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DERIVATION-LIKE MAPPINGS FOR NORMAL ELEMENTS IN PRIME RINGS WITH INVOLUTION

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Abstract: For a prime ring R with involution, we characterize additive maps $f : R \to Q_{ml}(R)$ such that $[f(u), u^*] + [u, f(u^*)] = 0$ whenever $[u, u^*] = 0$, that is, for all normal $u \in R$. It is shown that on K, the skew elements of R, such a map f is a sum of a derivation, a scalar map, and a map into the extended centroid of R.

One of the problems in algebra that continues to receive attention is that of characterizing maps that preserve certain algebraic properties. Some examples might be maps for which $f(x^n) = f(x)^n$ for a fixed n (preserve powers), or maps for which $ab = ba$ implies $f(a)f(b) = f(b)f(a)$ (preserves commutativity). Another example is maps for which u normal implies $f(u)$ is normal (preserves normality), where u is normal if $uu^* = u^*u$ where $*$ is an involution. In [1], Beidar et al. were able to characterize normal preserving bijective linear maps $f : A \to B$, where A, B are centrally closed prime algebras (with a few technical conditions). We refer the interested reader to that article for an excellent brief history of the normal preserver problem and a generous

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list of references. The problem we address in this article is not a "preserver" problem, but was motivated by the normal preserver problem in the following way. Normality can be expressed as $[u, u^*] = 0$, then f preserving normality means that $[f(u), f(u)^*] = 0$. In the case that f commutes with the involution, this is equivalent to $[f(u), f(u^*)] = 0$. Thus, f behaves like a Lie homomorphism on such $[u, u^*] = 0$. We are interested in maps that behave like a Lie derivation in the same situation.

Let R be a prime ring with involution, with center F, extended centroid C, and $Q = Q_{ml}(R)$ its maximal left ring of quotients (see [5, Chapter 2] for definitions and discussion of these terms). For any subset $A \subseteq R$, let $\langle A \rangle$ be the subring of R generated by A. Define the normal elements of R as $u \in R$ with $[u, u^*] = 0$. Let $f : R \to Q$ be an additive map such that $[f(u), u^*] + [u, f(u^*)] = 0$ for all normal $u \in R$. There are obvious candidates for f; that is, additive maps that would behave this way. Namely, derivations ($f(ab) = f(a)b + af(b)$), scalar maps ($f(a) = \lambda a$ for any fixed $\lambda \in C$), central maps ($f(a) \in C$), and sums of these. We are able to show, essentially, that these standard forms are the only possibilities. However, the example given in the last section shows that we should not expect in general to get a standard form for f on the whole ring. (This was also the case with normal preserving maps.) Th. 6 is the main result in which we are able to fully characterize such maps on K, the skew elements of R. Cors. 10, 11, and 12 provide stronger results (characterization on $\langle K \rangle$ or on all of R) in special cases.

1. Background

Functional identities (FI) play a key role in the proof, so we will give a brief introduction to that theory before quoting several results. Necessarily our comments will be less than thorough and we refer the interested reader to surveys [8] and [6] for a broader perspective and to technical articles [3], [4] for more precisely stated definitions and results.

Consider a subset $A \subseteq Q$ and write $\bar{x}_m = (x_1, \ldots, x_m)$ for elements of A^m, the cartesian product. Let $X_1, X_2, \ldots, X_m$ be noncommuting indeterminates and consider monomials $X_{i_1} X_{i_2} \cdots X_{i_n}$ where $\{i_1, i_2, \ldots, i_n\} \subseteq \{1, 2, \ldots, m\}$, which are linear in each of the indeterminates that appear. Note that each indeterminate or monomial can be considered a function on A^m in a natural way by $X_{i_1}X_{i_2}\cdots X_{i_n}(\bar{x}_m) = x_{i_1}x_{i_2}\cdots x_{i_n}$. Also, any function $f : A^{m-k} \to Q$ can be viewed as a function on A^m as long as it is clear which k entries of a typical $\bar{x}_m$ are excluded. This is done, with some abuse of notation, as a superscript on the map: for example $f^i(\bar{x}_m) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ $(k = 1)$, or $f^M(\bar{x}_m) = f(x_{j_1}, \dots, x_{j_{m-k}})$ where M is a monomial involving k indeterminates and the $x_{j_i}, i = 1, \dots, m-k$ correspond to the indeterminates not appearing in M. The names of maps often include subscripts corresponding to the indeterminates or monomials they appear with (thus it is common to see matching subscripts and superscripts). A multilinear quasi-polynomial of degree m is a sum of the form $\sum_L \lambda_L^L(\bar{x}_m)L(\bar{x}_m)$ where each L is a monomial and each $\lambda_L : A^k \to C$ where L involves $m-k$ indeterminates (if L involves all the x_i then λ_L is just an element of C; we also allow the trivial monomial 1 with $\lambda_1 = \lambda(x_1, \dots, x_m)$). For a fixed $n < m$ we are also interested in sums of the form $\sum_{M,N} M(\bar{x}_m)B_{MN}^{MN}(\bar{x}_m)N(\bar{x}_m)$ where M, N are monomials involving (disjointly) a total of $m-n$ indeterminates and each $B_{MN} : A^n \to Q$ is an unknown function. One possible functional identity on A would be an equation of the form $\sum_{M,N} M(\bar{x}_m)B_{MN}^{MN}(\bar{x}_m)N(\bar{x}_m) = \sum_L \lambda_L^L(\bar{x}_m)L(\bar{x}_m)$ that holds for all $\bar{x}_m \in A^m$ (we will encounter one such identity where all the B_{MN} are the same). Under certain conditions on the subset A we can conclude that the unknown maps B_{MN} must be multilinear quasi-polynomials. One of the earliest applications (and motivations) of this theory was in characterizing commuting maps (see [7]), which are the starting point for the proof of our main result.

In order to be able to apply the very powerful FI theory, we must consider a special condition, the d-freeness, of the sets of interest. For the sake of completeness, we include this fundamental definition here. Let $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \dots, m\}$ and let $E_i, F_j : A^{m-1} \to Q$, for $i \in \mathcal{I}, j \in \mathcal{J}$ be arbitrary maps. Consider the basic functional identities on A:

(1) $$\sum_{i\in\mathcal{I}} E_i^i(\bar{x}_m)x_i + \sum_{j\in\mathcal{J}} x_j F_j^j(\bar{x}_m) = 0,$$

(2) $$\sum_{i\in\mathcal{I}} E_i^i(\bar{x}_m)x_i + \sum_{j\in\mathcal{J}} x_j F_j^j(\bar{x}_m) \in C,$$

for all $\bar{x}_m \in A^m$. Suppose that there exist maps $\lambda_k : A^{m-1} \to C$ for each $k \in \mathcal{I} \cap \mathcal{J}$, and maps $p_{ij} : A^{m-2} \to Q$ for $i \in \mathcal{I}, j \in \mathcal{J}, i \neq j$, such that

$$(3)\qquad \begin{aligned} E_i^i(\bar{x}_m) &= \sum_{j\in\mathcal{J},i\neq j} x_j p_{ij}^{ij}(\bar{x}_m) + \lambda_i^i(\bar{x}_m),\ \ i\in\mathcal{I},\\ F_j^j(\bar{x}_m) &= \sum_{i\in\mathcal{I},i\neq j} p_{ij}^{ij}(\bar{x}_m)x_i + \lambda_j^j(\bar{x}_m),\ \ j\in\mathcal{J}, \end{aligned}$$

for all $\bar{x}_m \in A^m$ (assume $\lambda_k = 0$ for $k \notin \mathcal{I}\cap\mathcal{J}$). It is clear that (3) is a solution to both (1) and (2), which we will call a standard solution. Then, for a positive integer d, A is a d-free subset of Q if, whenever $\max\{|\mathcal{I}|,|\mathcal{J}|\} \le d$, the only solutions to any identity of form (1) are standard, and whenever $\max\{|\mathcal{I}|,|\mathcal{J}|\} \le d-1$, the only solutions to any identity of form (2) are standard.

The following result tells us about the d-freeness of certain subsets of R. Recall that the degree of a subset A of Q is the maximum degree of algebraicity over C of the elements of A.

Theorem 1 ([3, Th. 2.4]). *Let R be a prime ring with involution, $Q = Q_{ml}(R)$ its maximal left ring of quotients. Let S and K be the subsets of R of the symmetric and skew elements, respectively. Then*

1. *if $\deg(R) \ge d$, R is a d-free subset of Q;*
2. *if $\deg(R) \ge 2d+2$, both S and K are d-free subsets of Q.*

With that in hand we now present the key results on functional identities we will need.

Theorem 2 ([4, Th. 1.1]). *Suppose $P(\bar{x}_m)$ is a multilinear quasi-polynomial of degree m, which is zero on an $m+1$-free subset A of Q (m-free if $\lambda_1 = 0$). Then each coefficient map λ_L of P is zero on A.*

Theorem 3 ([4, Th. 1.2]). *Suppose $B : A^n \to Q$ is an n-additive map with $\sum a_{M,N} M(\bar{x}_m) B^{MN}(\bar{x}_m) N(\bar{x}_m) = P(\bar{x}_m)$ for all $\bar{x}_m \in A^m$, where P is a multilinear quasi-polynomial, and the $a_{M,N} \in C$. Suppose that either $\lambda_1 = 0$ and A is an m-free subset of Q, or that A is an $m+1$-free subset of Q. Then B is a multilinear quasi-polynomial of degree n.*

Corollary 4. *Suppose $T : A \to Q$ is the trace of an n-additive map such that $[T(x), x^{m-n}] = 0$ for all $x \in A$. If A is an m-free subset of Q and* char $(R) = 0$ *or $> n$, then there exist symmetric k-additive maps $\lambda_k : A^k \to C$, $k = 0,\dots,n$, such that $T(x) = \sum_{k=0}^{n} \lambda_k(x,\dots,x)x^{n-k}$ for all $x \in A$.*

The corollary follows from Th. 3 by fully linearizing $[T(x), x^{m-n}]$.

The following special result will also be used.

Theorem 5 ([4, Th. 2.9]). *Let A be a subset of Q, $f(x_1, \dots, x_m)$ a multilinear polynomial such that $f(a_1, \dots, a_m) \in A$ for all $a_i \in A$, and let map $B : A \times A \to Q$ be such that $B(a, b) = -B(b, a)$ for all $a, b \in A$ and*

$$B(f(a_1, \dots, a_m), b) = \sum_{i=1}^{m} f(a_1, \dots, B(a_i, b), \dots, a_m)$$

for all $a_i, b \in A$. If A is $2m$-free, then there exists $\rho \in C$ and map $\epsilon : A \times A \to C$ such that $B(a, b) = \rho[a, b] + \epsilon(a, b)$ for all $a, b \in A$.

Finally, we mention that the proof below will be directed towards showing the existence of a particular Lie derivation, whence we can apply the results in [2] on extensions of Lie maps to get related derivations.

2. Main Theorem

We now state the main result. The proof we give follows the basic outline of the proof of Th. 5.1 in [1] on normal preservers, though we are able to consider a more general situation.

Theorem 6. *Let R be a prime ring with involution, such that* char $(R) \neq 2$ *or* 3, *and* deg $(R) \geq 14$. *Let C be the extended centroid of R and $Q = Q_{ml}(R)$ its maximal left ring of quotients. Suppose $f : R \to Q$ is an additive map such that $[f(u), u^*] + [u, f(u^*)] = 0$ for all normal elements $u \in R$. Then $f(k) = \delta(k) + \gamma k + \phi(k)$ for all $k \in K$, for a derivation $\delta : \langle K \rangle \to Q$, $\gamma \in C$, and additive map $\phi : K \to C$.*

Proof. We proceed by a sequence of lemmas. Since deg $(R) \geq 14$, Th. 1 tells us that R is d-free for all $d \leq 14$ and K is d-free for all $d \leq 6$. These facts will allow us to apply Ths. 3 and 5, and Cor. 4. In fact, it is only in the application of Th. 5 in the proof of Lemma 9 that we need K to be 6-free.

For all $k \in K$, we know that k, k^2, and $k+k^2$ are normal elements, so evaluating $[f(u), u^*] + [u, f(u^*)] = 0$ with $u = k + k^2$, we get

$$(4) \qquad [f(k), k^2] + [k, f(k^2)] = [k, f(k^2) - f(k)k - kf(k)] = 0$$

for all $k \in K$. If we define a symmetric biadditive map $B : K \times K \to A$ by $B(k, l) = \frac{1}{2}\{f(k \circ l) - f(k) \circ l - k \circ f(l)\}$, we see that $T(k) = f(k^2) - f(k)k - kf(k) = B(k, k)$ is the trace of B. From Cor. 4 we have that $T(k) = \lambda k^2 + \mu(k)k + \nu(k, k)$ for all $k \in K$ where $\lambda \in C$,

$\mu : K \to C$ is additive, and $\nu : K \times K \to C$ is a symmetric biadditive map. Hence

$$f(k^2) = f(k)k + kf(k) + \lambda k^2 + \mu(k)k + \nu(k,k) \tag{5}$$

for all $k \in K$.

Define a map $g : R \to Q$ by $g(x) = f(x) + \lambda x$ and note that g is additive, $[g(u), u^*] + [u, g(u^*)] = 0$ for all normal $u \in R$, and using (5)

$$g(k^2) = g(k)k + kg(k) + \mu(k)k + \nu(k,k) \tag{6}$$

for all $k \in K$. Linearizing (6), we see that

$$g(kl + lk) = g(k)l + kg(l) + g(l)k + lg(k) + \mu(k)l + \mu(l)k + 2\nu(k,l) \tag{7}$$

for all $k, l \in K$.

Lemma 7. *There exist $\lambda_0 \in C$ and a symmetric biadditive map $\lambda_2 :$ $: K \times K \to C$ such that, for all $k, l \in K$, $g(klk) = g(k)lk + kg(l)k +$ $+ klg(k) + \lambda_0 klk + \frac{1}{2}\mu(k)(kl + lk) + \frac{1}{2}\mu(l)k^2 + \lambda_2(k,l)k - \frac{1}{2}\mu(klk)$.*

For $k \in K$, k^2, k^3, and $k^2 + k^3$ are normal so, as previously,

$$[g(k^3), k^2] + [k^3, g(k^2)] = 0 \tag{8}$$

for all $k \in K$. Using (6) gives

$$\begin{aligned}
&[g(k^3), k^2] + [k^3, g(k)k + kg(k) + \mu(k)k + \nu(k,k)] = \\
&= [g(k^3), k^2] + [k^2, g(k)k^2 + kg(k)k + k^2 g(k)] = \\
&= [g(k^3) - g(k)k^2 - kg(k)k - k^2 g(k), k^2] = 0.
\end{aligned}$$

Since $g(k^3) - g(k)k^2 - kg(k)k - k^2 g(k)$ can be viewed as the trace of a triadditive map, Cor. 4 gives

$$g(k^3) = g(k)k^2 + kg(k)k + k^2 g(k) + \lambda_0 k^3 + \lambda_1(k)k^2 + \lambda_2(k,k)k + \lambda_3(k,k,k) \tag{9}$$

for all $k \in K$, where $\lambda_0 \in C$, $\lambda_1 : K \to C$ is additive, $\lambda_2 : K \times K \to$ $\to C$ is symmetric biadditive, and $\lambda_3 : K \times K \times K \to C$ is symmetric triadditive. Linearizing (9) we have

$$\begin{aligned}
&g(k^2 l + klk + lk^2) = \\
&= g(k)kl + kg(k)l + k^2 g(l) + g(k)lk + kg(l)k + klg(k) + g(l)k^2 + lg(k)k + \\
&+ lkg(k) + \lambda_0(k^2 l + klk + lk^2) + \lambda_1(k)(kl + lk) + \lambda_1(l)k^2 + \\
&+ \lambda_2(k,k)l + 2\lambda_2(k,l)k + 3\lambda_3(k,k,l)
\end{aligned} \tag{10}$$

for all $k, l \in K$.

In order to characterize the λ_i, we proceed to compute $g(k^2lk + klk^2)$ in two ways, using $k^2lk + klk^2 = k \circ klk = \frac{1}{2}\{k \circ (k^2l + klk + lk^2) - k^3 \circ l\}$. From (7)

$$g(k \circ klk) = g(k)klk + kg(klk) + g(klk)k + klkg(k) + \mu(k)klk + \mu(klk)k + 2\nu(k, klk), \tag{11}$$

while

$$\begin{aligned} & g\Big(\frac{1}{2}\{k \circ (k^2l + klk + lk^2) - k^3 \circ l\}\Big) = \\ & = \frac{1}{2}\{g(k)(k^2l + klk + lk^2) + kg(k^2l + klk + lk^2) + \\ & + g(k^2l + klk + lk^2)k + (k^2l + klk + lk^2)g(k) + \\ & + \mu(k)(k^2l + klk + lk^2) + \mu(k^2l + klk + lk^2)k + \\ & + 2\nu(k, k^2l + klk + lk^2)\} - \frac{1}{2}\{g(k^3)l + k^3g(l) + \\ & + g(l)k^3 + lg(k^3) + \mu(k^3)l + \mu(l)k^3 + 2\nu(k^3, l)\} \end{aligned} \tag{12}$$

for all $k, l \in K$. Expanding (12) using (10) and (9), and equating with (11) yields

$$\begin{aligned} & k\{g(klk) - g(k)lk - kg(l)k - klg(k)\} + \\ & + \{g(klk) - g(k)lk - kg(l)k - klg(k)\}k = \\ & = \lambda_0 k^2lk + \lambda_0 klk^2 + \Big(\lambda_1(l) - \frac{1}{2}\mu(l)\Big)k^3 + \Big(\lambda_1(k) - \frac{1}{2}\mu(k)\Big)klk + \\ & + \frac{1}{2}\mu(k)k^2l + \frac{1}{2}\mu(k)lk^2 + 2\lambda_2(k,l)k^2 - \Big(\lambda_3(k,k,k) + \frac{1}{2}\mu(k^3)\Big)l + \\ & + \Big(3\lambda_3(k,k,l) + \frac{1}{2}\mu(k^2l - klk + lk^2)\Big)k + \\ & + \nu(k, k^2l - klk + lk^2) - \nu(k^3, l) \end{aligned} \tag{13}$$

for all $k, l \in K$.

Linearizing (13), we see it is of the form

$$k_1E(k_2,k_3,l) + E(k_2,k_3,l)k_1 + k_2E(k_1,k_3,l) + E(k_1,k_3,l)k_2 + k_3E(k_1,k_2,l) + E(k_1,k_2,l)k_3 = P(k_1,k_2,k_3,l) \tag{14}$$

for all $k_1, k_2, k_3, l \in K$, where $E(k_i, k_j, l) = g(k_ilk_j + k_jlk_i) - g(k_i)lk_j - k_ig(l)k_j - k_ilg(k_j) - g(k_j)lk_i - k_jg(l)k_i - k_jlg(k_i)$ and $P(k_1, k_2, k_3, l)$ is a multilinear quasi-polynomial. By Th. 3, E must be a multilinear quasi-polynomial; when this is substituted into (14) it leads to another multilinear quasi-polynomial $Q(k_1, k_2, k_3, l) = 0$. Using Th. 2, we have

$$\lambda_1(k_1) = \frac{3}{2}\mu(k_1),$$
$$\lambda_3(k_1,k_2,k_3) = -\frac{1}{12}\mu(k_1k_2k_3 + k_1k_3k_2 + k_2k_1k_3 + k_2k_3k_1 + k_3k_1k_2 + k_3k_2k_1),$$

and

$$E(k_1,k_2,l) = \lambda_0 k_1lk_2 + \lambda_0 k_2lk_1 + \frac{1}{2}\mu(k_1)k_2l + \frac{1}{2}\mu(k_1)lk_2 + \frac{1}{2}\mu(k_2)k_1l + \frac{1}{2}\mu(k_2)lk_1 + \frac{1}{2}\mu(l)k_1k_2 + \frac{1}{2}\mu(l)k_2k_1 + \lambda_2(k_1,l)k_2 + \lambda_2(k_2,l)k_1 - \frac{1}{2}\mu(k_1lk_2 + k_2lk_1),$$

for all $k_1, k_2, k_3, l \in K$. From the definition of E, we now have

$$g(k_1lk_2 + k_2lk_1) = g(k_1)lk_2 + k_1g(l)k_2 + k_1lg(k_2) + g(k_2)lk_1 + k_2g(l)k_1 + k_2lg(k_1) + \lambda_0(k_1lk_2 + k_2lk_1) + \frac{1}{2}\mu(k_1)(k_2l + lk_2) + \frac{1}{2}\mu(k_2)(k_1l + lk_1) + \frac{1}{2}\mu(l)(k_1k_2 + k_2k_1) + \lambda_2(k_1,l)k_2 + \lambda_2(k_2,l)k_1 - \frac{1}{2}\mu(k_1lk_2 + k_2lk_1) \tag{15}$$

for all $k_1, k_2, l \in K$, and

$$g(klk) = g(k)lk + kg(l)k + klg(k) + \lambda_0 klk + \frac{1}{2}\mu(k)(kl + lk) + \frac{1}{2}\mu(l)k^2\lambda_2(k,l)k - \frac{1}{2}\mu(klk) \tag{16}$$

for all $k, l \in K$. This completes the proof of Lemma 7.

Lemma 8. $\lambda_2(k,l) = 0$ *for all* $k, l \in K$.

Noting that $kl_1kl_2k + kl_2kl_1k = (kl_1k)l_2k + kl_2(kl_1k) = (kl_2k)l_1k + kl_1(kl_2k)$, we will compute $g(kl_1kl_2k + kl_2kl_1k)$ in two ways and compare. Using (15)

$$g((kl_1k)l_2k + kl_2(kl_1k)) =$$
$$= g(kl_1k)l_2k + kl_1kg(l_2)k + kl_1kl_2g(k) + g(k)l_2kl_1k + kg(l_2)kl_1k +$$

$$+ kl_2g(kl_1k) + \lambda_0(kl_1kl_2k + kl_2kl_1k) + \frac{1}{2}\mu(kl_1k)(kl_2 + l_2k)+$$
$$+ \frac{1}{2}\mu(k)(kl_1kl_2 + l_2kl_1k) + \frac{1}{2}\mu(l_2)(kl_1k^2 + k^2l_1k) + \lambda_2(kl_1k, l_2)k+$$
$$+ \lambda_2(k, l_2)kl_1k - \frac{1}{2}\mu(kl_1kl_2k + kl_2kl_1k)$$

which, using (16), expands to

$$g(kl_1kl_2k + kl_2kl_1k) =$$
$$= g(k)l_1kl_2k + \ldots + kl_1kl_2g(k) + g(k)l_2kl_1k + \ldots + kl_2kl_1g(k)+$$
$$+ 2\lambda_0(kl_1kl_2kkl_2kl_1k) + \frac{1}{2}\mu(k)(kl_1kl_2 + l_2kl_1k + kl_1l_2k+$$
$$+ l_1kl_2k + kl_2kl_1 + +kl_2l_1k) + \frac{1}{2}\mu(l_1)(kl_2k^2 + k^2l_2k)+$$
$$+ \frac{1}{2}\mu(l_2)(kl_1k^2 + k^2l_1k) + \lambda_2(kl_1k, l_2)k + 2\lambda_2(k, l_1)kl_2k+$$
$$+ \lambda_2(k, l_2)kl_1k - \frac{1}{2}\mu(kl_1kl_2k + kl_2kl_1k)$$

for all $k, l_1, l_2 \in K$. Computing $g(kl_1(kl_2k) + (kl_2k)l_1k)$ similarly, and subtracting, we find that

$$(17) \quad \lambda_2(k, l_1)kl_2k + \lambda_2(kl_1k, l_2)k - \lambda_2(k, l_2)kl_1k + \lambda_2(kl_2k, l_1)k = 0$$

for all $k, l_1, l_2 \in K$. Linearizing (17) yields a multilinear quasi-polynomial, which upon applying Th. 2, yields $\lambda_2(k, l) = 0$ for all $k, l \in K$. This completes the proof of Lemma 8.

Define an additive map $d : K \to Q$ by $d(k) = g(k) + \frac{1}{2}\lambda_0 k + \frac{1}{2}\mu(k)$ for $k \in K$. From (15), using Lemma 8,

$$(18) \quad \begin{aligned} d(k_1lk_2 + k_2lk_1) = d(k_1)lk_2 + k_1d(l)k_2 + k_1ld(k_2)+ \\ + d(k_2)lk_1 + +k_2d(l)k_1 + k_2ld(k_1) \end{aligned}$$

for all $k_1, k_2, l \in K$.

Lemma 9. $d([k, l]) - [d(k), l] - [k, d(l)] \in C$ *for all* $k, l \in K$.

Define a map $B : K \times K \to Q$ by $B(k, l) = d([k, l]) - [d(k), l] - - [k, d(l)]$ and note that $B(k, l) = -B(l, k)$. We proceed to compute $B(k_1k_2k_3 + k_3k_2k_1, l)$ for $k_1, k_2, k_3, l \in K$. Using the identity $[xyz, w] = = [x, w]yz + x[y, w]z + xy[z, w]$, along with (18), we have

$$
\begin{aligned}
(19)\ & B(k_1k_2k_3+k_3k_2k_1,l) = \\
&= d([k_1k_2k_3 + k_3k_2k_1, l]) - [d(k_1k_2k_3 + k_3k_2k_1), l] - \\
&\quad - [k_1k_2k_3 + k_3k_2k_1, d(l)] = \\
&= d([k_1,l]k_2k_3+k_3k_2[k_1,l])+d(k_1[k_2,l]k_3+k_3[k_2,l]k_1)+d(k_1k_2[k_3,l]+ \\
&\quad + [k_3,l]k_2k_1) - [d(k_1k_2k_3 + k_3k_2k_1), l] - [k_1k_2k_3 + k_3k_2k_1, d(l)] = \\
&= d([k_1,l])k_2k_3+[k_1,l]d(k_2)k_3+[k_1,l]k_2d(k_3)+\ldots+k_3k_2d([k_1,l])+\ldots \\
&\quad \ldots+d(k_1)k_2[k_3,l]+\ldots+[k_3,l]k_2d(k_3)-[d(k_1)k_2k_3+\ldots+k_3k_2d(k_1),l]- \\
&\quad - [k_1,d(l)]k_2k_3 - k_1[k_2,d(l)]k_3 - \ldots - k_3k_2[k_1,d(l)] = \\
&= \{d([k_1,l]) - [d(k_1),l] - [k_1,d(l)]\}k_2k_3 + k_1\{d([k_2,l]) - [d(k_2),l] - \\
&\quad - [k_2,d(l)]\}k_3 + \ldots + k_3k_2\{d([k_1,l]) - [d(k_1),l] - [k_1,d(l)]\} = \\
&= B(k_1,l)k_2k_3 + k_3k_2B(k_1,l) + k_1B(k_2,l)k_3+ \\
&\quad + k_3B(k_2,l)k_1 + k_1k_2B(k_3,l) + B(k_3,l)k_2k_1
\end{aligned}
$$

for all $k_1, k_2, k_3, l \in K$. (19) shows us that the map B, along with the multilinear polynomial $xyz + zyx$, satisfy the hypotheses of Th. 5, so there exist $\rho \in C$ and $\epsilon : K \times K \to C$ such that $B(k,l) = \rho[k,l] + \epsilon(k,l)$ for all $k, l \in K$. Thus, for all $k, l \in K$,

$$(20) \qquad d([k,l]) = [d(k),l] + [k,d(l)] + \rho[k,l] + \epsilon(k,l).$$

We can use the identity $[[x,y],z] = xyz + zyx - yxz - zxy$ to compute $d([[k,l],m])$ in two ways. First, using (20),

$$
\begin{aligned}
(21)\ & \\
d([[k,l],m]) &= [d([k,l]),m] + [[k,l],d(m)] + \rho[[k,l],m] + \phi([k,l],m) = \\
&= [[d(k),l] + [k,d(l)] + \rho[k,l] + \phi(k,l),m] + [[k,l],d(m)]+ \\
&\quad + \rho[[k,l],m] + \phi([k,l],m) = \\
&= [[d(k),l],m] + [[k,d(l)],m] + [[k,l],d(m)] + 2\rho[[k,l],m]+ \\
&\quad + \phi([k,l],m)
\end{aligned}
$$

for all $k, l, m \in K$. On the other hand, using (18),

$$
\begin{aligned}
d([[k,l],m]) &= d(klm + mlk) - d(lkm + mkl) = \\
(22) \qquad &= d(k)lm + \ldots + mld(k) - d(l)km - \ldots - mkd(l) = \\
&= [[d(k),l],m] + [[k,d(l)],m] + [[k,l],d(m)].
\end{aligned}
$$

Equating (21) and (22), we have $2\rho[[k,l],m] + \phi([k,l],m) = 0$ for all $k, l, m \in K$. Recognizing this as a multilinear quasi-polynomial, Th. 2 tells us that $\rho = 0$. This completes the proof of Lemma 9.

By [2, Th. 1.8], there exist a derivation $\delta : \langle K \rangle \to Q$ and an additive map $\tau : K \to C$ such that $d(k) = \delta(k) + \tau(k)$ for all $k \in$ $\in K$. Thus $f(k) = \delta(k) + (-\lambda - \frac{1}{2}\lambda_0)k + (\tau(k) - \frac{1}{2}\mu(k))$ for all $k \in K$, completing the proof of Th. 6. ◊

3. Special cases

We can strengthen the conclusion of Th. 6 in several special situations.

Corollary 10. *In* Th. 6, *suppose additionally that* $*$ *is of the first kind and* $f : R \to R_C$ ($R_C = RC + C$ *the central closure of* R). *Then there exist a derivation* $\delta : \langle K \rangle \to Q$, *elements* $\gamma_1, \gamma_2 \in C$, *and an additive map* $\psi : \langle K \rangle \to C$, *such that* $f(x) = \delta(x) + \gamma_1 x + \gamma_2 x^* + \psi(x)$ *for all* $x \in \langle K \rangle$.

Proof. For each $k \in K$, we have $[f(k) + f(k)^*, k] = 0$. Then, by Cor. 4, there exist $\lambda_K \in C$ and additive map $\mu_K : K \to C$ such that $f(k) + f(k)^* = \lambda_K k + \mu_K(k)$ for all $k \in K$. Since $*$ is on the first kind, applying $*$ we find that $f(k)^* = -f(k) + \mu_K(k)$ for all $l \in K$. In a similar way we find that $f(s)^* = f(s)$ for all $s \in S$. Extending μ_K to all of R by defining $\mu_K(s) = 0$ for all $s \in S$, we can define $\bar{f} : R \to R_C$ by $\bar{f}(x) = f(x) - \frac{1}{2}\mu_K(x)$. It is now easy to see that $[\bar{f}(u), u^*] + [u, \bar{f}(u^*)] = 0$ for all normal $u \in R$ and that $\bar{f}(x^*) = \bar{f}(x)^*$ for all $x \in R$.

Applying Th. 6 to $\bar{f}$, we see that the maps $\mu, \tau : K \to C$ in that proof are identically zero. Thus $\bar{f}(k) = \delta(k) - (\lambda + \frac{1}{2}\lambda_0)k$ for all $k \in K$. Also, for each $k \in K$,

$$\begin{aligned}\bar{f}(k^2) &= g(k^2) - \lambda k^2 = \\ &= g(k)k + kg(k) + \nu(k,k) - \lambda k^2 = \\ &= \left(\delta(k) - \frac{1}{2}\lambda_0 k\right) k + k\left(\delta(k) - \frac{1}{2}\lambda_0 k\right) + \nu(k,k) - \lambda k^2 = \\ &= \delta(k)k + k\delta(k) - (\lambda + \lambda_0)k^2 + \nu(k,k) = \\ &= \delta(k^2) - (\lambda + \lambda_0)k^2 + \nu(k,k),\end{aligned}$$

or $\bar{f}(k^2) - \delta(k^2) + (\lambda + \lambda_0)k^2 \in C$. Since $\langle K \rangle = K + K \circ K$ and $K \circ K$ is spanned additively by all k^2 with $k \in K$, we can define an additive map $\phi : \langle K \rangle \to C$ by $\phi(k) = 0$ for $k \in K$ and $\phi(s) = f(s) - \delta(s) + (\lambda + \lambda_0)s$ for $s \in K \circ K$. Thus, for all $x \in \langle K \rangle$, $f(x) = \bar{f}(x) + \frac{1}{2}\mu_K(x) = \delta(x) +$

$+(-\lambda-\frac{3}{4}\lambda_0)x+(\lambda-\frac{1}{4}\lambda_0)x^*+(\phi+\frac{1}{2}\mu_K)(x)$. This completes the proof of Cor. 10. ◊

Corollary 11. *In Th. 6, suppose additionally that R is centrally closed ($C = F$), $*$ is of the second kind, and f is linear. If $\deg(R) \geq 6$, then there exist a derivation $\delta : R \to Q$, $\gamma \in C$, and an additive map $\phi :$ $: R \to C$, such that $f(x) = \delta(x) + \gamma x + \phi(x)$ for all $x \in R$.*

Proof. Since $*$ is of the second kind, there exist $\epsilon \in C$ such that $\epsilon^* = -\epsilon$, and $R = S + K = S + \epsilon S$.

Let $s \in S$ and $\lambda \in C$, then $[s^2+\lambda s, (s^2+\lambda s)^*] = [s^2+\lambda s, s^2+\lambda^* s] =$ $= 0$ so $s^2+\lambda s$ is normal. By assumption, and using the linearity of f,

$$\begin{aligned} 0 &= [f(s^2+\lambda s), s^2+\lambda^* s] + [s^2+\lambda s, f(s^2+\lambda^* s)] = \\ &= [f(s^2)+\lambda f(s), s^2+\lambda^* s] + [s^2+\lambda s, f(s^2)+\lambda^* f(s)] = \\ &= [f(s^2), s^2] + [s^2, f(s^2)] + \lambda([f(s), s^2] + [s, f(s^2)]) + \\ &\quad + \lambda^*([f(s^2), s] + [s^2, f(s)]) + \lambda\lambda^*([f(s), s] + [s, f(s)]) = \\ &= (\lambda^* - \lambda)\left\{[f(s^2), s] + [s^2, f(s)]\right\}. \end{aligned}$$

Letting $\lambda = \epsilon$ then yields

$$[f(s^2), s] + [s^2, f(s)] = 0 \tag{23}$$

for all $s \in S$. Linearizing (23) we get

$$[f(t^2), s] + [t^2, f(s)] + [f(s \circ t), t] + [s \circ t, f(t)] = 0 \tag{24}$$

for all $s, t \in S$.

Let $x \in R$; then $x = s + \epsilon t$ for some $s, t \in S$. Using (23) and (24) we can calculate

$$\begin{aligned} &[f((s+\epsilon t)^2), s+\epsilon t] + [(s+\epsilon t)^2, f(s+\epsilon t)] = \\ &= [f(s^2) + \epsilon f(s \circ t) + \epsilon^2 f(t^2), s+\epsilon t] + \\ &\quad + [s^2 + \epsilon(s \circ t) + \epsilon^2 t^2, f(s) + \epsilon f(t)] = \\ &= [f(s^2), s] + [s^2, f(s)] + \epsilon\{[f(s^2), t] + [s^2, f(t)] + \\ &\quad + [f(s \circ t), s] + [s \circ t, f(s)]\} + \\ &\quad + \epsilon^2\{[f(t^2), s] + [t^2, f(s)] + [f(s \circ t), t] + [s \circ t, f(t)]\} + \\ &\quad + \epsilon^3\{[f(t^2), t] + [t^2, f(t)]\} = 0. \end{aligned}$$

Therefore, for all $x \in R$, we have

$$[f(x^2), x] + [x^2, f(x)] = 0. \tag{25}$$

Similarly, starting with $s^3 + \lambda s^2$, we get

$$[f(x^3), x^2] + [x^3, f(x^2)] = 0 \tag{26}$$

for all $x \in R$.

Comparing (25) and (26) to (4) and (8), we see that the proof of Th. 6 carries through with R in place of K (though a direct proof would be considerable shorter since R is closed under squares), noting that we only need $\deg(R) \geq 6$ since then, by Th. 1, R is 6-free. An application of [2, Th. 1.3] shows the existence of the appropriate derivation, completing the proof of the corollary. ◊

Corollary 12. *Suppose R is a centrally closed simple algebra with involution, and* $\deg(R) \geq 14$, $\operatorname{char}(R) \neq 2, 3$. *Then any linear map satisfying the conditions of Th. 6 is of the form $f(x) = \delta(x) + \lambda_1 x + \lambda_2 x^* + \phi(x)$ for all $x \in R$, where δ is a derivation, $\lambda_1, \lambda_2 \in F$, and $\phi : R \to F$.*

This follows from Cors. 10 and 11, noting that in the case of an involution of the first kind, $\langle K \rangle = R$ [9, Th. 2]. ◊

The following example, adapted from an example given in [1], shows that we should not expect to be able to characterize such maps, in general, on the whole ring. Let $A = F\langle x, y\rangle$, the free algebra over a field F in two indeterminates, equipped with the involution given by $x^* = x, y^* = y, \lambda^* = \lambda$ for $\lambda \in F$ (thus $(xy)^* = yx$, etc). Noting that $A = F \oplus U \oplus V$ as an F-vector space, where U is spanned by the nonzero powers of x and of y, and V is spanned by the monomials which involve both x and y. In this context the normal elements are $\lambda + u + v$ where either $v = v^*$ or $u = 0$ and $v^* = -v$. Let the linear map $' : V \to A$ be the derivation defined by $\lambda' = 0, x' = 1, y' = 1$. We find that the map $f : A \to A$ given by $f(\lambda + u + v) = \lambda + u + v'$ satisfies the required condition on normal elements. But it is clear that f is not of standard form on A, though it is already a derivation on $V \supset \langle K \rangle$.

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