = 0$ and $a'(0) = 0$ are immediate consequences of the fact that a is additive and the neutral element of the hypergroup is zero. $\Diamond$

It is obvious that the unique solution a_λ of the boundary value problem (6) is $\lambda\, a_1$, where a_1 is the unique solution of (6) with $\lambda = 1$. This means that all additive functions of a Sturm–Liouville hypergroup are constant multiples of a fixed nonzero additive function. We call a_1 the **generating additive function** of the given Sturm–Liouville hypergroup.

It turns out that the boundary value problem (6) can be solved explicitly. Namely, we have the following theorem (see [8]).

Theorem 5.2. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. Then the generating additive function of the hypergroup K is given by*

$$a_1(x) = \int_0^x \int_0^y \frac{A(t)}{A(y)}\, dt\, dy \tag{7}$$

for each nonnegative x. Hence any additive function of the hypergroup K is given by

$$a_\lambda(x) = \lambda \int_0^x \int_0^y \frac{A(t)}{A(y)}\, dt\, dy \tag{8}$$

for each nonnegative x, where λ is an arbitrary complex number.

Proof. The proof is obvious using standard methods from the theory of linear differential equations. Another way of proving the statement is direct verification and using the uniqueness theorem. $\Diamond$

As an illustration we compute the additive functions on the Bessel–Kingman hypergroup, which is a special Chébli–Trimèche hypergroup. Here $A(x) = x^\alpha$ for all nonnegative x with some positive number α. In this case we have

$$a_1(x) = \int_0^x \int_0^y \frac{t^\alpha}{y^\alpha}\, dt\, dy = \frac{x^2}{2(\alpha+1)}$$

and

$$a_\lambda(x) = \lambda \int_0^x \int_0^y \frac{t^\alpha}{y^\alpha}\, dt\, dy = \frac{\lambda x^2}{2(\alpha+1)}$$

for each nonnegative x and complex number λ.

Another example is given here for a special Levitan hypergroup, the square hypergroup, where $A(x) = (1+x)^2$ for all nonnegative x. From the above formulas we have

$$a_1(x) = \int_0^x \int_0^y \frac{(1+t)^2}{(1+y)^2}\, dt\, dy = \frac{x^3+3x^2}{6(x+1)}$$

and

$$a_\lambda(x) = \lambda \int_0^x \int_0^y \frac{(1+t)^2}{(1+y)^2}\, dt\, dy = \frac{\lambda(x^3+3x^2)}{6(x+1)}$$

for each nonnegative x and complex number λ.

6. Moment functions on Sturm–Liouville hypergroups

Let K be a Sturm–Liouville hypergroup. In this section we describe all generalized moment sequences of second order defined on K. We remark that in [4] and [5] the general form of generalized moment functions on polynomial hypergroups is given.

Theorem 6.1. *Let K be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. The continuous functions $f_0, f_1, f_2 : \mathbb{R}_0 \to \mathbb{C}$ form a generalized moment sequence of second order on the hypergroup K if and only if they are C^∞ and there are complex numbers c_0, c_1, c_2 such that*

$$\begin{aligned} f_0(x) &= \varphi(x, c_0)\,, \\ f_1(x) &= c_1 \partial_2 \varphi(x, c_0)\,, \\ f_2(x) &= c_2 \partial_2 \varphi(x, c_0) + c_1^2 \partial_2^2 \varphi(x, c_0) \end{aligned} \tag{9}$$

holds for each positive x.

Proof. First we prove the necessity of the given condition. By assumption the functions $f_0, f_1, f_2 : \mathbb{R}_0 \to \mathbb{C}$ satisfy the functional equations

$$
(10) \qquad \begin{aligned}
f_0(x*y) &= f_0(x)f_0(y)\,,\\
f_1(x*y) &= f_1(x)f_0(y)+f_0(x)f_1(y)\,,\\
f_2(x*y) &= f_2(x)f_0(y)+2f_1(x)f_1(y)+f_0(x)f_2(y)
\end{aligned}
$$

holds for all x, y in $\mathbb{R}_0$. Substituting $y = 0$ in the second and third equation of (10) we get $f_1(0) = f_2(0) = 0$. Similarly, differentiating the second and third equations of (10) with respect to the variable x, then using the second equation of (1) we get $f_1'(0) = f_2'(0) = 0$. As f_0 is not identically zero, it is an exponential, hence, by Th. 4.1, there exists a complex number c_0 such that the first equation of (9) holds for each x in $\mathbb{R}_0$. By the definition of the hypergroup the second equation of (10) implies

$$
\left(f_1''(x)+\frac{A'(x)}{A(x)}\,f_1{}'(x)\right)f_0(y)+\left(f_0''(x)+\frac{A'(x)}{A(x)}\,f_0{}'(x)\right)f_1(y)=
$$

$$
=\left(f_1''(y)+\frac{A'(y)}{A(y)}\,f_1{}'(y)\right)f_0(x)+\left(f_0''(y)+\frac{A'(y)}{A(y)}\,f_0{}'(y)\right)f_1(x)
$$

for all positive x, y. This implies

$$
\left(f_1''(x)+\frac{A'(x)}{A(x)}\,f_1{}'(x)\right)f_0(y)+c_0f_0(x)f_1(y)=
$$

$$
=\left(f_1''(y)+\frac{A'(y)}{A(y)}\,f_1{}'(y)\right)f_0(x)+c_0f_0(y)f_1(x)\,,
$$

or

$$
\left(f_1''(x)+\frac{A'(x)}{A(x)}\,f_1{}'(x)-c_0f_1(x)\right)f_0(y)=
$$

$$
=\left(f_1''(y)+\frac{A'(y)}{A(y)}\,f_1{}'(y)-c_0f_1(y)\right)f_0(x)
$$

for all positive x, y.

It follows that

$$
(11) \qquad f_1''(x)+\frac{A'(x)}{A(x)}\,f_1{}'(x)=c_0f_1(x)+c_1f_0(x)
$$

holds for each positive x with some complex number c_1. In particular, f_1 is C^∞. On the other hand, differentiating the first equation of (5) with respect to λ and then substituting $\lambda = c_0$ it follows

$$\partial_1^2\partial_2\,\varphi(x,c_0)+\frac{A'(x)}{A(x)}\,\partial_1\partial_2\,\varphi(x,c_0)=\varphi(x,c_0)+c_0\partial_2\,\varphi(x,c_0)$$

for each positive x. Further, obviously $\partial_2\,\varphi(0,c_0)=\partial_1\partial_2\,\varphi(0,c_0)=0$, as a consequence of (5). This means that the function $g:\mathbb{R}_0\to\mathbb{C}$ defined by $g(x)=f_1(x)-c_1\partial_2\,\varphi(x,c_0)$ for each x in $\mathbb{R}_0$ satisfies

$$g''(x)+\frac{A'(x)}{A(x)}\,g'(x)=0\,,$$
$$g(0)=0,\quad g'(0)=0$$

for each positive x, and hence, by uniqueness, $g\equiv 0$ and we have the second equation of (9).

Now we derive the third equation of (9). Using similar argument like before, by the definition of the hypergroup the third equation of (10) implies

$$\left(f_2''(x)+\frac{A'(x)}{A(x)}\,f_2{}'(x)-c_0f_2(x)-2c_1f_1(x)\right)f_0(y)=$$
$$=\left(f_2''(y)+\frac{A'(y)}{A(y)}\,f_2{}'(y)-c_0f_2(y)-2c_1f_1(y)\right)f_0(x)$$

for all positive x,y. It follows that

$$f_2''(x)+\frac{A'(x)}{A(x)}\,f_2{}'(x)=c_0f_2(x)+2c_1^2\partial_2\,\varphi(x,c_0)+c_2f_0(x)$$

holds for each positive x with some complex number c_2. In particular, f_2 is C^∞. On the other hand, differentiating two times the first equation of (5) with respect to λ and then substituting $\lambda=c_0$ it follows

$$\partial_1^2\partial_2^2\,\varphi(x,c_0)+\frac{A'(x)}{A(x)}\,\partial_1\partial_2^2\,\varphi(x,c_0)=2\partial_2\,\varphi(x,c_0)+c_0\partial_2^2\,\varphi(x,c_0)$$

for each positive x. Further, obviously $\partial_2^2\,\varphi(0,c_0)=\partial_1\partial_2^2\,\varphi(0,c_0)=0$, as a consequence of (5). This means that the function $g:K\to\mathbb{C}$ defined by $g(x)=f_2(x)-c_1^2\partial_2^2\,\varphi(x,c_0)$ for each x in $\mathbb{R}_0$ satisfies

$$g''(x)+\frac{A'(x)}{A(x)}\,g'(x)=c_0g(x)+c_2f_0(x),$$
$$g(0)=0,\quad g'(0)=0$$

for each positive x. Applying the same argument we used in the case of equation (11) we arrive at

$$g(x) = c_2 \partial_2 \varphi(x, c_0)$$

for each x in $\mathbb{R}_0$, which implies the third equation of (9) and the necessity of the given condition is proved.

To prove the converse we suppose that the C^∞-functions $f_0, f_1, f_2 : \mathbb{R}_0 \to \mathbb{C}$ satisfy (9) for each x in $\mathbb{R}_0$ with some complex numbers c_0, c_1, c_2. Then f_0 is an exponential of K, and the first equation of (10) is satisfied. Let for all x, y in $\mathbb{R}_0$

$$v(x, y) = f_1(x) f_0(y) + f_1(y) f_0(x).$$

Then for all positive x, y we have $\partial_1 v(0, y) = \partial_2 v(x, 0) = 0$, $v(x, 0) = f_1(x)$, $v(0, y) = f_1(y)$. Finally, an easy calculation shows that $(L_A)_x v(x, y) = (L_A)_y v(x, y)$ holds for all positive x, y, that is, v is a solution of the boundary value problem (1) defining the Sturm–Liouville hypergroup K. This means, $f_1(x * y) = v(x, y)$ and the second equation of (10) is satisfied. For the proof of the third equation of (10) we apply a similar argument for the function $w : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{C}$ defined by

$$w(x, y) = f_2(x) f_0(y) + 2 f_1(x) f_1(y) + f_0(x) f_2(y)$$

for all $x, y \in \mathbb{R}_0$ to get $w(x, y) = f_2(x * y)$, and the theorem is proved. $\diamond$

For instance, on the Bessel–Kingman hypergroup $A(x) = x^\alpha$ holds for all nonnegative x with some positive number α and we have the general form of moment sequences of second order:

$$\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= \frac{c_1 x^2}{2(\alpha + 1)}, \\ f_2(x) &= \frac{c_1^2 x^4}{4(\alpha + 1)(\alpha + 3)} + \frac{c_2 x^2}{2(\alpha + 1)} \end{aligned}$$

for each nonnegative x. In the other example, on the square hypergroup $A(x) = (1 + x)^2$ holds for all nonnegative x, and we have the general form of moment sequences of second order:

$$\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= c_1 \frac{x^3 + 3x^2}{6(x + 1)}, \\ f_2(x) &= \frac{1}{10} c_1^2 \frac{x^5 + 5x^4}{6(x + 1)} + c_2 \frac{x^3 + 3x^2}{6(x + 1)} \end{aligned}$$

for each nonnegative x.

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