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COMPOSITIONS ON THE RING OF POLYNOMIALS IN TWO INDETERMINATES

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Abstract: The ring of polynomials in two or more indeterminates is not endowed with a canonical composition. Several compositions are proposed and for each a number of first properties are determined.

For a ring R, the polynomial ring $(R[x], +, \cdot)$ is endowed with a natural nearring multiplication, namely the usual composition $f(x) \circ \circ\, g(x) = f(g(x))$. This nearring $(R[x], +, \circ)$ has been studied extensively – also in the setting as the composition ring $(R[x], +, \cdot, \circ)$. On the other hand, the ring of polynomials in two indeterminates $(R[x, y], +, \cdot)$ does not admit a natural composition. That is, unless one wants to construct the polynomials in two indeterminates by iterating the one indeterminate construction. To wit, for any ring S we have the polynomial composition ring $(S[x], +, \cdot, \circ)$. Replacing the ring S with the ring $(R[y], +, \cdot)$ will give the composition ring $(R[x, y], +, \cdot, \circ) = = ((R[y])[x], +, \cdot, \circ)$ with composition $f(x, y) \circ g(x, y) = f(g(x, y), y)$. But this approach has the major disadvantage that the constant part (or "foundation" in the composition ring terminology) is exactly the elements of $R[y]$, the polynomials in the one indeterminate y.

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In Clay [3] the composition $f(x,y) \circ g(x,y) = f(g(x,y), g(x,y))$ on $R[x,y]$ was suggested and this was further developed by Gutierrez and de Velasco [4]. This composition does not admit an identity, but it does have left identities. But these two compositions are not the only possibilities. Below we will present many more and for each we will discuss some of its first properties. In particular, we will show that there are compositions possible which do admit identities and which have the constant part the underlying ring R. For compositions with one-sided identities, we will describe the subset of $R[x,y]$ on which they are identities and then we will also determine the units.

1. Preliminaries

R will always be a commutative ring with identity 1. The polynomial ring $(R[x], +, \cdot)$ has identity 1 and $f(x) = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n$ is a unit if and only if f_0 is a unit of R and $f_1, f_2, \ldots, f_n$ $(n \geq 1)$ are nilpotent elements of R. On the other hand, the polynomial nearring $(R[x], +, \circ)$ has identity x and a polynomial is a unit in this nearring if and only if it is of the form $f(x) = f_0 + f_1x$ where $f_0 \in R$ and f_1 is a unit of R. $R_0[x]$ will denote the 0-symmetric part of the nearring $(R[x], +, \circ)$, i.e. all the polynomials of the form $f(x) = f_1x + f_2x^2 + \cdots + f_nx^n, n \geq 1$. For more information on nearrings, Pilz [6] or Clay [3] can be consulted. The ring of polynomials in the two commuting indeterminates x and y will be denoted by $(R[x,y], +, \cdot)$. A typical element $f(x,y)$ of $R[x,y]$ will be written as $f(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} f_{ij}x^iy^j$ where $n \geq 0$ (and also $i, j \geq 0$) and $f_{ij} \in R$. For a given $f(x,y)$, we will use f_k to denote $f_k = \sum_{i+j=k} f_{ij}, k = 0, 1, 2, \ldots, n$. As usual, the degree of $f(x,y)$ is given by $\deg f(x,y) = \max\{i+j \mid f_{ij} \neq 0\}$.

We will rely heavily on the Substitution Rule:

Given $f(x,y), g(x,y), h(x,y) \in R[x,y]$, then $f(g(x,y), h(x,y))$ will denote the polynomial f where each occurrence of x has been replaced by $g(x,y)$ and each occurrence of y has been replaced by $h(x,y)$. In particular, if $g(x,y) = r$ and $h(x,y) = s$ where r and s are fixed elements of R, then $f(r,s)$ is just the evaluation of $f(x,y)$ in the point $(x,y) = (r,s)$.

To define the different compositions and to avoid unnecessary repetitions, we will use two functions $\alpha_1, \alpha_2 : R[x,y] \to R[x,y]$ to define the composition $f(x,y) \circ g(x,y)$. For $f(x,y) \in R[x,y]$, we will denote $\alpha_i(f(x,y))$ by $\alpha_i(f(x,y)) = \alpha_i(f)(x,y) \in R[x,y]$. Care should be taken to distinguish between the functions α_i and the polynomials $\alpha_i(f)$. For example, let α_1 be the constant function $\alpha_1(f(x,y)) = y$ for all $f(x,y) \in R[x,y]$. Then $\alpha_1(g(h(x,y), k(x,y)) = y$ by the definition of α_1 while $\alpha_1(g)(h(x,y), k(x,y)) = k(x,y)$ by the substitution rule. For another example, suppose α_1 is the function $\alpha_1(f(x,y)) = f(y,x)$ for all $f(x,y) \in R[x,y]$, i.e. $\alpha_1(f)(x,y)$ is the polynomial which is the same as $f(x,y)$ except that all occurrences of x and y are interchanged. Then $\alpha_1(g(h(x,y), k(x,y)) = g(h(y,x), k(y,x))$ by the definition of α_1 while $\alpha_1(g)(h(x,y), k(x,y)) = g(k(x,y), h(x,y))$ by the substitution rule.

For a given pair α_1 and α_2, define a composition $f(x,y) \circ g(x,y)$ on $R[x,y]$ by $f(x,y) \circ g(x,y) = f(\alpha_1(g(x,y)), \alpha_2(g(x,y)))$. For various choices of α_1 and α_2 we will get a nearring multiplication on $R[x,y]$. In fact, for all our choices given below, we will get a composition ring structure on $R[x,y]$. Recall, a composition ring C is a quadruple $C = (C, +, \cdot, \circ)$ where $(C, +, \cdot)$ is a ring, $(C, +, \circ)$ is a nearring and the composition $\circ$ distribute from the right over the multiplication (i.e. $ab \circ c = (a \circ c)(b \circ c)$ for all $a, b, c \in C$). The foundation of a composition ring is $\mathrm{Found}(C) = \{c \in C \mid c \circ 0 = c\}$. This subset is a subcomposition ring of C, but usually by the foundation is meant the ring $(\mathrm{Found}(C), +, \cdot)$. More about composition rings can be found in Adler [1].

The composition defined above (using the functions α_1 and α_2) distributes from the right over both the addition as well as over the multiplication, but in general it need not be associative. For this we have to impose a suitable condition on the functions α_1 and α_2, namely:

(A) For all $i = 1, 2$ and $g(x,y), h(x,y) \in R[x,y]$,

$$\alpha_i(g)(\alpha_1(h(x,y)), \alpha_2(h(x,y))) = \alpha_i(g(\alpha_1(h(x,y)), \alpha_2(h(x,y)))).$$

This condition can briefly be stated as $\alpha_i(g) \circ h = \alpha_i(g \circ h)$. To demonstrate the requirements of this condition, we give an example. If $\alpha_1(f(x,y)) = f(y,x)$ and $\alpha_2(f(x,y)) = f(x,x)$ for all $f(x,y) \in R[x,y]$, then condition (A) is not satisfied. Indeed,

$$\begin{aligned} \alpha_1(g)(\alpha_1(h(x,y)), \alpha_2(h(x,y))) &= g(\alpha_2(h(x,y)), \alpha_1(h(x,y)))) = \\ &= g(h(x,x), h(y,x)), \quad \text{but} \\ \alpha_1(g(\alpha_1(h(x,y), \alpha_2(h(x,y))) &= \alpha_1(g(h(y,x), h(x,x)) = \\ &= g(h(x,y), h(y,y)). \end{aligned}$$

Proposition 1. *Suppose the functions $\alpha_1, \alpha_2 : R[x,y] \to R[x,y]$ satisfy condition (A). Then $(R[x,y], +, \cdot, \circ)$ is a composition ring with respect to the composition $f(x,y) \circ g(x,y) = f(\alpha_1(g(x,y)), \alpha_2(g(x,y)))$.*

As usual, we will use $R_0[x,y]$ to denote the 0-symmetric part of $R[x,y]$. Pointwise addition and multiplication induced by that on R makes $(M(R^2,R),+,\cdot)$ a ring where $M(R^2,R) = \{f \mid f$ is a function $f : R^2 \to R\}$; here R^2 denotes $R \oplus R$. Then we have a ring homomorphism $\eta : R[x,y] \to M(R^2,R)$ defined by $\eta(f(x,y)) = \overline{f}$ where $\overline{f}$ is the function $\overline{f} : R^2 \to R$ with $\overline{f}(r,s) = f(r,s)$ for all $r,s \in R$. If, for a given α_1 and α_2, $R[x,y]$ is a composition ring, η need not be a composition ring homomorphism, since no composition need to be defined on $M(R^2,R)$. One would like to use the functions α_i to define a composition on $M(R^2,R)$, but this may not be possible. For example, if α_1 denotes the formal partial derivative of $f(x,y)$ with respect to x, then $\alpha_1(h)$ need not be defined for all $h \in M(R^2,R)$.

We will say a function $\alpha : R[x,y] \to R[x,y]$ is extendable if there exists a function $\overline{\alpha} : M(R^2,R) \to M(R^2,R)$ such that for all $f(x,y) \in \in R[x,y]$, $\overline{\alpha}(\eta(f(x,y)) = \eta(\alpha(f(x,y)))$. This means $\overline{\alpha(f)} = \eta(\alpha(f)(x,y)) = = \eta(\alpha(f(x,y)) = \overline{\alpha}(\eta(f(x,y)) = \overline{\alpha}(\overline{f})$ for all $f(x,y) \in R[x,y]$. If both α_1 and α_2 are extendable, then we can define a composition on $M(R^2,R)$ by $f \circ g : R^2 \to R$, $(f \circ g)(r,s) := f(\overline{\alpha_1}(g)(r,s), \overline{\alpha_2}(g)(r,s))$ for all $r,s \in R, f,g \in M(R^2,R)$. Of course, for a given α_1 and α_2, the extensions $\overline{\alpha_1}$ and $\overline{\alpha_2}$ need not be uniquely determined, but once they have been chosen, they are fixed for the definition of the composition on $M(R^2,R)$.

Two extendable functions α_1 and α_2 are said to satisfy condition (EA) (with respect to the extensions $\overline{\alpha_1}$ and $\overline{\alpha_2}$) if:

(EA) For all $i = 1,2$ and $g,h \in M(R^2,R)$,

$$\overline{\alpha}_i(g)(\overline{\alpha}_1(h)(r,s), \overline{\alpha}_2(h)(r,s)) = \overline{\alpha}_i(g \circ h)(r,s) \quad \text{for all } r,s \in R.$$

If α_1 and α_2 are extendable, then condition (EA) will follow from (A) provided the function η has certain properties, but in general it need not be the case.

Proposition 2. *Suppose the functions $\alpha_1, \alpha_2 : R[x,y] \to R[x,y]$ are extendable and satisfy conditions (A) and (EA). Then $(M(R^2,R), +, \cdot, \circ)$ is a composition ring with respect to $(f \circ g)(r,s) := f(\overline{\alpha_1}(g)(r,s), \overline{\alpha_2}(g)(r,s))$ and η is a composition ring homomorphism.*

We next remark on the relationship between $\overline{R[x,y]} := \eta(R[x,y])$ and $M(R^2,R)$. Using Lagrange's interpolation (or the fact that a finite field is 2-polynomially complete) we see that it is the same as for the one indeterminate case:

Proposition 3. *$\eta : R[x,y] \to M(R^2, R)$ defined by $\eta(f(x,y)) = \overline{f}$ is a surjection if and only if R is a finite field.*

The kernel of the map $\eta, \ker\eta := \{f(x,y) \in R[x,y] \mid f(r,s) = 0\}$, is of much interest; in particular if $\ker\eta = 0$ then the only "vanishing" is the zero polynomial. We also need a result from Aczél [2]. Suppose $f(z) = f_0 z^n + f_1 z^{n-1} + \dots + f_{n-1}z + f_n$ is a polynomial in one indeterminate over R and $(R,+)$ has no nonzero elements of finite order. If $f(r) = 0$ for all $r \in R$, then $f(z) = 0$, i.e. $f_0 = f_1 = \dots = f_n = 0$. The argument given by Aczél is as follows: $f(z)$ can be written as $f(z) = f_0 z^n + P_{n-1}(z)$ where $P_{n-1}(z)$ is a polynomial in z with degree at most $n-1$. Let $F_0(z) := f(z)$. Then $F_1(z) := F_0(z+1) - F_0(z) = nf_0 z^{n-1} + P_{n-2}(z)$ where $P_{n-2}(z)$ is a polynomial in z with degree at most $n-2$ and $F_1(r) = 0$ for all $r \in R$. Repeat this process to ultimately get $F_n(z) = n!f_0$ with $F_n(r) = 0$ for all $r \in R$. From the assumption on R we may conclude that $f_0 = 0$. This holds for any n, so we can apply this technique as many times as necessary to get $f(z) = 0$. As we shall see in the next proposition, this result extends to $R[x,y]$.

Proposition 4. *Let $\eta : R[x,y] \to M(R^2, R)$ be defined by $\eta(f(x,y)) = \overline{f}$. If η is injective, then R must be an infinite ring. For a converse, if $(R,+)$ has no nonzero elements of finite order, then η is injective.*

Proof. Suppose η is injective. Then $R[x,y] \simeq \eta(R[x,y]) \leq M(R^2, R)$. If R is finite, then so is $M(R^2, R)$ and thus also $R[x,y]$ which is not possible. Hence R is infinite.

Suppose $(R,+)$ has no nonzero elements of finite order. Then it can be shown that η is injective by applying Aczél's result twice to $R[x,y] = (R[x])[y]$. $\Diamond$

In each of the sections to follow, we will consider a composition $f(x,y) \circ g(x,y) = f(\alpha_1(g(x,y)), \alpha_2(g(x,y)))$ defined by two prescribed functions α_1 and α_2. Amongst others, our functions α_i will exhaust all possibilities of the form $\alpha_i(g(x,y)) = x$ (or y) or, for $s,t \in \{x,y\}$ fixed, $\alpha_i(g(x,y)) = g(s,t)$ subject to the two functions α_1 and α_2 fulfilling condition (A). Every choice will have a "dual" which we will not consider. For example, we will consider the composition $f(x,y) \circ g(x,y) = f(g(x,y),y)$ (Sec. 2), but not its "dual" $f(x,y) \circ g(x,y) = f(x, g(x,y))$ which will have similar properties. The functions α_1 and α_2 will always be extendable fulfilling condition (EA) with respect to the canonical extensions. By the canonical extensions we mean, for example, the following. If $\alpha_i(g(x,y)) = x$ for all $g(x,y) \in R[x,y]$, then $(\overline{\alpha_i}(h))(u,v) = u$ for all $h \in M(R^2,R)$ and $u,v \in R$. For another example, if $\alpha_i(g(x,y)) = g(x,x)$ for all $g(x,y) \in R[x,y]$, then $(\overline{\alpha_i}(h))(u,v) = h(u,u)$ for all $h \in M(R^2,R)$ and $u,v \in R$. Thus we get two composition rings $(R[x,y],+,\cdot,\circ)$ and $(M(R^2,R),+,\cdot,\circ)$. For each $R[x,y]$ we will then investigate the following:

(1) The ideal structure of composition rings in general, and in particular the relation between maximal ideals of the foundation and those of the composition ring, have been discussed earlier, see for example Adler [1], Peterson and Veldsman [5] and Veldsman [7]. We will thus not say much about the ideals of $(R[x,y],+,\cdot,\circ)$, but mainly remark on two aspects, both inspired by the one indeterminate case. If C is a composition ring with foundation K, an ideal J of the ring K is called an *C-ideal* of C if $c \circ (k+j) - c \circ k \in J$ for all $c \in C, k \in K$ and $j \in J$. A composition ring is called *compatible* if every ideal of the foundation is an C-ideal. Any $(R[x],+,\cdot,\circ)$ is compatible, so it is natural to ask whether this is true for the two indeterminate case. Another property enjoyed by $R[x]$ is that any ideal of the ring $(R[x],+,\cdot)$ is a left ideal of the nearring $(R[x],+,\circ)$ – we will comment on this property for $R[x,y]$. We use $\triangleleft$ to denote ideal and if endowed with a subscript, it will denote the indicated one-sided ideal. Ideal of C means ideal of the composition ring. Often in the literature, these ideals are called full ideals.

(2) We will determine the identity or one-sided identities (with respect to the composition). If $e(x,y)$ is a one-sided identity, then we will describe the semigroup $(D_e,\circ)$ where $D_e = \{f(x,y) \in R[x,y] \mid f(x,y) \circ e(x,y) = f(x,y) = e(x,y) \circ f(x,y)\}$. This semigroup has identity $e(x,y)$ and we will determine the group of units $(U(D_e),\circ)$. To facilitate this aspect of our investigations, note the following: For any composition ring C, if e is an identity, then $D_e = \{c \in C \mid c \circ e = c = e \circ c\}$ coincides with C; if e is a left identity, then $D_e = C \circ e$ which is a subcomposition ring of C and the map $C \to C \circ e$ defined by $c \mapsto c \circ e$ is a surjective composition ring homomorphism with kernel $\{u \in C \mid u \circ C = 0\}$. If e is a right identity, then $D_e = e \circ C$ which, in general, is only a semigroup with respect to the composition.

(3) If $f(x,y) \in U(D_e)$ with inverse $g(x,y)$, a "geometric" relationship between f and g will be given by describing a transformation

T from the "graph" of f, $\operatorname{graph} f = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} | c = \overline{f}(a,b), a, b \in R \right\}$ to $\operatorname{graph} g = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} | c = \overline{g}(a,b), a, b \in R \right\}$.

We should mention that for a given choice of α_1 and α_2, there is often a symmetric version of the composition on $R[x,y]$ possible by interchanging α_1 and α_2. Also, many of the compositions we will consider can be extended to polynomials in more than two commuting indeterminates. We will refrain from discussing any of these cases.

2. $f(x, y) \circ g(x, y) = f(g(x, y), y)$

For all $f(x,y) \in R[x,y]$, let $\alpha_1(f(x,y)) = f(x,y)$ be the identity function and $\alpha_2(f(x,y)) = y$ a constant function. The resulting composition ring $(R[x,y], +, \cdot, \circ)$ with composition as above, has foundation $R[y]$, identity $e(x,y) = x$ and thus $D_e = R[x,y]$. Moreover, $R[x,y]$ is compatible and every ideal of the ring $(R[x,y], +, \cdot)$ is a left ideal of the nearring $(R[x,y], +, \circ)$. All this is hardly surprising, since $R[x,y]$ is just the composition ring in one indeterminate x over the ring $(R[y], +, \cdot)$. This means the units of $R[x,y]$ are of the form $f(x,y) = f_0 + f_1 x$ where $f_0 \in R[y]$ and $f_1 \in U(R[y])$, i.e. $f(x,y) = f_{00} + f_{01}y + f_{02}y^2 + \cdots + f_{0m}y^m + (f_{10} + f_{11}y + f_{12}y^2 + \cdots + f_{1n}y^n)x$ where $f_{ij} \in R$ with $f_{10} \in U(R)$ and $f_{11}, f_{12}, \ldots, f_{1n}$ nilpotent. In fact, as is well-known, we have a group isomorphism $(U(R[x,y], \circ) \simeq (R[y], +) +_\theta (U(R[y]), \circ)$ where $+_\theta$ denotes semi-direct sum. If $f(x,y) \in U(R[x,y])$ with inverse $g(x,y)$, then $x = g(f(x,y),y)$. So, if $c = \overline{f}(a,b)$, then $a = \overline{e}(a,b) = \overline{g}(\overline{f}(a,b),b) = \overline{g}(c,b)$. Thus a transformation $T : \operatorname{graph} f \to \operatorname{graph} g$ can be defined by

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This is just reflection in $z = x$, an invertible linear transformation (independent of f and e).

3. $f(x, y) \circ g(x, y) = f(g(x, x), g(y, y))$

For this composition we have $\alpha_1(f(x,y)) = f(x,x)$ and $\alpha_2(f(x,y)) = f(y,y)$ for all $f(x,y) \in R[x,y]$. The motivation for this composition, as well as the one in the next section, is that in f, all occurances of x should be replaced by something involving only g and x and similarly for y. The composition ring $(R[x,y],+,\cdot,\circ)$ has foundation R, no identity but right identities. A polynomial $e(x,y)$ is a right identity if and only if it is of the form $e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij}x^iy^j$ for $n \geq 1$ and with $e_1 := := e_{10} + e_{01} = 1$ and $e_k := \sum_{i+j=k} e_{ij} = 0$ for all $k = 2,3,\ldots,n$. $R[x,y]$ is compatible and an ideal I of $(R[x,y],+,\cdot)$ is a left ideal with respect to the composition if and only if $f(x,x) \in I$ and $f(y,y) \in I$ for all $f(x,y) \in I$.

Next we determine D_e for the right identity

$$e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij}x^iy^j.$$

We know that $D_e = e(x,y) \circ R[x,y]$, so a typical element of D_e is of the form

$$e(x,y) \circ g(x,y) = e(g(x,x), g(y,y)) =$$

$$= \sum_{k=1}^{n} \sum_{i+j=k} e_{ij}(g_0 + g_1x + \cdots + g_mx^m)^i(g_0 + g_1y + \cdots + g_my^m)^j$$

where $g(x,y) = \sum_{k=0}^{m} \sum_{i+j=k} g_{ij}x^iy^j$. Thus we have

Proposition 5. *Let $e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij}x^iy^j$ be a right identity in $(R[x,y],+,\circ)$. Then $D_e = \{h(x,y) = \sum_{k=0}^{mn} \sum_{i+j=k} h_{ij}x^iy^j \in R[x,y] \mid m \geq$ ≥ 0 and there are $g_0, g_1, \ldots, g_m$ in R such that for all $0 \leq s,t \leq mn$ with $s+t \leq mn$, $h_{st} = \sum_{k=1}^{n} \sum_{i+j=k} \sum_{(a_i)} \sum_{(b_j)} \frac{i!j!}{i_0!i_1!\ldots i_m!j_0!j_1!\ldots j_m!} e_{ij} g_0^{i_0+j_0} g_1^{i_1+j_1} \ldots$ $\ldots g_m^{i_m+j_m}$ where (a_i) denotes the sum over all $i_0, i_1, \ldots, i_m$ with $0 \leq$ $\leq i_p \leq i, i_0 + i_1 + \cdots + i_m = i, i_1 + 2i_2 + \cdots + mi_m = s$ and (b_j) denotes the sum over all $j_0, j_1, \ldots, j_m$ with $0 \leq j_p \leq j, j_0 + j_1 + \cdots + j_m =$ $= j, j_1 + 2j_2 + \cdots + mj_m = t\}$.*

Proposition 6. *Let $e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij}x^iy^j$ be a right identity in $(R[x,y],+,\circ)$. Then $e(x,y) \circ f(x,y) \in U(D_e)$ if and only if $f(x,y) =$*

$= \sum_{k=0}^{m} \sum_{i+j=k} f_{ij}x^i y^j$ *where* $m \geq 1$, $f_1 := f_{10} + f_{01} \in U(R)$ *and* $f_2 = f_3 = \cdots = f_m = 0$. *Moreover,* $(U(D_e), \circ) \simeq (R, +) +_\theta (U(R), \cdot)$ *where* $+_\theta$ *denotes the semi-direct sum with* $(r, u)(s, v) = (r + us, uv)$.

Proof. Let $e(x, y) \circ f(x, y) \in U(D_e)$ with inverse $e(x, y) \circ g(x, y)$, say $f(x, y) = \sum_{k=0}^{m} \sum_{i+j=k} f_{ij}x^i y^j$ and $g(x, y) = \sum_{k=0}^{p} \sum_{i+j=k} g_{ij}x^i y^j$. Since $\psi : R[x, y] \to R[x]$ defined by $\psi(h(x, y)) = h(x, x)$ is a surjective composition ring homomorphism, $f(x, x)$ is a unit in $R[x]$ with inverse $g(x, x)$. This means $x = (f_0 + f_1 x + \cdots + f_m x^m)(g_0 + g_1 x + \cdots + g_p x^p)$ from which we get $f_1 g_1 = 1$, $g_0 = \frac{-f_1}{f_0}$ and $f_2 = f_3 = \cdots = f_m = g_2 = \cdots = g_p = 0$ as required.

Conversely, suppose $f(x, y) = \sum_{k=0}^{m} \sum_{i+j=k} f_{ij}x^i y^j$ with $m \geq 1$, $f_1 := f_{10} + f_{01} \in U(R)$, say the inverse of f_1 is u, and $f_2 = f_3 = \cdots = f_m = 0$. Let $g(x, y) = g_{00} + g_{10}x + g_{01}y$ where $g_{00} = \frac{-f_0}{f_1}$, $g_{10} = u$ and $g_{01} = -f_{10}$. Then $e(x, y) \circ f(x, y) \in U(D_e)$ with inverse $e(x, y) \circ g(x, y)$.

Lastly define $\eta : U(D_e) \to R +_\theta U(R)$ by $\eta(e(x, y) \circ f(x, y)) = (f_0, f_1)$ where $f(x, y) = \sum_{k=0}^{m} \sum_{i+j=k} f_{ij}x^i y^j$ with $m \geq 1$, $f_1 \in U(R)$ and $f_2 = f_3 = \cdots = f_m = 0$. It can be verified that η is a group isomorphism which completes the proof. ◊

We conclude our discussion on this composition by describing the transformation T : graph $f \to$ graph g for $f(x, y) = e(x, y) \circ h(x, y) \in U(D_e)$ with inverse $g(x, y)$. If $c = \overline{f}(a, b)$ then, from $e(x, y) = f(x, y) \circ g(x, y) = g(f(x, x), f(y, y))$, we get

$$\begin{aligned} c = \overline{f}(a, b) &= \overline{e}(\overline{h}(a, a), \overline{h}(b, b)) = \\ &= \overline{g}(\overline{f}(\overline{h}(a, a)), \overline{h}(a, a)), \overline{f}(\overline{h}(b, b), \overline{h}(b, b))) = \\ &= \overline{g}(h_0 + h_0 h_1 + h_1^2 a, h_0 + h_0 h_1 + h_1^2 b) \end{aligned}$$

since $\overline{f}(\overline{h}(a, a)), \overline{h}(a, a)) = \overline{e}(h_0 + h_1 h(a, a), h_0 + h_1 h(a, a)) = h_0 + h_0 h_1 + h_1^2 a$ and likewise $\overline{f}(\overline{h}(b, b), \overline{h}(b, b)) = h_0 + h_0 h_1 + h_1^2 b$. Thus, we may define

$$T \begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} h_0 + h_0 h_1 + h_1^2 a \\ h_0 + h_0 h_1 + h_1^2 b \\ c \end{pmatrix} = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} h_0 + h_0 h_1 \\ h_0 + h_0 h_1 \\ 0 \end{pmatrix}$$

which is an invertible affine transformation (depending on f).

4. $f(x, y) \circ g(x, y) = f(g(x, 0) - g_{00}, g(0, y) - g_{00})$

For the next composition we take $\alpha_1(f(x,y)) = f(x,0) - f_{00}$ and $\alpha_2(f(x,y)) = f(0,y) - f_{00}$ for all $f(x,y) \in R[x,y]$. The motivation for this composition is in the spirit of the previous one – the argument being that in defining $f(x,y)\circ g(x,y)$, the x in $f(x,y)$ must be replaced by something involving g and x. Using only $g(x,0)$ (and $g(0,y)$ in the place of y) we do not get an associative composition; hence the proposed composition to overcome this problem. With respect to this composition, $(R[x,y], +, \cdot, \circ)$ is a compatible composition ring with foundation R. An ideal I of $R[x,y]$ with respect to multiplication is a left ideal with respect to composition if and only if $f(x,0)-f_{00} \in I$ and $f(0,y)-f_{00} \in I$ for all $f(x,y) \in I$. Note that $(0:R)_2 := \{f(x,y) \in R[x,y] \mid f(x,y) \circ \circ R = 0\} = R_0[x,y] \lhd R[x,y]$ and the quotient composition ring $\frac{R[x,y]}{R_0[x,y]}$ is isomorphic to the composition ring $(R, +, \cdot, *)$ where $*$ is the constant multiplication $a * b = a$ for all $a, b \in R$. $R[x,y]$ does not have an identity, but it has right identities. An element $e(x,y) \in R[x,y]$ is a right identity if and only if it is of the form $e(x,y) = e_{00} + x + y + m_e(x,y)$ where $m_e(x,y) \in R[x,y]$ with $m_e(x,0) = 0 = m_e(0,y)$.

Proposition 7. *Let $e(x,y)=e_{00}+x+y+m_e(x,y)$ be a right identity in $R[x,y]$. Then $D_e = \{f(x,y) \in R[x,y] \mid f(x,y) = e_{00} + g(x) + h(y) + + m_e(g(x), h(y))$ for $g(x) \in R_0[x]$ and $h(y) \in R_0[y]\}$ with $(D_e, \circ) \simeq \simeq (R_0[x],\circ)\oplus(R_0[y],\circ)$. Furthermore, $U(D_e)=\{f(x,y)\in R[x,y] \mid f(x,y)= = e_{00} + ux + vy + m_e(ux, vy)$ for $u, v \in U(R)\}$ and $(U(D_e), \circ) \simeq \simeq (U(R), \cdot) \oplus (U(R), \cdot)$.*

Proof. It is straightforward to determine D_e; we only determine the units of D_e. Let $f(x,y) = e_{00} + h(x) + k(y) + m_e(h(x), k(y))$ be a unit of D_e with inverse $g(x,y) = e_{00} + p(x) + q(y) + m_e(p(x), q(y))$. Suppose that $h(x) = h_1x + h_2x^2 + \cdots + h_nx^n$ and $p(x) = p_1x + p_2x^2 + \cdots + p_mx^m$. From $e_{00}+x+y+m_e(x,y) = e(x,y) = f(x,y)\circ g(x,y) = e_{00} + h(p(x)) + + k(q(y)) + m_e(h(p(x)), k(q(y)))$ we get, by comparing the coefficients of the powers of x, $1 = h_1p_1$, $0 = h_1p_2 = h_1p_3 = \cdots = h_1p_m$. Thus $h_1, p_1 \in U(R)$ and $p_2 = p_3 = \cdots = p_m = 0$. But also $h_ip_1^ix^i = 0$ for all $i = 2, 3, \ldots, n$; hence $h_2 = h_3 = \cdots = h_n = 0$. A similar argument will show that $k(y) = k_1y$ for some $k_1 \in U(R)$. Thus $f(x,y)$ is of the required form. $\Diamond$

Note that here we also have D_e and $U(D_e)$ independent of which particular identity $e(x,y)$ we choose in the sense that for any two right identities e_1 and e_2 we have $D_{e_1} \simeq D_{e_2}$ and $U(D_{e_1}) \simeq U(D_{e_2})$. The relationship between $f(x,y) = e_{00} + ux + vy + m_e(ux, vy) \in U(D_e)$ and its inverse $g(x,y) = e_{00} + u^{-1}x + v^{-1}y + m_e(u^{-1}x, v^{-1}y)$ can be described by the transformation $T : \operatorname{graph} f \to \operatorname{graph} g$ defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} u^2 a \\ v^2 b \\ c \end{pmatrix} = \begin{bmatrix} u^2 & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Indeed, if $c = \overline{f}(a,b)$, then $\overline{g}(u^2 a, v^2 b) = c$. Note that T is an invertible linear transformation (which depends on f).

5. $f(x, y) \circ g(x, y) = f(pg(x, y), qg(x, y))$

Let p and q be fixed elements of R and define α_1 and α_2 by $\alpha_1(f(x,y)) = pf(x,y)$ and $\alpha_2(f(x,y)) = qf(x,y)$ for all $f(x,y) \in$ $\in R[x,y]$. With $p = q = 1$, this is just the composition proposed by Clay [3] and which has been investigated by Gutierrez and de Velasco [4] (actually, they considered the more general case with 2 or more indeterminates).

For α_1 and α_2 as defined above, the resulting composition ring on $R[x,y]$ has foundation R, it is compatible and any ideal of $R[x,y]$ with respect to multiplication is a left ideal with respect to the composition. We do not have an identity, but $e(x,y)$ is a left identity if and only if it is of the form $e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij} x^i y^j$, $n \geq 1$, with $e_{10}p + e_{01}q = 1$ and $\sum_{i+j=k} e_{ij} p^i q^j = 0$ for all $k = 2, 3, \ldots, n$. For a given ring R and $p, q \in R$, there may not be any left identities (e.g. if $R = \mathbb{Z}$ with p and q not relatively prime). Interesting to note is that if $p = q = 1$, then these left identities are exactly the right identities with respect to the composition $f(x,y) \circ g(x,y) = f(g(x,x), g(y,y))$ which we considered in Sec. 3.

Proposition 8. *Let* $e(x,y) = \sum_{k=1}^{n} \sum_{i+j=k} e_{ij} x^i y^j$ *be a left identity. Then* $D_e = R[x,y] \circ e(x,y) = \{r_0 + r_1 e(x,y) + r_2 e(x,y)^2 + \cdots + r_m e(x,y)^m \mid$ $\mid m \geq 0, r_1, r_2, \ldots, r_m \in R\}$ *with* $(D_e, +, \cdot, \circ) \simeq (R[e], +, \cdot, \circ)$ *where* $e = e(x,y)$. *Moreover,* $U(D_e) = \{r_0 + r_1 e(x,y) \mid r_0 \in R, r_1 \in U(R)\}$

with $(U(D_e),\circ) \simeq (R,+) +_\theta (U(R),\cdot)$.

Proof. For any $f(x,y)\circ e(x,y) \in R[x,y]\circ e(x,y)$ we then have

$$\begin{aligned} f(x,y)\circ e(x,y) = f_{00} + (f_{10}p + f_{01}q)e(x,y) + \\ + (f_{20}p^2 + f_{11}pq + f_{02}q^2)e(x,y)^2 + \\ \cdots + (f_{m0}p^m + f_{m-1,1}p^{m-1}q + \cdots + f_{0m}q^m)e(x,y)^m \end{aligned}$$

which is in

$$\{r_0 + r_1e(x,y) + r_2e(x,y)^2 + \cdots + r_me(x,y)^m \mid m \geq 0, r_1, r_2, \ldots, r_m \in R\}.$$

Conversely, we know $e_{10}p + e_{01}q = 1$. This means, for any $r \in R$ there are $a, b \in R$ with $r = ap + bq$. More generally, for any $k \geq 1$ and $r \in R$, there are $a_0, a_1, \ldots, a_k \in R$ with $r = a_0p^k + a_1p^{k-1}q + + \cdots + a_{k-1}pq^{k-1} + a_kq^k$ since $(e_{10}p + e_{01}q)^k = 1$. This means any $r_0 + r_1e(x,y) + r_2e(x,y)^2 + \cdots + r_me(x,y)^m$ can be written in the form $f(x,y)\circ e(x,y)$.

The mapping $r_0 + r_1e(x,y) + r_2e(x,y)^2 + \cdots + r_me(x,y)^m \mapsto \mapsto r_0 + r_1e + r_2e^2 + \cdots + r_me^m$ from D_e to $R[e]$ is clearly a surjective composition ring homomorphism, we only remark on the injectivity.

If $r_0 + r_1e(x,y) + r_2e(x,y)^2 + \cdots + r_me(x,y)^m = 0$, then $r_0 = r_1 = = \cdots = r_m = 0$. Indeed, $r_0 = 0$ is clear and from $r_1(e_{10}x + e_{01}y + \cdots + + e_{0n}y^n) + r_2(e_{10}x + e_{01}y + \cdots + e_{0n}y^n)^2 + \cdots + r_m(e_{10}x + e_{01}y + \cdots + +e_{0n}y^n)^m = 0$, we get $r_1e_{10} = 0 = r_1e_{01}$. Thus $r_1 = r_1(pe_{10} + qe_{01}) = 0$. Also, $r_2e_{10}^2x^2 = r_2e_{10}e_{01}xy = r_2e_{01}^2y^2 = 0$ implies $r_2e_{10}^2 = r_2e_{10}e_{01} = = r_2e_{01}^2 = 0$. Then $r_2 = r_2(pe_{10} + qe_{01})^2 = 0$. Continue in this way to get $r_0 = r_1 = \cdots = r_m = 0$. ◊

Let $f(x,y) = f_0 + f_1e(x,y) \in U(D_e), f_1 \in U(R)$, with inverse $g(x,y) = g_0 + g_1e(x,y)$ where $f_1g_1 = 1$ and $g_0 = -g_1f_0$. If $c = \overline{f}(a,b)$, then $\overline{e}(a,b) = f_1^{-1}(c - f_0)$. Thus $\overline{g}(a,b) = -f_0f_1^{-1} + f_1^{-2}(c - f_0)$. A transformation T : graph $f \to$ graph g can be defined by

$$\begin{aligned} T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} a \\ b \\ f_1^{-2}(c - f_0(1 + f_1)) \end{pmatrix} = \\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f_1^{-2} \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -f_1^{-2}f_0(1 + f_1) \end{pmatrix}. \end{aligned}$$

This is an invertible affine transformation (which depends of f).

6. $f(x, y) \circ g(x, y) = f(g_{10}x, g_{01}y)$

One could, of course, take the composition $f(x,y) \circ g(x,y)$ to be trivial in the sense of defining $f(x,y) \circ g(x,y) = 0$ for all $f(x,y), g(x,y) \in R[x,y]$ or $f(x,y) \circ g(x,y) = f(x,y)$ for all $f(x,y), g(x,y) \in R[x,y]$. In both cases we do get a composition ring on $R[x,y]$, but this is not very interesting. In the latter case, with the composition constant, every element of $R[x,y]$ is a right identity. The more right identities e there are, the smaller D_e should be, as it is in this case since $D_e = \{e\}$. To further illustrate this point, we next consider an "almost constant" composition.

For every $f(x,y) \in R[x,y]$, let $\alpha_1(f(x,y)) = f_{10}x$ and $\alpha_2(f(x,y)) = f_{01}y$. This gives a compatible composition ring on $R[x,y]$ with foundation R. An ideal I of $(R[x,y], +, \cdot)$ is a left ideal of $(R[x,y], +, \circ)$ if and only if $i_{10}x \in I$ and $i_{01}y \in I$ for all $i(x,y) \in I$. We have $e(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} e_{ij}x^iy^j$ a right identity of $R[x,y]$ if and only if $n \geq 1$ and $e_{10} = e_{01} = 1$. We will write $e(x,y)$ as $e(x,y) = e_{00} + x + y + m_e(x,y)$ where $m_e(x,y) = \sum_{k=2}^{n} \sum_{i+j=k} e_{ij}x^iy^j$. This then gives:

Proposition 9. *For a right identity* $e(x,y) = e_{00} + x + y + m_e(x,y)$, $D_e = \{f(x,y) \in R[x,y] \mid f(x,y) = e_{00} + ax + by + m_e(ax, by), a, b \in R\}$ *with* $(D_e, \circ) \simeq (R, \cdot) \oplus (R, \cdot)$. *Moreover,* $(U(D_e), \circ) \simeq (U(R), \cdot) \oplus (U(R), \cdot)$ *where* $U(D_e) = \{f(x,y) \in R[x,y] \mid f(x,y) = e_{00} + ux + vy + m_e(ux, vy), u, v \in U(R)\}$.

Let $f(x,y) = e_{00} + ux + vy + m_e(ux, vy) \in U(D_e)$ with inverse $g(x,y) = e_{00} + u^{-1}x + v^{-1}y + m_e(u^{-1}x, v^{-1}y)$. Then we can define $T : \text{graph}\, f \to \text{graph}\, g$ by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} u^2a \\ v^2b \\ c \end{pmatrix} = \begin{bmatrix} u^2 & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This is an invertible linear transformation (which depends on f).

All the above should look familiar: In Sec. 4 we considered the composition $f(x,y) \circ g(x,y) = f(g(x,0) - g_{00}, g(0,y) - g_{00})$. The right identities with respect to this composition are very similar to the ones in this section (in fact, the right identities of Sec. 4 will be right identities with respect to the composition of this section) and the units for both compositions coincide as does the transformation T.

7. $f(x, y) \circ g(x, y) = f(g(x, x), g(x, x))$

Taking $\alpha_1(f(x,y)) = \alpha_2(f(x,y)) = f(x,x)$ for all $f(x,y) \in R[x,y]$ gives a composition ring on $R[x,y]$. However, it does not seem to be very interesting. Although it is compatible with foundation R, there are no one-sided identities and an ideal I of $R[x,y]$ with respect to the multiplication will be a left ideal with respect to the composition if and only if $i(x,x) \in I$ for all $i(x,y) \in I$.

8. $f(x, y) \circ g(x, y) = f(g(x, y), g(y, x))$

The structure of the last two compositions we consider is much more interesting. Firstly we take $\alpha_1(f(x,y)) = f(x,y)$ and $\alpha_2(f(x,y)) = f(y,x)$ for all $f(x,y) \in R[x,y]$. Then we get a compatible composition ring on $R[x,y]$ with foundation R and identity $e(x,y) = x$. Recall that $e(x,y) = x$ is also the identity for the composition $f(x,y) \circ g(x,y) = f(g(x,y),y)$ considered in Sec. 2. But these two composition rings cannot be isomorphic since the foundation of the composition ring in Sec. 2 is $R[y]$.

We will discuss the ideal structure of this composition ring in more detail.

Proposition 10. *Let I be an ideal of $(R[x,y],+,\cdot)$. Then:*

(i) $I \triangleleft_l (R[x,y],+,\circ)$ *if and only if* $i(y,x) \in I$ *for all* $i(x,y) \in I$.

(ii) $I \triangleleft (R[x,y],+,\cdot,\circ)$ *if and only if* $I \triangleleft_r (R[x,y],+,\circ)$.

Proof. (i) Suppose $I \triangleleft_l (R[x,y],+,\circ)$ and let $i(x,y) \in I$. Then $f(x,y) \circ (i(x,y)+0) - f(x,y) \circ 0 \in I$ for any $f(x,y) \in R[x,y]$. In particular for $f(x,y) = y$, we get $i(y,x) = f(x,y) \circ (i(x,y)+0) - f(x,y) \circ 0 \in I$. Conversely, suppose the condition holds and let $f(x,y), g(x,y) \in R[x,y], i(x,y) \in I$. Then

$$\begin{aligned} & f(x,y) \circ (i(x,y)+g(x,y)) - f(x,y) \circ g(x,y) = \\ & = f_{10}i(x,y) + f_{01}i(y,x) + f_{20}i(x,y)^2 + 2f_{20}i(y,x)g(x,y) + \\ & \quad + f_{11}i(x,y)i(y,x) + f_{11}i(x,y)g(y,x) + f_{11}g(x,y)i(y,x) + \dots \\ & \quad \dots + f_{1,n-1}i(y,x)g(y,x)^{n-1} \in I. \end{aligned}$$

(ii) Suppose $I \triangleleft_r (R[x,y],+,\circ)$. For any $i(x,y) \in I$ and $f(x,y) = y$, $i(y,x) = i(f(x,y),f(y,x)) = i(x,y) \circ f(x,y) \in I$. Using (i) we may conclude that $I \triangleleft_l (R[x,y],+,\cdot,\circ)$. $\Diamond$

Ideals of $(R[x,y],+,\cdot)$ which are also left ideals of $(R[x,y],+,\circ)$ need not be ideals of the composition ring. Define $\gamma : R[x,y] \to R[x]$ by $\gamma(f(x,y)) = f(x,x)$. Then γ is a surjective composition ring homomorphism with $\ker\gamma = \{f(x,y) \in R[x,y] \mid f(x,x) = 0\} \lhd R[x,y]$.

If $I \lhd R[x,y]$ with $I \cap \ker\gamma = 0$, then $i(x,y) = i(y,x)$ for all $i(x,y) \in I$. Indeed, if $i(x,y) \in I$, then $i(x,y) - i(y,x) \in I \cap \ker\gamma = 0$. Let $h(x,y) \in R[x,y]$ with $h(x,y) = h(y,x)$. Then $I := R[x,y]h(x,y)$ is an ideal of $(R[x,y],+,\cdot)$ and a left ideal of $(R[x,y],+,\circ)$, but it need not be an ideal of $R[x,y]$.

Because $R[x,y]$ is compatible, any ideal J of R is an C-ideal of R and so $(J:R)_2 := \{f(x,y) \in R[x,y] \mid f(x,y) \circ r \in J \text{ for all } r \in R\}$ is an ideal of $R[x,y]$. In fact, $(J:R)_2$ is the kernel of the surjective composition ring homomorphism $\eta : R[x,y] \to M(R/J, R/J)$ defined by $\eta(f(x,y)) = \widehat{f}$ and $\widehat{f}(r+J) = \overline{f}(r,r) + J$. Then $\ker\eta \subseteq (0:R)_2 \subseteq (J:R)_2$ and the first inclusion is in general sharp. For example, if $R = \mathbb{Z}_4$ and $f(x,y) = x + y + x^2 + y^2$, then $f(x,x) \neq 0$, but $\overline{f}(r,r) = 0$ for all $r \in R$.

Proposition 11. *Let $L \lhd_l (R[x,y],+,\circ)$ where R is a ring with $\frac{1}{2} \in R$ (i.e. $2 := 1+1$ is invertible). If $L \cap U(R) \neq \emptyset$, then $L = R[x,y]$.*

Proof. Note firstly that $L \cap U(R) \neq \emptyset$ implies $R \subseteq L$. Indeed, for $u \in L \cap U(R)$ and any $r \in R$, $r = (ru^{-1})u \in RL \subseteq L$; this latter inclusion follows from $rl(x,y) = rx \circ l(x,y) \in R_0[x,y] \circ L \subseteq L$. Thus $a := \frac{1}{2} \in R \subseteq L$ and also $a^2 \in L$. Then from $x + a^2 = x^2 \circ (x+a) - x^2 \circ x \in L$ we get $x \in L$. This means $R_0[x,y] \subseteq L$ since for any $f(x,y) \in R_0[x,y]$, $f(x,y) = f(x,y) \circ x \in R_0[x,y] \circ L \subseteq L$. Thus $R[x,y] = R + R_0[x,y] \subseteq L + L = L$. $\Diamond$

Proposition 12. *Let R be a field with more than 2 elements. Let L be a left ideal of $(R[x,y],+,\circ)$ with $L \cap R \neq 0$. Then $L = R[x,y]$.*

Proof. If $\operatorname{char} R \neq 2$, then $\frac{1}{2} \in R$ and the result follows from the previous proposition. Suppose $\operatorname{char} R = 2$ and choose $u \in R \setminus \{0,1\}$. Then both u and $w := u - 1$ are invertible. As in the proof of the previous proposition, we get $R \subseteq L$ and $RL \subseteq L$. From $x^2 + x + 1 = x^3 \circ (x+1) - x^3 \circ x \in L$ we get $x^2 + x \in L$. Let $p := u^{-1}x^3$. Since $u \in R \subseteq L$, we have $x^2 + ux + u^2 = p \circ (x+u) - p \circ x \in L$. But then $x^2 + x + ux + u^2 - x \in L$ and so $wx = ux - x \in L$. Hence $x = w^{-1}(wx) \in RL \subseteq L$. Thus $R_0[x,y] \subseteq L$ and $L = R[x,y]$ follows. $\Diamond$

Proposition 13. *Let R be a ring with $\frac{1}{2}$ and let $I \lhd_l (R[x,y],+,\circ)$. Then $I \lhd (R[x,y],+,\cdot)$.*

Proof. As in the proof of Prop. 11, we have $RI \subseteq I$. Let $i(x,y) \in I$ and $f(x,y) \in R[x,y]$. Then $f(x,y)i(x,y) = \frac{1}{2}[(x^2 \circ (f(x,y)+i(x,y)) - - x^2 \circ f(x,y)) - (x^2 \circ i(x,y))] \in I$. ◊

Next we investigate the units. Since $R[x,y] \to R[x]$ defined by $f(x,y) \mapsto f(x,x)$ is a surjective composition ring homomorphism, we know that if $f(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} f_{ij}x^iy^j$ is a unit in $R[x,y]$, then $f(x,x) = \sum_{k=0}^{n} f_kx^k$ is a unit in $R[x]$ (with respect to composition). This means $f_1 = f_{10} + f_{01} \in U(R)$ and $f_2 = f_3 = \cdots = f_n = 0$. Although these conditions are necessary for $f(x,y)$ to be a unit, they are in general not sufficient. Note that in the next result, and also in the last part of this section, we deviate from our canonical notation for the coefficients in $f(x,y)$.

Proposition 14. *Any $f(x,y) = f_0 + uy + f_1(x-y) + f_2(x-y)^2 + + f_4(x-y)^4 + \cdots + f_{2n}(x-y)^{2n}$ with $n \geq 0, u$ and $2f_1 - u$ in $U(R)$ and $f_2, f_4, \ldots, f_{2n} \in R$ is a unit in $R[x,y]$.*

Proof. Let $f(x,y) = f_0 + uy + f_1(x-y) + f_2(x-y)^2 + f_4(x-y)^4 + + \cdots + f_{2n}(x-y)^{2n}$ be of the specified form. Put $w = u^{-1}, g_0 = = -wf_0$, $g_1 = wf_1(2f_1 - u)^{-1}$ and $g_{2i} = -wf_{2i}(2f_1 - u)^{-1}$ for $i = = 1, 2, \ldots, n$. We will show that

$$g(x,y) = g_0 + wy + g_1(x-y) + g_2(x-y)^2 + g_4(x-y)^4 + \cdots + g_{2n}(x-y)^{2n}$$

is the inverse of $f(x,y)$.

Note firstly that $(2f_1 - u)(2g_1 - w) = 1$ and if we write g for $g(x,y)$ and $\widehat{g}$ for $g(y,x)$, then $g - \widehat{g} = (2g_1 - w)(x-y)$. Hence

$$\begin{aligned} f(x,y) \circ g(x,y) &= f(g(x,y), g(y,x)) = \\ &= f_0 + u\widehat{g} + f_1(g - \widehat{g}) + \sum_{i=1}^{n} f_{2i}(g - \widehat{g})^{2i} = \\ &= f_0 + u(g_0 + wx + g_1(y-x) + \sum_{i=1}^{n} g_{2i}(x-y)^{2i}) + f_1(2g_1 - w)(x-y) + \\ &\qquad + \sum_{i=1}^{n} f_{2i}(2g_1 - w)^{2i}(x-y)^{2i} = \\ &= (f_0 + ug_0) + x + (x-y)(-ug_1 + f_1(2g_1 - w)) + \\ &\qquad + \sum_{i=1}^{n} (x-y)^{2i}(g_{2i}u + f_{2i}(2g_1 - w)^{2i}) = x \end{aligned}$$

since $f_0 + ug_0 = 0$, $-ug_1 + f_1(2g_1 - w) = 0$ and $g_{2i}u + f_{2i}(2g_1 - w)^{2i} = 0$.

Likewise it can be shown that $g(x,y) \circ f(x,y) = x$. ◊

Whether the units mentioned in the above result are the only possible units is still not clear. We conclude with a description of a transformation T : graph $f \to$ graph g for the unit $f(x,y) = f_0 + uy + f_1(x-y) + \sum_{i=1}^{n} f_{2i}(x-y)^{2i}$ with inverse $g(x,y)$.

Suppose $c = \overline{f}(a,b)$. Then $c = f_0 + ub + f_1(a-b) + \sum_{i=1}^{n} f_{2i}(a-b)^{2i}$ and if $\overline{c} = \overline{f}(b,a)$, then $\overline{c} = f_0 + ua + f_1(b-a) + \sum_{i=1}^{n} f_{2i}(b-a)^{2i} = c + (b-a)(2f_1 - u) = c + (b-a)t$ where $t = (2f_1 - u)$.

From $x = e(x,y) = g(x,y) \circ f(x,y) = g(f(x,y), g(y,x))$ we then have $a = \overline{g}(\overline{f}(a,b), \overline{f}(b,a)) = \overline{g}(c, \overline{c})$. We may thus define T by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} c \\ \overline{c} \\ a \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -t & t & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

which is an invertible linear transformation (which depends on f).

9. $f(x,y) \circ g(x, y) = f(g(x, y), g(y,y))$

For our last composition, we take $\alpha_1(f(x,y)) = f(x,y)$ and $\alpha_2(f(x,y)) = f(y,y)$ for all $f(x,y) \in R[x,y]$. This gives a compatible composition ring on $R[x,y]$ with foundation R and identity $e(x,y) = x$. As for the previous composition, it can be shown that

Proposition 15. *Let I be an ideal of $(R[x,y],+,\cdot)$. Then:*

(i) *$I \lhd_l (R[x,y],+,\circ)$ if and only if $i(y,y) \in I$ for all $i(x,y) \in I$.*

(ii) *$I \lhd (R[x,y],+,\cdot,\circ)$ if and only if $I \lhd_r (R[x,y],+,\circ)$.*

Next we determine the units of $R[x,y]$ and we do so for R an integral domain.

Proposition 16. *Let R be an integral domain. A unit of $R[x,y]$ is of the form $f(x,y) = f_0 + f_{10}x + f_{01}y$ with f_{10} and $f_1 = f_{10} + f_{01}$ in $U(R)$. The inverse of $f(x,y)$ is $g(x,y) = g_0 + g_{10}x + g_{01}y$ where $g_1 = g_{10} + g_{01}$ is the inverse of f_1, $f_{10}g_{10} = 1, g_0 = \frac{-f_0}{f_1}$ and $g_{01} = \frac{-f_{01}}{f_{10}f_1}$.*

Proof. Let $f(x,y) = \sum_{k=0}^{m} \sum_{i+j=k} f_{ij}x^i y^j$ be a unit in $R[x,y]$ with inverse $g(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} g_{ij}x^i y^j$. Suppose $m \geq 2$ and $n \geq 2$. Since $R[x,y] \to R[x]$ defined by $h(x,y) \mapsto h(x,x)$ is a surjective composition ring homomorphism, we know that $f(x,x) = \sum_{k=0}^{m} f_k x^k$ is a unit in $R[x]$

with inverse $g(x,x) = \sum_{k=0}^{n} g_k x^k$. This means $f_1 = f_{10} + f_{01} \in U(R)$ with inverse $g_1 = g_{10} + g_{01}$ and $f_2 = f_3 = \cdots = f_m = g_2 = g_3 = \cdots = g_n = 0$. Let us write g for $g(x,y)$ and $\widehat{g}$ for $g(y,y)$. Then $x = e(x,y) = f(g,\widehat{g}) = \sum_{k=0}^{m} \sum_{i+j=k} f_{ij} g^i \widehat{g}^j$. We will refer to this equation throughout the remainder of the proof, in particular to compare terms to ultimately get $f_{ij} = 0$ for all $i+j \geq 2$.

Suppose $f_{m0} \neq 0$. The coefficient of x^{mn} on the right-hand side of the above equation is $f_{m0} g_{n0}^m$ and on the left-hand side it is 0, hence $g_{n0} = 0$. Suppose $g_{n0} = g_{n-1,1} = \cdots = g_{n-(t-1),t-1} = 0$ for $1 \leq t \leq n-1$. We show $g_{n-t,t} = 0$. Consider the term containing $x^{m(n-t)} y^{mt}$. This term has degree mn and will appear with f_{m0}. Thus $0 = f_{m0} \left(\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_m j_m} \right) x^{m(n-t)} y^{mt}$ where the sum is over all indices $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m \in \{0,1,2,\ldots,n\}$ with $i_1 + i_2 + \cdots + i_m = mn - mt, j_1 + j_2 + \cdots + j_m = mt$ and $i_p + j_p \leq n$ for all $p = 1,2,\ldots,m$. Every i_p is of the form $i_p = n - r_p$ for some $0 \leq r_p \leq n$ where $r_1 + r_2 + \cdots + r_m = mt = j_1 + j_2 + \cdots + j_m$. Then $n \geq i_p + j_p = n - r_p + j_p$ for all p. This means $r_p \geq j_p$ for all p. If $r_{p_0} > j_{p_0}$ for some p_0, then $mt = j_1 + j_2 + \cdots + j_m < j_1 + \cdots + r_{p_0} + \cdots + j_m \leq r_1 + r_2 + \cdots + r_m = mt$; a contradiction. Therefore $j_p = r_p$ and $i_p = n - j_p$ for all p. If $j_p \leq t-1$ for some p, then $g_{n-j_p, j_p} = 0$ by assumption and thus $g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_p j_p} \cdots g_{i_m j_m} = 0$. So, for possible nonzero terms in $\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_m j_m}$, suppose $j_p \geq t$ for all p. But from $j_1 + j_2 + \cdots + j_m = mt$ we then have $j_p = t$ *for* all p. Hence $0 = f_{m0} \left(\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_m j_m} \right) x^{m(n-t)} y^{mt} = f_{m0}(0 + 0 + \cdots + g_{n-t,t} g_{n-t,t} \cdots g_{n-t,t} + \cdots + 0) x^{m(n-t)} y^{mt}$ and so $g_{n-t,t} = 0$. Thus $g_{n0} = g_{n-1,1} = \cdots = g_{01} = 0$.

Next we look at terms containing $x^{m((n-1)-t)} y^{mt}$, i.e. terms of degree $m(n-1)$. Using similar arguments as above, we get $g_{n-1,0} = g_{(n-1)-1,1} = \cdots = g_{0,n-1} = 0$. We may repeat this process to get $g_{n-k,0} = g_{(n-k)-1,1} = \cdots = g_{0,n-k} = 0$ for $k = 0,1,2,\ldots$ as long as $n - k \geq 2$. So, suppose $g_{(n-k)-t,t} = 0$ for all $t = 0,1,2,\ldots,n-k; k = 0,1,2,\ldots,n-2$. Then the equation above becomes $x = f_0 + f_{10}(g_0 + g_{10}x + g_{01}y) + f_{01}(g_0 + g_1 y) + f_{20}(g_0 + g_{10}x + g_{01}y)^2 + f_{11}(g_0 + g_{10}x + g_{01}y)(g_0 + g_1 y) + f_{02}(g_0 + g_1 y)^2 + \cdots + f_{m0}(g_0 + g_{10}x + g_{01}y)^m + f_{m-1,1}(g_0 + g_{10}x + g_{01}y)^{m-1}(g_0 + g_1 y) + \cdots + f_{0m}(g_0 + g_1 y)^m$.

Comparing the coefficient of x^m, we see that $f_{m0} g_{10}^m = 0$ which gives the contradiction $g_{10} = 0$. Thus $f_{m0} = 0$.

Suppose now $f_{m-1,1} \neq 0$. Then

$$0 = f_{m-1,1}\Big(\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_{m-1} j_{m-1}}\Big) x^{n(m-1)} g_1 y$$

where the sum is over all indices $i_1, i_2, \ldots, i_{m-1}, j_1, j_2, \ldots, j_{m-1}$ with $i_1 + i_2 + \cdots + i_{m-1} = n(m-1), j_1 + j_2 + \cdots + j_{m-1} = 0$ and $i_p + j_p \leq n$ for all $p = 1, 2, \ldots, m-1$. This means $i_1 = i_2 = \cdots = i_{m-1} = n$ and so $0 = f_{m-1,1} g_{n0}^{m-1} x^{n(m-1} g_1 y$. Since $g_1 \neq 0$ we get $g_{n0} = 0$. Suppose $g_{n-j,j} = 0$ for all $j = 0, 1, 2, \ldots, t-1; 0 \leq t < n$. Then $f_{m-1,1}(\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_{m-1} j_{m-1}} x^{(m-1)(n-t)} y^{(m-1)t}) g_1 y = 0$ where $i_1 + i_2 + \cdots + i_{m-1} = (m-1)(n-t), j_1 + j_2 + \cdots + j_{m-1} = (m-1)t$, $i_p + j_p \leq n$. As in the previous case (for $f_{m0} \neq 0$) we get $0 = f_{m-1,1} g_1 g_{n-t,t} x^{(m-1)(n-t)} y^{(m-1)t}) y$ and thus $g_{n-t,t} = 0$. Thus $g_{n0} = g_{n-1,1} = \cdots = g_{01} = 0$. Next consider the terms of degree $(m-1)(n-1)+1$. They will appear with $f_{m-1,1}$ and we get $0 = f_{m-1,1}(\sum g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_{m-1} j_{m-1}}) x^{(m-1)(n-1)-t} y^{(m-1)t} g_1 y$ where $i_1 + i_2 + \cdots + i_{m-1} = (m-1)(n-1) - t, j_1 + j_2 + \cdots + j_{m-1} = (m-1)t$, $i_p + j_p \leq n$. As before, we will get $g_{(n-1)-t,t} = 0, 0 \leq t \leq n-1$, and we may continue until we get the contradiction $g_{10} = 0$. Thus $f_{m-1,1} = 0$.

Using similar arguments, we successively get $f_{m-1,0} = 0, f_{m-2,2} = 0, \ldots$ till eventually we have $f(x,y) = f_0 + f_{10}x + f_{01}y$ (remember, $f_2 = f_3 = \cdots = f_m = 0$).

Thus $x = f_0 + f_{10} g + f_{01} \widehat{g}$. Since $f_{10} \neq 0$ (we need $f_{10} \neq 0$ to get an x term on the right-hand side), we get $g_{ij} = 0$ for all $i + j \geq 2$ which concludes the proof. ◇

This composition ring is not isomorphic to the one considered in the previous section (compare units). Finally we describe the transformation T for the unit $f(x,y) = f_0 + f_{10}x + f_{01}y$ with inverse $g(x,y) = g_0 + g_{10}x + g_{01}y$. Let $c = \overline{f}(a,b)$. Then $a = \overline{g}(\overline{f}(a,b), \overline{f}(b,b)) = \overline{g}(c, f_0 + f_1 b)$. But $c = f_0 + f_{10}a + f_{01}b$ implies $c + f_{10}b - f_{10}a = f_0 + f_1 b$. We may thus define T by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} c \\ c - f_{10}a + f_{10}b \\ a \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -f_{10} & f_{10} & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This is an invertible linear transformation (which depends on f).

In conclusion, we should mention that there are still many matters to explore for both the nearrings $(R[x,y], +, \circ)$ and $(M(R^2, R), +, \circ)$ with respect to the various compositions introduced above. For example, one would like to know their ideal structure; in particular whether they are simple when R is a field (finite or infinite). One would also like to know their radicals.

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