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ASYMPTOTIC ANALYSIS OF IMPLICIT TOLERANCE PROBLEMS

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Abstract: This paper investigates a number of relations between geometric objects in Euclidean $\mathbb{R}^3$ from the viewpoint of tolerance zones and error propagation. Our investigations are based on an certain inequality concerning the linearization error useful for quadratic constraint problems. By collecting numerical data and looking at limit cases we investigate the influence of the choice of coordinate system on the tolerance analysis of a collection of quadratic constraint equations, which represent geometric problems in Euclidean space.

1. Introduction

Geometric constraint solving means the problems which arise when the location of geometric objects is described via relations between them. Issues important in applications of this concept are solvability of constraint problems and their sensitivity to errors [2]. Many methods have been proposed for geometric constraint solving: based on dependency graphs, rule-based and numerical ones, and methods based on symbolic computing. For the literature on these topics in the context of Computer-Aided design, see [8]. This paper is concerned with the propagation of errors through implicit constraints, based on the concept of tolerance zone [3, 4, 6, 7]. The present paper is a sequel of [8], which describes a general analysis of the propagation of tolerance zones through implicit constraints, with a focus on geometric constructions. A more detailed version comprises part of the thesis [9].

We assume that a certain number of geometric objects is given imprecisely – each of them is known to be contained in a certain tolerance zone. Other geometric objects are located via constraints, and we want to give tolerance zones for them. This is done by linearizing the system of constraints and estimating the linearization error. For each configuration, this works only up to a certain maximum size of tolerance zones, dependent on the particular instance of the constraint problem we wish to analyze, on the number of objects and constraints involved, and on the behaviour of the constraints' derivatives. Estimating the linearization error in the way presented here is most efficient if the constraints are quadratic polynomials. Conveniently, it is hard to think of geometric relations which are not expressible via quadratic polynomials. A short discussion of the relation of this work and tolerance zones in general to interval arithmetic can be also found in the introductions to [7] and [8].

2. Preliminaries

We consider two kinds of entities: the fixed variables $x = (x_1, \dots, x_n)$, and the moving variables $y = (y_1, \dots, y_m)$ with $x_i, y_j \in \mathbb{R}$. The constraints which are assume to hold are collected in a C^2 function F as follows:

$$F : U \times V \to W : F(x,y) = (F_1(x,y), \dots, F_m(x,y)) \quad (U = \mathbb{R}^n, V = W = \mathbb{R}^m), \tag{1}$$

where each component $F_i(x,y)$ represents a constraint. Solving the constraint problem means finding y for given x such that $F(x,y) = 0$.

We shortly discuss solvability and uniqueness of a solution: Suppose that $F(u,v) = 0$. A local solution of the constraint problem which extends the solution (u,v) is a function $G : U \to V$, defined in a connected neighbourhood of u such that $F(x, G(x)) = 0$ for all x where G is defined. It follows from the inverse function theorem that such a local solution exists if $F_{,y}(u,v)$ is nonsingular. If we are interested in only one y_j, we write $y_j = G_j(x)$.

2.1. Linear and bilinear mappings: notation

For the convenience of the reader we repeat some facts concerning linear and bilinear operators, their norms, and their relation to the Taylor expansion in Sec. 2.1–Sec. 2.4.

We use the symbols U, V, W for linear spaces. $L(U,W)$ and $B(U,V,W)$ denote the spaces of linear mappings from U to W and and bilinear mappings from $U \times V$ to W, respectively. We employ the notation "$\alpha \cdot u$" and "$\beta[u,v]$" to indicate that we apply α to u and β to the pair (u,v). "$\alpha(u)$" is a linear mapping which depends on u. For each $\beta \in B(U,V,W)$ there are associated mappings $\beta^{\phi} \in L(U, L(V,W))$, $\beta^{\psi} \in \in L(V, L(U,W))$, with $\beta[u,v] = \beta^{\phi}(u) \cdot v = \beta^{\psi}(v) \cdot u$. Subscripts indicate coefficients of vectors with respect to previously defined bases: $\alpha \in \in L(U,W)$ and $\beta \in B(U,V,W)$ have the coordinate representations $[\alpha \cdot u]_r = \sum_i \alpha_{ri} u_i$ and $\beta[u,v]_r = \sum_{i,j} \beta_{rij} u_i v_j$, respectively. The *coordinate matrix* of α contains the coefficients α_{ri}. It is elementary that the coordinate matrices of the linear mappings $\beta^{\phi}(u)$ and $\beta^{\psi}(v)$ consist of $[\beta^{\phi}(u)]_{rj} = \sum_i u_i \beta_{rij}$ and $[\beta^{\psi}(v)]_{ri} = \sum_j v_j \beta_{rij}$, respectively.

2.2. Taylor expansion of the constraints

Derivatives of the function F of (1) with respect to x and y at (u,v) are the linear mappings $F_{,x}(u,v) \in L(U,W)$, $F_{,y}(u,v) \in L(V,W)$ $(U = \mathbb{R}^n, V = W = \mathbb{R}^m)$, whose coefficients are given by the partial derivatives $\partial F_r/\partial x_i$ and $\partial F_r/\partial y_i$, respectively. Second derivatives of F are the bilinear mappings $F_{,xx} \in B(U,U,W)$, $F_{,xy} \in B(U,V,W)$, $F_{,yy} \in \in B(V,V,W)$ $(U = \mathbb{R}^n, V = W = \mathbb{R}^m)$, whose coefficients are the second partial derivatives $\partial^2 F_r/\partial x_i \partial x_j$, $\partial^2 F_r/\partial x_i \partial y_j$, and $\partial^2 F_r/\partial y_i \partial y_j$ (in that order). Taylor's theorem says that for any $(u,v), (h,k) \in \mathbb{R}^n \times \mathbb{R}^m$ there is $\theta \in [0,1]$ with $F([\begin{smallmatrix}u\\v\end{smallmatrix}] + [\begin{smallmatrix}h\\k\end{smallmatrix}]) = F([\begin{smallmatrix}u\\v\end{smallmatrix}]) + F_{,x}([\begin{smallmatrix}u\\v\end{smallmatrix}]) \cdot h + F_{,y}([\begin{smallmatrix}u\\v\end{smallmatrix}]) \cdot k + + \frac{1}{2} F_{,xx}([\begin{smallmatrix}u\\v\end{smallmatrix}] + \theta[\begin{smallmatrix}h\\k\end{smallmatrix}])[h,h] + F_{,xy}([\begin{smallmatrix}u\\v\end{smallmatrix}] + \theta[\begin{smallmatrix}h\\k\end{smallmatrix}])[h,k] + \frac{1}{2} F_{,yy}([\begin{smallmatrix}u\\v\end{smallmatrix}] + \theta[\begin{smallmatrix}h\\k\end{smallmatrix}])[k,k]$. Here we employed column vector notation "$[\begin{smallmatrix}u\\v\end{smallmatrix}]$" for $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$.

2.3. Computing norms of linear and bilinear mappings

We assume that $\alpha \in L(U,W)$, $\beta \in B(U,V,W)$, and that the linear spaces U, V, W are equipped with norms. We are going to use the L^p norms in $p = 1, 2, \infty$: $\|x\|_p := (\sum |x_i|^p)^{1/p}$ for $1 \le p < \infty$ and $\|x\|_\infty = \max_i |x_i|$. In any case, $\|\alpha\|_{L(U,W)} := \sup_{\|u\|_U \le 1} \|\alpha \cdot u\|_W$,

$\|\beta\|_{B(U,V,W)} := \sup_{\|u\|_U,\|v\|_V\le 1}\|\beta[u,v]\|_W$. For computing norms in $L(U,W)$, see e.g. [1]. In general, if the unit sphere S_U in U is a convex polyhedron with vertices x_i, then for any normed space X and linear mapping $\alpha : U \to X$, we have $\|\alpha\|_{L(U,X)} = \max_i \|\alpha \cdot x_i\|_X$. This applies to the 1-norm and the ∞-norm in U. As to bilinear mappings, it is not difficult to show that $\|\beta\|_{B(U,V,W)} = \|\beta^\phi\|_{L(U,L(V,W))} = \|\beta^\psi\|_{L(V,L(U,W))}$. This means that in case either S_V or S_U is a polyhedron, we are able to compute $\|\beta\|_{B(U,V,W)}$. We write $\|\beta\|_{p,q,r}$ in order to indicate that the spaces U, V, W use the p-, q-, and r-norms, resp. A case not handled by the polyhedral approach is $\|\beta\|_{2,2,\infty}$, which equals the maximum singular value of the $\dim W$ matrices $(\beta_{rij})_{i=1,\ldots,\dim U}^{j=1,\ldots,\dim V}$. Further, there is the inequality $\|\beta\|_{2,2,2} \le \sqrt{\dim W}\|\beta\|_{2,2,\infty}$. For more details, see also [8].

2.4. Norms of derivatives

The three vector spaces U, V, W involved in the definition of F in (1) and its second derivatives are assumed to be equipped with norms. $V = W$ as a linear space, but V and W may be different as normed linear spaces. We are going to consider only solutions of the constraint problems where there are upper bounds of the following form

(2)
$$\|F_{,xx}(u,v)\| \le \alpha,\ \|F_{,xy}(u,v)\| \le \beta,\ \|F_{,yy}(u,v)\| \le \gamma \quad (\alpha^2+\beta^2+\gamma^2 > 0).$$

Upper bounds as required by (2) are particularly simple to give if F is a quadratic function, because then $F_{,xx}$, $F_{,xy}$, and $F_{,yy}$ depend neither on x nor on y. Later we need the following function:

(3)
$$\Delta(s,t) := (\alpha\, s^2 + 2\beta\, st + \gamma\, t^2)/2.$$

3. Tolerance zones and implicit equations

This section sums up results of [8]. We first discuss local solutions of an implicit equations and later apply a linearized local solution to tolerance zones. Th. 1 below yields an upper bound for the error we make in this process, provided tolerance zones are small enough. The range of validity of Th. 1 is the subject of Sec. 5 below.

3.1. Local solutions

Geometric tolerance analysis means that we are dealing with imprecisely defined geometric objects $p_1, p_2, \ldots$, each of which is contained in its tolerance zone $P_1, P_2, \ldots$. Geometric objects $q_1, q_2, \ldots$ depend on the p_i's, and we want to find tolerance zones $Q_1, Q_2, \ldots$ for the $q_1, q_2, \ldots$ such that whenever $p_i \in P_i$ for all i, we can be sure that $q_j \in Q_j$ for all j. We treat this problem by introducing coordinates for all geometric entities involved, such that each p_i is represented by a group of fixed variables, and each q_i is given by a group of moving variables:

(4)
$$x = (\underbrace{x_1, \ldots, x_{r_1}}_{p_1}, \underbrace{x_{r_1+1}, \ldots, x_{r_1+r_2}}_{p_2}, \ldots, x_n), \quad y = (\underbrace{y_1, \ldots, y_{s_1}}_{q_1}, \ldots, y_m).$$

If $p_i \in P_i$ for all i, then the vector x, which actually constitutes coordinates for $p_1, p_2, \ldots$, is contained in the set $P_1 \times P_2 \times \ldots \in \mathbb{R}^{r_1} \times \times \mathbb{R}^{r_2} \times \ldots$. Suppose $F(u, v) = 0$ as above, such that $x = u, y = v$ represents a particular solution of the constraint problem, then the local solution $y = G(x)$ leads to a tolerance zone $G(P_1 \times P_2 \times \ldots)$ for the vector y. We define the functions $G^{(j)}$ as those coordinates of G, which belong to the geometric object q_j:

$$(5) \quad G(x) = (\underbrace{G_1(x), \ldots, G_{s_1}(x)}_{q_1 = G^{(1)}(x)}, \underbrace{G_{s_1+1}(x), \ldots, G_{s_1+s_2}(x)}_{q_2 = G^{(2)}(x)}, \ldots, G_m(x)).$$

Thus a tolerance zone of the geometric entity q_j is given by $G^{(j)}(P_1 \times \times P_2 \times \ldots)$. It is customary to consider only such tolerance zones P_i which have the topology of a ball. For computations one usually chooses simple shapes, such as convex ones.

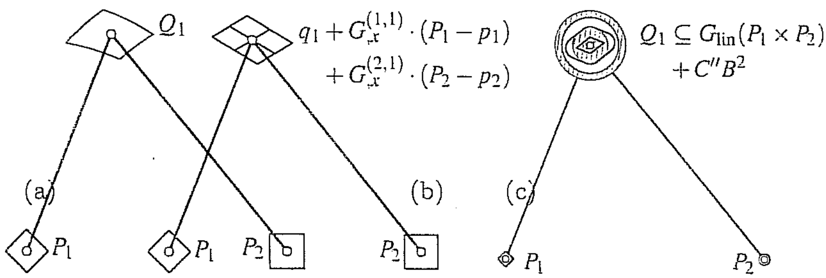

Fig. 1. (a) Exact and (b) linearized tolerance zones.
(c) Upper bound of linearization error.

As an example, we consider the case $n = 4$, $m = 2$, and $F_1(x, y) = = (x_1 - y_1)^2 + (x_2 - y_2)^2 - 2900$, $F_2(x, y) = (x_3 - y_1)^2 + (x_4 - - y_2)^2 - 4100$). A particular solution is $(u, v) = (0, 0, 60, 0, 20, 50)$.

This constraint problem has the following interpretation: The points $p_1 = (x_1, x_2)$, $p_2 = (x_2, x_4)$, $q_1 = (y_1, y_2)$ are constrained by the conditions $\|p_1 - q_1\|^2 = 2900$ and $\|p_2 - q_1\|^2 = 4100$. Fig. 1 (a) illustrates tolerance zones P_1, P_2, and the tolerance zone $Q_1 = G^{(1)}(P_1 \times P_2)$, where $y = G(x)$ is a local solution of the equation $F(x, y) = 0$ in a neighbourhood of $x = u, y = v$.

3.2. Linearizing constraints

We linearize the local solution $y = G(x)$ in a neighbourhood of a particular solution u, v with $F(u, v) = 0$:

$$\begin{aligned} &G_{\text{lin}}(u+h) = G(u) + G_{,x}(u) \cdot h, \\ &\text{where } G_{,x}(u) = -F_{,y}(u,v)^{-1} F_{,x}(u,v) \in L(U,V). \end{aligned} \tag{6}$$

The matrix $G_{,x}$ can be partitioned into column groups which correspond to the variables contribute to a particular geometric entity p_i, and into row groups which contribute to a particular entity q_j. Thus we get the following block matrix decomposition with numbers r_i and s_j from (4), and a first order approximation for tolerance zones Q_j:

$$G_{,x} = \begin{bmatrix} G_{,x}^{(1,1)} & G_{,x}^{(2,1)} & \cdots \\ G_{,x}^{(1,2)} & G_{,x}^{(2,2)} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} \}s_1 \\ \}s_2 \\ \end{matrix} \Longrightarrow \tag{7}$$

$$\underbrace{}_{r_1} \quad \underbrace{}_{r_2}$$

$$\Longrightarrow \begin{cases} Q_j \approx G_{\text{lin}}^{(j)}(P_1 \times P_2 \times \ldots) \\ \quad = q_j + \sum_i G_{,x}^{(i,j)} \cdot (P_i - p_i). \end{cases}$$

This Minkowski sum of affinely transformed tolerance zones P_i is particularly simple to compute if $s_i \leq 2$ in (4). We continue the example above, which is illustrated by Fig. 1: Here $G_{,x} = \left[G_{,x}^{(1,1)} \mid G_{,x}^{(2,1)}\right] = \frac{1}{30}\begin{bmatrix} 10 & 25 & 20 & -25 \\ 8 & 20 & -8 & 10 \end{bmatrix}$. The resulting linearized tolerance zone is shown in Fig. 1 (b). Both $G_{,x}^{(1,1)}$ and $G_{,x}^{(2,1)}$ are singular, and $G_{,x}^{(i,1)} \cdot (P_i - p_i)$ is a straight line segment. It follows that we approximate the tolerance zone Q_1 by a parallelogram.

3.3. Estimating the linearization error

The linearization error is the difference between an exact local solution G and the linearized one, G_{lin}. Following [8], we use function Δ defined in (3).

If F is linear, then the norms $\|F_{,xx}\|, \ldots$ are zero and linearization is exact. For our purposes it is essential that $\Delta(s,t)$ is non-zero if $s,t > 0$. Therefore we require that $\alpha^2 + \beta^2 + \gamma^2 > 0$.

Theorem 1. *Consider a solution (u,v) of the constraint problem $F(x,y) = 0$, and assume that $\Delta(s,t)$ is defined according to (3). Further assume that there is a local solution G with $v = G(u)$ and the corresponding linearized solution G_{lin}. Choose $C, C', C_{\max}$ such that*

(8)

$$C_{\max} = \frac{\|G_{,x}(u)\|}{\|F_{,y}(u,v)^{-1}\| \cdot \Delta(1, 2\|G_{,x}(u)\|)}, \quad C < C_{\max}, \quad C' = \|G_{,x}(u)\|\, C.$$

A perturbation in u causes v to move with $G(u+h) = v + k$. The linearization of this equation is $G_{\text{lin}}(u+h) = v + k_{\text{lin}}$. The linearization error obeys the following inequalities:

(9)

$$\|h\| \le C \implies \|k\| < 2C', \quad \|k - k_{\text{lin}}\| \le \|F_{,y}(u,v)^{-1}\| \cdot \Delta(C, 2C') < C'.$$

[8] gives examples which use Th. 1 in order to give an upper bound for the linearization error. Fig. 1 illustrates an offset of the linearized tolerance zone, where the exact tolerance zone Q_1 is known to be contained in.

Th. 1 gives an answer to the question of maximal size of the tolerance zone of the moving variables such that a tolerance zone of the corresponding moving variables can be computed with linear analysis plus an estimate for the linearization error. Conversely, assume that the tolerance zone of the moving variables is prescribed as a ball of radius C^* with $C^* < C^*{}_{\max} = \|G_{,x}\| C_{\max}$ (in the notation of Th. 1). Then the choice of

$$C = \frac{1}{2\|F_{,y}^{-1}\|\Delta(1, 2\|G_{,x}\|)} \times \left(\sqrt{\|G_{,x}(u)\|^2 + 4C^* \cdot \|F_{,y}^{-1}\| \cdot \Delta(1, 2\|G_{,x}\|)} - \|G_{,x}\|\right) \tag{10}$$

ensures that $\|h\| \le C$ implies $\|k\| < C^*$.

3.4. Balancing the constraint equations

Obviously the local solutions do not change if we multiply some constraints by factors, but the computation of $C_{\max}$ is affected by it. A rule of thumb might be that all variables should have values of the same order of magnitude. The same holds true for the choice of coordinate system, especially the choice of unit length. Some of the coordinates may reflect length, or length squared, or might have no dimension. The coordinate vector of a plane, for instance, contains a unit vector together with a coordinate whose geometric meaning is length. By choosing the unit length appropriately it is easy to achieve any magnitude of that single coefficient. A general answer to the balancing question appear to be difficult.

It is an aim of this paper to investigate several geometric constructions in Euclidean $\mathbb{R}^3$ in order to gain insight in the behaviour of $C_{\max}$ and the norms of derivatives needed when changing the coordinate system.

4. Coordinates and relations

This section sums up elementary properties of points, lines and planes in Euclidean space.

4.1. Coordinates for geometric objects

A point $(x_1, x_2, x_3) \in \mathbb{R}^3$ naturally is given the coordinates x_1, x_2, x_3. The plane with equation $\langle u, x\rangle + u_0 = 0$ such that $u = (u_1, u_2, u_3)$ has given the coordinates (u_0, u_1, u_2, u_3). We normalize the equation such that $u_1^2 + u_2^2 + u_3^2 = \langle u, u\rangle = 1$. Actually such coordinates represent an oriented plane, i.e., a plane together with a side of the plane where the normal vector u points to. A line parallel to the vector $l = (l_1, l_2, l_3)$ with $l_1^2 + l_2^2 + l_3^2 = 1$ is uniquely characterized by the moment vector $\bar{l} = x \times l$, if x is a point on the line, and the line is reconstructed as the solution set of the three equations $x \times l = \bar{l}$, if vectors l and $\bar{l}$ with $\langle l, \bar{l}\rangle = 0$ are given [5]. Thus we coordinatize the set of straight lines in $\mathbb{R}^3$ by the six coordinates $(l, \bar{l}) = (l_1, \ldots, l_6)$ with the side conditions $\langle l, l\rangle = 1$ and $\langle l, \bar{l}\rangle = 0$. Actually any such coordinate vector means an oriented line, and $(-l, -\bar{l})$ means the same line, but equipped with the reverse orientation.

4.2. Relations between geometric objects

We summarize relations between geometric objects in Table 1 and Table 2. We use the symbols p, q for points, $L = (l, \bar{l})$, $G = (g, \bar{g})$, $H = (h, \bar{h})$ for lines, and $U = (u_0, u)$, $V = (v_0, v)$ for planes. First comes a relation which involves points only: the distance constraint. Next are relations between a point and a line. The incidence relation $p \in L$ either uses only two out of the three equations $\bar{l} = p \times l$, or the condition that $\langle l, \bar{l} \rangle = 0$ has to be dropped. This is indicated by the canceling stroke in the right hand column. We further consider the case that Q is the pedal point of P on L, which means that $Q \in L$ and the line $P \vee Q$ is orthogonal to L. For the pedal point we give two formulas: One in Table 1 (see Sec. 6), and another one in Table 2, which introduces as a new variable the distance of P's pedal point Q from the origin's pedal point $l \times \bar{l}$. The oriented distance of points on a line, denoted by the symbol $\overrightarrow{\text{dist}}_L(P, Q)$, is negative, if the vector $\overrightarrow{PQ}$ does not point in the same direction as l. Next come relations between points and planes, which are straightforward. Relations between lines include parallelity, distance of parallel lines, and distance of skew lines G, H. The latter constraint can be made quadratic by introducing both sine and cosine of the angle $\sphericalangle(G, H)$ as new variables. Relations between a line and a plane are orthogonality (two cases), parallelity and incidence ($L \subset U$). A relations between planes given here is parallelity. As the line given as intersection of two planes has coordinates proportional to $(u \times v, u_0 v - v_0 u)$, also this results in a quadratic relation. It it easy to add more relations to this table.

4.3. Changing the coordinate system

It is an aim of this paper to study the influence of translation, rotation, and scaling of the underlying coordinate system on the local tolerance analysis via Th. 1. The choice of a different unit length (i.e., a scaling of the coordinate system with a factor $s > 0$), translation by $t \in \mathbb{R}^3$, and rotation by a matrix $A \in SO_3$ transform coordinates according to

$$(11) \quad p \longrightarrow sp, \quad (l, \bar{l}) \longrightarrow (l, s\bar{l}), \qquad (u_0, u) \longrightarrow (su_0, u).$$

$$(12) \quad p \longrightarrow p + t, (l, \bar{l}) \longrightarrow (l, \bar{l} + t \times l), \quad (u_0, u) \longrightarrow (u_0 - \langle u, t \rangle, u).$$

$$(13) \quad x \longrightarrow Ax, \quad (l, \bar{l}) \longrightarrow (Al, A\bar{l}) \qquad (u_0, u) \longrightarrow (u_0, Au).$$

The value $C_{\max}$ as computed by Th. 1 or the formulas following it means the maximum size of tolerance zone of the fixed variables "x"

around a local solution $x = u, y = v$ of the constraint problem $F(x, y) = 0$. When changing the unit length so that coordinates of points get multiplies by a factor $s > 0$, $C_{\max}$ will usually change.

If the fixed variables consist only of points, then an optimal method for local tolerance analysis would result in $C_{\max}$ gets multiplied by s. If the different parts of x as described by (4) have also other meanings, such a simple statement is no longer possible. For lines and planes, for instance, not all of its coordinates are scaled. While it would be nice if $C_{\max}$ would get bigger if all coordinates are multiplied by s, we cannot expect this to be the case.

As all three type of geometric entities considered in detail in this paper contain at least one coordinate which is scaled with s, we do the following: We scale with s according to (11), and have a look at $C_{\max}/s$, which in the case of points means the size of tolerance zone with respect to the coordinate system before scaling.

5. Examples

In this section, we collect the most useful constraints in geometric constraint solving problems and show the influence of translation, rotation, and scaling on the value of $C_{\max}$ computed via Th. 1. In the remain content, we use 1-norm in the fixed variable space and 2-norm in the moving variable space, and other norms are only illustrated in the data of the tables. When investigating the influence of translations and rotations we randomly select the translation vectors as $t(\tau) = (\tau, \tau, \tau)$ and the rotation about the x axis for demonstration.

5.1. Pedal points

Consider the geometric relation $q_1 = \text{pedal}_L(p_1)$, where $p_1 = (x_1, x_2, x_3)$ is a fixed point, $q_1 = (y_1, y_2, y_3)$ is a moving point, $L = (x_4, \ldots, x_9)$ is a line. According to Table 2, we add a variable $\lambda = y_4$. We get the following constraint problem $F(x, y) = 0$, where

$$(14) \qquad F(x, y) = \begin{bmatrix} x_4x_1 + x_5x_2 + x_6x_3 - y_4 \\ x_5x_9 - x_6x_8 + y_4x_4 - y_1 \\ x_6x_7 - x_4x_9 + y_4x_5 - y_2 \\ x_4x_3 - x_5x_7 + y_4x_6 - y_3 \end{bmatrix}.$$

Formally, we let $L = p_2$. As a particular solution, we consider $p_1 = (100, 100, 100)$, $L = (-1, 1, 1, 0, -100, 100)/\sqrt{3}$, $q_1 = (100, 200, 200)/3$, and $\lambda = y_4 = 100\sqrt{6}/3$. Experimental data are shown in Table 3.

When scaling with a factor $s > 0$, $F_{,y}$ does not depend on s. So the bilinear mappings $B_1 := F_{,y}^{-1} F_{,xx}$ and $B_2 := F_{,y}^{-1} F_{,xy}$ are constant. $F_{,yy}$ is zero. $G_{,x}$ expands to

$$\begin{bmatrix} -x_4^2 & -x_4x_5 & -x_4x_6 & (-x_4x_1-y_4)s & (-x_4x_2-x_9)s & (-x_4x_3+x_8)s & 0 & x_6 & -x_5 \\ -x_4x_5 & -x_5^2 & -x_5x_6 & (-x_5x_1+x_9)s & (-x_5x_2-y_4)s & (-x_5x_3-x_7)s & -x_6 & 0 & -x_4 \\ -x_4x_6 & -x_5x_6 & -x_6^2 & (-x_6x_1-x_8)s & (-x_6x_2+x_7)s & (-x_6x_3-y_4)s & x_5 & -x_4 & 0 \\ -x_4 & -x_5 & -x_6 & -x_1s & -x_2s & -x_3s & 0 & 0 & 0 \end{bmatrix}.$$

It is obvious that both $M_0 := \lim_{s\to 0} G_{,x}$ and $\lim_{s\to\infty}(G_{,x}/s)$ depend only on x. Thus we get the following expressions for $C_{\max}$: $C_{\max} = 2\|G_{,x}\|/(\|B_1\| + 4\|B_2\|\|G_{,x}\|)$, and

(15)
$$\lim_{s\to 0} C_{\max} = 2\|M_0\|/(\|B_1\| + 4\|M_0\|\|B_2\|), \quad \lim_{s\to\infty} C_{\max} = 1/(2\|B_2\|).$$

It follows that the graph of $\eta = \ln(C_{\max}/s)$ over $\xi = \ln s$ has asymptotes of the form $\eta = -\xi + \ln C$ for both $s \to 0$ and $s \to \infty$, where $\ln C$ is the logarithm of either of the two values in (15). Experimental data for the change of $C_{\max}$ when changing the coordinate system is also shown in Fig. 2.

The pedal point in a plane is much easier to analyze, because in that case $F_{,xx} = 0$ and $F_{,yy} = 0$, so so, $C_{\max}(u, v) = (2\|F_{,y}^{-1}F_{,xy}\|)^{-1}$ and in view of (11) does not depend on the choice of unit length.

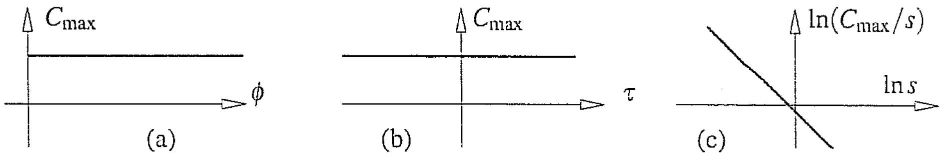

Fig. 2. (a) Diagram of the change of $C_{\max}$ over the rotation angle ϕ in the constraint problem of Sec. 5.1 while rotating the coordinate system. (b) the same for translating the coordinate system. (c) Logarithmic diagram of $C_{\max}/s$ over a scaling factor s.

5.2. The line spanned by two points

We consider two points $p_1 = (x_1, x_2, x_3)$, $p_2 = (x_4, x_5, x_6)$ as fixed variables, and the coordinates of the line $L = (l, \bar{l}) = (y_1, \ldots, y_6)$ spanned by them as moving variables. Table 1 contains two different ways of expressing the condition that $p_1 \in L$. Because the four equations $\bar{l} = p_i \times l$ plus $\langle l, \bar{l} \rangle = 0$ are not independent, each incidence condition can use only three of them. For reasons of symmetry, it is preferable that we drop $\langle l, \bar{l} \rangle = 0$, but we can do that only once – for the other incidence constraint, one of the three equations of $\bar{l} = p_i \times l$ has to go also. Thus we get the following six equations for $y_1, \dots, y_6$: $y_1^2 + y_2^2 + y_3^2 - 1 = y_3x_2 - y_2x_3 - y_4 = y_1x_3 - y_3x_1 - y_5 = y_2x_1 - y_1x_2 - y_6 = y_3x_5 - y_2x_6 - y_4 = y_1x_6 - y_3x_4 - y_5 = 0$.

The particular solution for which we display experimental data in Table 4 and Fig 3, a–c is $p_1 = (40, 30, 70)$, $p_2 = (30, 40, -70)$, $L = \sqrt{22}(-1/2, 1/2, -7, -245, 245, 35)/33$.

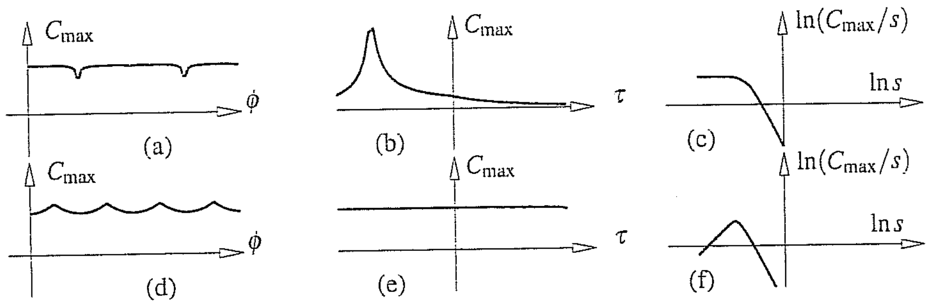

Fig. 3. (a)–(c) analogous to Fig. 2, but for the constraint problem of Sec. 5.2 (first variant). (d)–(f): second variant.

It is elementary to compute the following derivatives:

$$F_{,y} = \begin{bmatrix} 2y_1 & 2y_2 & 2y_3 & 0 & 0 & 0 \\ 0 & -x_3 & x_2 & -1 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & -1 & 0 \\ -x_2 & x_1 & 0 & 0 & 0 & -1 \\ 0 & -x_6 & x_5 & -1 & 0 & 0 \\ x_6 & 0 & -x_4 & 0 & -1 & 0 \end{bmatrix},$$

$$F_{,x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_3 & -y_2 & 0 & 0 & 0 \\ -y_3 & 0 & y_1 & 0 & 0 & 0 \\ y_2 & -y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & -y_3 & 0 & y_1 \end{bmatrix}.$$

Further, $F_{,xx} = 0$, $F_{1,xy} = 0$, $F_{r,yy} = 0$ for $r = 2, \dots, 6$, $F_{r,xy} = \begin{bmatrix} K_r & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$ for $r = 2, 3, 4$, $F_{r,xy} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ K_{r-3} & 0_{3\times3} \end{bmatrix}$ for $r = 5, 6$, and

$F_{1,yy} = \begin{bmatrix} 2E_3 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}$, where we have used the abbreviations

$$K_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When scaling the coordinates with a factor s, we get we get $G_{,x}(s) = = \begin{bmatrix} \frac{1}{s}M_{3\times 6} \\ N_{3\times 6} \end{bmatrix}$, with certain matrices $M_{3\times 6}$ and $N_{3\times 6}$. The second derivatives have the following behaviour: $[F_{,y}^{-1}F_{,xy}(s)]_r$ equals $1/s$ times a constant for $r = 1, 2, 3$, and is scale invariant for $r = 4, 5, 6$. So is $[F_{,y}^{-1}F_{,yy}(s)]_r$ for $r = 1, 2, 3$. $[F_{,y}^{-1}F_{,yy}(s)]_r$ equals a constant times s for $r = 4, 5, 6$.

We consider the limits

(16)
$$B_0 = \lim_{s\to 0}(sG_{,x}(s)), \quad C_0 = \lim_{s\to 0}(sF_{,y}^{-1}F_{,xy}(s)), \quad D_0 = \lim_{s\to 0}(F_{,y}^{-1}F_{,yy}(s)),$$
$$B_\infty = \lim_{s\to\infty}(G_{,x}(s)), \; C_\infty = \lim_{s\to\infty}(F_{,y}^{-1}F_{,xy}(s)), \; D_\infty = \lim_{s\to\infty}(F_{,y}^{-1}F_{,yy}(s)/s).$$

The formula of Th. 1 now shows that and get

$$\lim_{s\to 0}\frac{C_{\max}(s)}{s} = \frac{1}{2(\|C_0\| + \|B_0\|\|D_0\|)}, \qquad (17)$$
$$\lim_{s\to\infty}(sC_{\max}(s)) = \frac{1}{2\|B_\infty\|\|D_\infty\|}.$$

The graph of $\eta = \ln(C_{\max}/s)$ over $\xi = \ln s$ then has the similar asymptotes to that of Sec. 5.3.

By introducing the oriented distance $d = \overrightarrow{\mathrm{dist}}_L(p_1, p_2)$ of the points p_1 and p_2, we get a set of equations different from the previous one: $\|l\|^2 = 1$, $\bar{l} = p_1 \times l$ and $p_2 = p_1 + dl$. Experimental data are shown in Table 5 and Fig. 3, d–f. The limit case of scaling in the constraint is similar to that of Sec. 5.5 and we don't want to study it in detail. We notice the following facts: Introduction of an auxiliary variable did not diminish the size of $C_{\max}$ overmuch, and it did improve the behaviour with respect to translations. However, it is apparently more important to choose the right scaling factor s than it was with the first variant.

5.3. The plane spanned by three points

Consider the three points $p_1 = (x_1, x_2, x_3)$, $p_2 = (x_4, x_5, x_6)$, $p_3 = (x_7, x_8, x_9)$ as fixed variables and the coordinates of the plane $U = (u_0, u) = (y_1, \ldots, y_4)$ as moving variables. The condition that $p_1, p_2, p_3 \in U$ is expressed by the three constraints $\langle p_i, u\rangle + u_0 = 0$ together with the normalization $\|u\|^2 = 1$. Experimental data for the particular solution $p_1 = (100, 0, 0)$, $p_2 = (0, 100, 0)$, $p_3 = (0, 0, 100)$. and $U = (-100, 1, 1, 1)/\sqrt{3}$ are shown in Table 6 and Fig. 4.

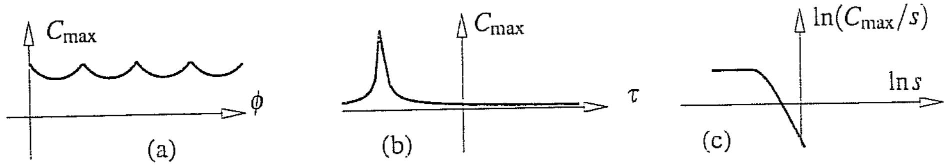

Fig. 4. (a)–(c) analogous to Fig. 2, but for the constraint problem of Sec. 5.3.

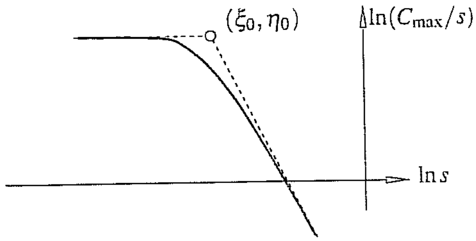

Fig. 5. Detail of Fig. 4 (c) (asymptotes).

We demonstrate the influence of the choice of unit length via the following detailed computations. Obviously, $F_{,xx} = 0$, so $C_{\max} = 1/[2(\|F_{,y}^{-1}F_{,xy}\| + \|G_{,x}\|\,\|F_{,y}^{-1}F_{,yy}\|)]$. We have

$$F(x,y) = \begin{bmatrix} y_2^2+y_3^2+y_4^2-1 \\ y_1+y_2x_1+y_3x_2+y_4x_3 \\ y_1+y_2x_4+y_3x_5+y_4x_6 \\ y_1+y_2x_7+y_3x_8+y_4x_9 \end{bmatrix},$$

$$F_{,y} = \begin{bmatrix} 0 & 2y_2 & 2y_3 & 2y_4 \\ 1 & x_1 & x_2 & x_3 \\ 1 & x_4 & x_5 & x_6 \\ 1 & x_7 & x_8 & x_9 \end{bmatrix}, \quad F_{,x} = \begin{bmatrix} 0_{1\times3} & 0_{1\times3} & 0_{1\times3} \\ M & 0_{1\times3} & 0_{1\times3} \\ 0_{1\times3} & M & 0_{1\times3} \\ 0_{1\times3} & 0_{1\times3} & M \end{bmatrix},$$

where $M = [y_2, y_3, y_4]$. As to the inverse of $F_{,y}$, we let $m = \det(F_{,y})$

and define coefficients n_{ij} by $F_{,y}^{-1} = (\frac{n_{ij}}{m})_{i,j=1,\ldots,4}$. We further use the abbreviation $N = [0_{3\times 1}\ E_3]$.

When scaling with a factor $s > 0$, we get the following dependencies on s: Coordinates $(x_1, \ldots, x_9)$ change to $(sx_1, \ldots, sx_9)$ and $(y_1, \ldots, y_4)$ becomes (sy_1, y_2, y_3, y_4), according to (11). We have $F_{,xx} = 0$ for all s. With exponents $\alpha_1 = 0, \alpha_2 = \cdots = \alpha_4 = 1$ we can write the dependence of $G_{,x}$ and the component matrices of $F_{,y}^{-1}F_{,xy}(s)$ and $F_{,y}^{-1}F_{,yy}(s)$ in the form

$$G_{,x}(s) = \frac{1}{sm}\begin{bmatrix} sn_{12}M & sn_{13}M & sn_{14}M \\ n_{22}M & n_{23}M & n_{24}M \\ n_{32}M & n_{33}M & n_{34}M \\ n_{42}M & n_{43}M & n_{44}M \end{bmatrix},$$

$$[F_{,y}^{-1}F_{,xy}(s)]_r = \frac{1}{s^{\alpha_r}m}\begin{bmatrix} n_{r2}N \\ n_{r3}N \\ n_{r4}N \end{bmatrix}$$

$$[F_{,y}^{-1}F_{,yy}(s)]_r = \frac{2sn_{r1}}{s^{\alpha_r}m}\mathrm{diag}(0,1,1,1).$$

With the limits from (16), we can compute the limit behaviour of $C_{\max}(s)/s$ with (17). The graph of $\eta = \ln(C_{\max}/s)$ over $\xi = \ln s$ has exactly the same behaviour as the respective graph in Sec. 5.2 (first variant), as is also illustrated by Fig. 4.

5.4. Intersection of two planes

We consider the intersection line $L = (l, \bar{l}) = (y_1, \ldots, y_6)$ of two planes $U = (u_0, u) = (x_1, \ldots, x_4)$ and $V = (v_0, v) = (x_5, \ldots, x_8)$, where the planes are fixed and the line is moving. The constraints $F(x, y) = 0$ are defined by the relation $L = U \cap V$ according to Table 2. By introducing the auxiliary variable $\lambda = y_7$, we get

$$F(x,y) = \begin{bmatrix} y_1^2+y_2^2+y_3^2-1 \\ x_3x_8-x_4x_7-y_1y_7 \\ x_4x_6-x_2x_8-y_2y_7 \\ x_2x_7-x_3x_6-y_3y_7 \\ x_1x_6-x_2x_5-y_4y_7 \\ x_1x_7-x_3x_5-y_5y_7 \\ x_1x_8-x_4x_5-y_6y_7 \end{bmatrix} \Rightarrow$$

$$F_{,y}^{-1}=\frac{-1}{2y_7}\begin{bmatrix}-y_1y_7 & 2(1-y_1^2) & -2y_1y_2 & -2y_1y_3 & 0 & 0 & 0\\ -y_2y_7 & -2y_1y_2 & 2(1-y_2^2) & -2y_2y_3 & 0 & 0 & 0\\ -y_3y_7 & -2y_1y_3 & -2y_2y_3 & 2(1-y_3^2) & 0 & 0 & 0\\ -y_4y_7 & -2y_1y_4 & -2y_2y_4 & -2y_3y_4 & 2 & 0 & 0\\ -y_5y_7 & -2y_1y_5 & -2y_2y_5 & -2y_3y_5 & 0 & 2 & 0\\ -y_6y_7 & -2y_1y_6 & -2y_2y_6 & -2y_3y_6 & 0 & 0 & 2\\ y_7^2 & 2y_1y_7 & 2y_2y_7 & 2y_3y_7 & 0 & 0 & 0\end{bmatrix},$$

$$G_{,x}=\begin{bmatrix}0 & R_1y_1 & x_8+R_2y_1 & -x_7+R_3y_1 & 0 & R_4y_1 & -x_4+R_5y_1 & x_3+R_6y_1\\ 0 & -x_8+R_1y_2 & R_2y_2 & R_3y_2 & 0 & x_4+R_4y_2 & R_5y_2 & -x_2+R_6y_2\\ 0 & x_7+R_1y_3 & -x_6+R_2y_3 & R_3y_3 & 0 & -x_3+R_4y_3 & x_2+R_5y_3 & iR_6y_3\\ x_6 & -x_5+R_1y_4 & R_2y_4 & R_3y_4 & -x_2 & x_1+R_4y_4 & R_5y_4 & R_6y_4\\ x_7 & R_1y_5 & -x_5+R_2y_5 & R_3y_5 & -x_3 & R_4y_5 & x_1+R_5y_5 & R_6y_5\\ x_8 & R_1y_6 & R_2y_6 & -x_5+R_3y_6 & -x_4 & R_4y_6 & R_5y_6 & x_1+R_6y_6\\ 0 & -R_1y_7 & -R_2y_7 & -R_3y_7 & 0 & -R_4y_7 & -R_5y_7 & -R_6y_7\end{bmatrix}.$$

where $R_1 = x_8y_2 - x_7y_3$, $R_2 = x_6y_3 - x_8y_1$, $R_3 = x_7y_1 - x_6y_2$, $R_4 = = x_3y_3 - x_4y_2$, $R_5 = x_4y_1 - x_2y_3$, $R_6 = x_2y_2 - x_3y_1$. Second derivatives have the form $[F_{,y}^{-1}F_{,xx}]_r = \frac{1}{y_7}\cdot\begin{bmatrix}0_{4\times4} & -S_r\\ S_r & 0_{4\times4}\end{bmatrix}$ for $r = 1,\dots,7$, where

$$S_1=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -y_1y_3 & y_1y_2\\ 0 & y_1y_3 & 0 & 1-y_1^2\\ 0 & -y_1y_2 & y_1^2-1 & 0\end{bmatrix},\quad S_2=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -y_2y_3 & y_1(y_2^2-1)\\ 0 & y_2y_3 & 0 & -y_1y_2\\ 0 & y_1(1-y_2^2) & y_1y_2 & 0\end{bmatrix},$$

$$S_3=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1-y_3^2 & y_1y_2y_3\\ 0 & y_3^2-1 & 0 & -y_1y_3\\ 0 & -y_1y_2y_3 & y_1y_3 & 0\end{bmatrix},\quad S_4=\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & -y_3y_4 & y_2y_4\\ 0 & y_3y_4 & 0 & -y_1y_4\\ 0 & -y_2y_4 & y_1y_4 & 0\end{bmatrix},$$

$$S_5=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & -y_3y_5 & y_2y_5\\ -1 & y_3y_5 & 0 & -y_1y_5\\ 0 & -y_2y_5 & y_1y_5 & 0\end{bmatrix},\quad S_6=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -y_3y_6 & y_2y_6\\ 0 & y_3y_6 & 0 & -y_1y_6\\ -1 & -y_2y_6 & y_1y_6 & 0\end{bmatrix},$$

$$S_7=y_7\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & y_3 & -y_2\\ 0 & -y_3 & 0 & y_1\\ 0 & y_2 & -y_1 & 0\end{bmatrix}.$$

$F_{,xy}$ is zero. $F_{,y}^{-1}F_{,yy}$ has the form

$$[F_{,y}^{-1}F_{,yy}]_r = \begin{bmatrix} \mathrm{diag}(y_r y_7, y_r y_7, y_r y_7, 0,0,0) & N_r \\ N_r^T & 0 \end{bmatrix} / y_7$$

for $r = 1, \ldots, 6$ and

$$[F_{,y}^{-1}F_{,yy}]_7 = \begin{bmatrix} \mathrm{diag}(-y_7, -y_7, -y_7, 0,0,0) & N_7 \\ N_7^T & 0 \end{bmatrix},$$

where the column vectors N_i are defined by

$$[N_1, \ldots, N_7] = \begin{bmatrix} 1-y_1^2 & -y_1y_2 & -y_1y_3 & -y_1y_4 & -y_1y_5 & -y_1y_6 & y_1 \\ -y_1y_2 & 1-y_2^2 & -y_2y_3 & -y_2y_4 & -y_2y_5 & -y_2y_6 & y_2 \\ -y_1y_3 & -y_2y_3 & 1-y_3^2 & -y_3y_4 & -y_3y_5 & -y_3y_6 & y_3 \\ 0_{3\times 3} & & E_{3\times 3} & & 0_{3\times 1} & & \end{bmatrix}.$$

When scaling with a factor s, $(x_1, x_5, y_4, y_5, y_6)$ are replaced by $s(x_1, x_5, y_4, y_5, y_6)$. The other variables are scale-independent. We consider the limit cases $s \to 0$ and $s \to \infty$. In a way analogous to previous constraint problems, we consider the limits

$$B_0 = \lim_{s\to 0} G_{,x}(s), \quad C_0 = \lim_{s\to 0} F_{,y}^{-1}F_{,xx}(s),$$

$$D_0 = \lim_{s\to 0} F_{,y}^{-1}F_{,yy}(s),$$

$$B_\infty = \lim_{s\to\infty}(G_{,x}(s)/s), \quad C_\infty = \lim_{s\to\infty}(F_{,y}^{-1}F_{,xx}(s)/s),$$

$$D_\infty = \lim_{s\to\infty}(F_{,y}^{-1}F_{,yy}(s)/s).$$

The formula for $C_{\max}$ from Th. 1 shows that

$$\lim_{s\to 0} C_{\max}(s) = \frac{2\|B_0\|}{\|C_0\| + 4\|B_0\|^2\|D_0\|},$$

$$\lim_{s\to\infty}(s^2 C_{\max}(s)) = \frac{1}{2\|B_\infty\|\|D_\infty\|}.$$

Thus the graph of $\eta = \ln(C_{\max}/s)$ over $\xi = \ln s$ has the asymptotes $\eta = -\xi + \ln(2\,\|B_0\|) - \ln(\|C_0\| + 4\|B_0\|^2\|D_0\|)$ as $\xi \to -\infty$, and and $\eta = -3\xi - \ln(2\|B_\infty\|\,\|D_\infty\|)$ as $\xi \to \infty$. They intersect at $\xi = \ln s_0$, where $s_0^2 = (\|C_0\| + 4\,\|B_0\|^2\,\|D_0\|)/\,(4\,\|B_0\|\,\|B_\infty\|\,\|D_\infty\|)$. Experimental data for the particular solution $U = (100, -1, -1, -1)/\sqrt{3}$, $V = (100, -1, -1, 1)/\sqrt{3}$, $L = (-1, 1, 0, 0, 0, 100)/\sqrt{2}$, and $\lambda = 2\sqrt{2}/3$ are shown in Table 7 and Fig. 6.

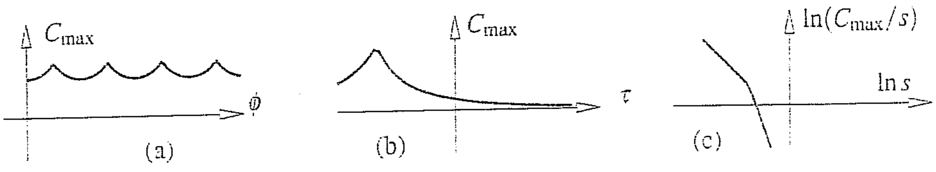

Fig. 6. (a)–(c) analogous to Fig. 2, but for the constraint problem of Sec. 5.4.

5.5. Two points determine a unit vector

This is a constraint problem not contained in the tables above. We have the fixed variables $p_1 = (x_1, x_2, x_3)$, $p_2 = (x_4, x_5, x_6)$ and the moving variables $q_1 = (y_1, y_2, y_3) \in \mathbb{R}^3$, $y_4 \in \mathbb{R}$ with the constraints $\|q_1\|^2 = 1$, $p_1 - p_2 = y_4 q_1$ (y_4 is the distance of p_1 from p_2).

The particular solution $p_1 = (40, 30, 70)$, $p_2 = (30, 40, -70)$, $q_1 = (p_2 - p_1)/y_4$, $y_4 = \|p_2 - p_1\|$ is illustrated in Table 8 and Fig. 7.

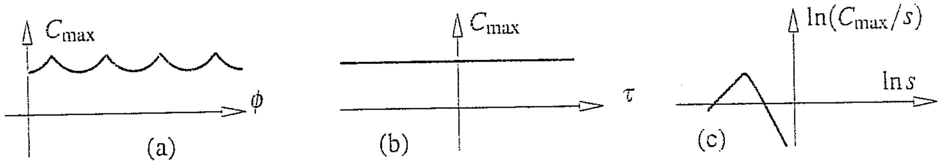

Fig. 7. (a)–(c) analogous to Fig. 2, but for the constraint problem of Sec. 5.5.

We have $F_{,xx} = 0$ and $F_{,xy} = 0$, so we get

$$C_{\max}(s) = 1/(2\|G_{,x}(s)\|\|F_{,y}^{-1}F_{,yy}(s)\|).$$

An elementary computation shows that

$$G_{,x}(s) = \frac{1}{sy_4}\begin{bmatrix} \widetilde{G}(s) \\ \widetilde{g}(s) \end{bmatrix}, \tag{18}$$

where $\widetilde{g}(s) = \begin{bmatrix} sy_4y_1 & sy_4y_2 & sy_4y_3 & -sy_4y_1 & -sy_4y_2 & -sy_4y_3 \end{bmatrix}$,

$$\widetilde{G}(s) = \begin{bmatrix} 1-y_1^2 & -y_2y_1 & -y_1y_3 & y_1^2-1 & y_2y_1 & y_1y_3 \\ -y_2y_1 & 1-y_2^2 & -y_3y_2 & y_2y_1 & y_2^2-1 & y_3y_2 \\ -y_1y_3 & -y_3y_2 & 1-y_3^2 & y_1y_3 & y_3y_2 & y_3^2-1 \end{bmatrix}.$$

We define $\widetilde{M}(v_1, v_2, v_3, v_4) = \begin{bmatrix} v_1 & & & v_2 \\ & v_1 & & v_3 \\ & & v_1 & v_4 \\ v_2 & v_3 & v_4 & \end{bmatrix}$ and get $F_{,y}^{-1}F_{,yy}(s) = \frac{1}{sy_4}B_s$, where $B_s \in B(\mathbb{R}^4, \mathbb{R}^4, \mathbb{R}^4)$ has the following coordinates:

$$[B_s]_1 = \widetilde{M}(sy_4y_1, 1 - y_1^2, -y_2y_1, -y_1y_3),$$

$$[B_s]_2 = \widetilde{M}(sy_4y_2, -y_2y_1, 1 - y_2^2, -y_3y_2),$$

$$[B_s]_3 = \widetilde{M}(sy_4y_3, -y_1y_3, -y_3y_2, 1-y_3^2),$$
$$[B_s]_4 = \widetilde{M}(-s^2y_4^2y_3, sy_4y_1, sy_4y_2, sy_4y_3).$$

Limits for $s \to 0$ and $s \to \infty$ are the following:

$$\text{(19)} \quad \begin{aligned} L_0 &:= \lim_{s\to 0} sG_{,x}(s) = \frac{1}{y_4}\begin{bmatrix} \widetilde{G}(s) \\ \hline 0_{1\times 6}\end{bmatrix}, \\ L_\infty &:= \lim_{s\to\infty} G_{,x}(s) = \begin{bmatrix} O_{3\times 6} \\ \hline y_1 \; y_2 \; y_3 \; -y_1 \; -y_2 \; -y_3 \end{bmatrix}. \end{aligned}$$

Further, $\lim_{s\to 0} s\, F_{,y}^{-1}F_{,yy}(s) = \frac{1}{y_4}\overline{B}_0$, where $\overline{B}_0$ has the following coordinates:

$[\overline{B}_0]_1 = \widetilde{M}(0, 1-y_1^2, -y_2y_1, -y_1y_3)$, $[\overline{B}_0]_2 = \widetilde{M}(0, -y_2y_1, 1-y_2^2, -y_3y_2)$,
$[\overline{B}_0]_3 = \widetilde{M}(0, -y_1y_3, -y_3y_2, 1-y_3^2)$, $[\overline{B}_0]_4 = 0_{4\times 4}$.

The limit $\lim_{s\to\infty} \frac{1}{s}F_{,y}^{-1}F_{,yy}(s)$ is denoted by $\overline{B}_\infty$ and expands to

$$[\overline{B}_\infty]_r = 0_{4\times 4} \text{ for } r = 1,2,3; \quad [\overline{B}_\infty]_4 = \operatorname{diag}(-y_4, -y_4, -y_4, 0).$$

Thus

$$\text{(20)} \quad \lim_{s\to 0}\left(\frac{1}{s^2}C_{\max}(s)\right) = \frac{1}{2\|L_0\|\|\overline{B}_0\|}, \quad \lim_{s\to\infty}(sC_{\max}(s)) = \frac{1}{2\|L_\infty\|\|\overline{B}_\infty\|}.$$

The graph of $\eta = \ln(C_{\max}(s)/s)$ over $\xi = \ln s$ has the asymptotes

$$\eta = \xi - \ln(2\|L_0\|\|\overline{B}_0\|) \qquad (\xi \to -\infty),$$
$$\eta = -2\xi - \ln(2\|L_\infty\|\|\overline{B}_\infty\|) \quad (\xi \to \infty).$$

They intersect in the point

$$\text{(21)} \quad (\xi_0, \eta_0) = \frac{1}{3}\left(\ln \frac{\|L_0\|\|\overline{B}_0\|}{\|L_\infty\|\|\overline{B}_\infty\|}, -\ln(8\|L_0\|^2\|\overline{B}_0\|^2\|L_\infty\|\|\overline{B}_\infty\|)\right).$$

We have $\xi_0 = \ln s_0$, where $s_0^3 = \|L_0\|\|\overline{B}_0\|/(\|L_\infty\|\|\overline{B}_\infty\|)$. This is illustrated in Fig. 8.

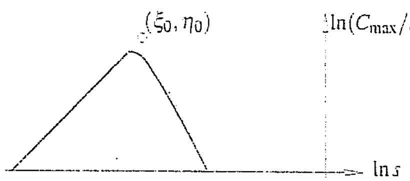

Fig. 8. Detail of Fig. 7 (c) (asymptotes).

6. Tables

geometric relation	number and nature of constraints involving more than one geometric entity		number and nature of constraints involving only one geometric entity	
$\operatorname{dist}(p,q)=d$	1	$\lVert p-q\rVert^2=d$	0	
$P\in L$	2	two of $\bar{l}=p\times l$	2	$\lVert l\rVert^2=1, \langle l,\bar{l}\rangle=0$
$P\in L$	3	$\bar{l}=p\times l$	1	$\lVert l\rVert^2=1$, ~~$\langle l,\bar{l}\rangle=0$~~
$Q=\operatorname{pedal}_L(P)$	4	$q\times l=\bar{l}, \langle p-q,l\rangle=0$	1	$\lVert l\rVert^2=1$, ~~$\langle l,\bar{l}\rangle=0$~~
$\overrightarrow{\operatorname{dist}}(P,U)=d$	1	$u_0+\langle u,p\rangle=d$	1	$\lVert u\rVert^2=1$
$P\in U$	1	$u_0+\langle u,p\rangle=0$	1	$\lVert u\rVert^2=1$
$\sphericalangle(G,H)=\theta$	1	$\langle g,h\rangle=\cos\theta$	4	$\lVert g\rVert^2=1, \lVert h\rVert^2=1,$ $\langle g,\bar{g}\rangle=0, \langle g,\bar{h}\rangle=0$
$G\parallel H,$	3	$g=\pm h,$	3	$\lVert g\rVert^2=1$, ~~$\lVert h\rVert^2=1$~~ $\langle g,\bar{g}\rangle=0, \langle g,\bar{h}\rangle=0$
$G\cap H\neq\{\}$	1	$\langle g,\bar{h}\rangle+\langle\bar{g},h\rangle=0$	4	$\lVert g\rVert^2=1, \lVert h\rVert^2=1$ $\langle g,\bar{g}\rangle=0, \langle g,\bar{h}\rangle=0$
$L\subset U$	3	$u\times\bar{l}=u_0 l$	3	$\lVert u\rVert^2=\lVert l\rVert^2=1, \langle l,\bar{l}\rangle=0$
$U\perp L$	3	$u=\pm l$	2	$\lVert u\rVert^2=1$, ~~$\lVert l\rVert^2=1$~~, $\langle l,\bar{l}\rangle=0$
$U\parallel V$	3	$u=\pm v$	1	$\lVert u\rVert^2=1$, ~~$\lVert v\rVert^2=1$~~

Table 1. Relations between points p,q, lines $L=(l,\bar{l})$, $G=(g,\bar{g})$, $H=(h,\bar{h})$, and planes $U=(u_0,u)$, $V=(v_0,v)$ (cf. Sec. 4.2).

geometric relation	number and nature of constraints involving more than one geometric entity		number and nature of constraints involving only one geometric entity	
$Q=\operatorname{pedal}_L(P)$ $[\lambda=\langle l,p\rangle]$	4	$\langle l,p\rangle=\lambda,$ $l\times\bar{l}+\lambda l=q$	2	$\lVert l\rVert^2=1, \langle l,\bar{l}\rangle=0$
$Q=\operatorname{pedal}_U(P)$ $[\lambda=\overrightarrow{\operatorname{dist}}_U(P,Q)]$	4	$u_0+\langle q,u\rangle=0$ $p-q=\lambda u,$	1	$\lVert u\rVert^2=1$
$G\parallel H,$ $\operatorname{dist}(G,H)=d$	4	$g=\pm h, \lVert\bar{g}\mp\bar{h}\rVert^2=d^2$	3	$\lVert g\rVert^2=1$, ~~$\lVert h\rVert^2=1$~~ $\langle g,\bar{g}\rangle=0, \langle h,\bar{h}\rangle=0$
$\operatorname{dist}(G,H)=d$ $[\lambda_2=\cos\sphericalangle(G,H)]$	2	$\langle g,h\rangle=\lambda_2,$ $\langle g,\bar{h}\rangle+\langle\bar{g},h\rangle=d\lambda_1$	5	$\lVert g\rVert^2=\lVert h\rVert^2=\lambda_1^2+\lambda_2^2=1$ $\langle g,\bar{g}\rangle=0, \langle g,\bar{h}\rangle=0$
$L=U\cap V$	6	$\lambda(l,\bar{l})=(u\times v, u_0 v-v_0 u)$	1	$\lVert l\rVert^2=1$, ~~$\langle l,\bar{l}\rangle=0$~~

Table 2. Relations becoming quadratic with new variables (cf. Sec. 4.2).

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	7.46	1.00	0.00	31.73	4.72	149.8
∞	1	24.39	3.00	0.00	76.86	1.62	124.8
∞	2	12.63	1.73	0.00	43.82	2.77	121.5
1	∞	1.00	1.00	0.00	13.94	4.91	68.5
1	1	2.73	1.00	0.00	29.71	4.89	145.2
1	2	1.41	1.00	0.00	18.10	4.90	88.8
2	∞	1.15	1.00	0.00	17.45	4.92	85.8
2	1	4.62	2.00	0.00	43.87	2.47	108.2
2	2	2.31	2.00	0.00	25.07	2.47	62.0

Table 3. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.1, where $s = 0.1$.

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	0.00	28.57	2.98	15.22	6.76	102.8
∞	1	0.00	31.43	1.21	33.51	6.94	232.6
∞	2	0.00	24.74	2.45	21.45	6.49	139.3
1	∞	0.00	7.14	2.98	7.07	17.70	125.2
1	1	0.00	8.47	1.21	8.39	26.84	225.1
1	2	0.00	7.16	2.44	7.11	20.42	145.2
2	∞	0.00	14.25	2.98	10.03	11.32	113.5
2	1	0.00	18.35	1.21	18.94	12.11	229.3
2	2	0.00	24.62	2.45	10.06	10.17	102.4

Table 4. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.2 (first variant), where $s = 0.001$.

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	0.00	2.00	4.43	2.98	6.45	19.23
∞	1	0.00	2.00	2.34	9.25	4.14	38.32
∞	2	0.00	1.73	3.77	4.66	5.08	23.65
1	∞	0.00	1.00	4.43	1.39	13.74	19.04
1	1	0.00	1.00	2.34	2.34	15.11	35.40
1	2	0.00	1.00	3.77	1.50	14.73	22.08
2	∞	0.00	1.41	4.43	1.97	9.69	19.05
2	1	0.00	2.65	2.34	4.33	7.67	33.20
2	2	0.00	2.65	3.77	2.03	9.51	19.32

Table 5. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.2 (second variant), where $s = 0.0051$.

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	0.00	4.00	1.73	2.31	6.25	14.43
∞	1	0.00	3.00	2.31	5.20	3.33	17.32
∞	2	0.00	2.89	1.15	2.89	8.04	23.20
1	∞	0.00	0.67	1.73	0.38	37.50	14.43
1	1	0.00	1.67	2.31	0.96	12.86	12.37
1	2	0.00	0.89	1.15	0.51	34.02	17.32
2	∞	0.00	1.73	1.73	1.00	14.44	14.43
2	1	0.00	1.81	2.31	1.81	8.35	15.11
2	2	0.00	1.63	1.15	1.00	17.94	17.94

Table 6. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.3, where $s = 0.01$.

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	5.66	0.00	5.66	3.27	26.4	86.4
∞	1	20.01	0.00	2.43	7.53	26.4	198.7
∞	2	8.53	0.00	4.17	3.70	31.2	115.5
1	∞	1.06	0.00	5.66	0.82	101.1	82.6
1	1	1.82	0.00	2.43	1.84	106.2	195.1
1	2	1.06	0.00	4.17	1.06	106.9	113.4
2	∞	1.06	0.00	5.66	1.41	61.1	86.4
2	1	7.44	0.00	2.43	3.21	59.7	191.7
2	2	2.81	0.00	4.17	1.41	78.1	110.5

Table 7. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.4, where $s =$ $= 0.001$.

$\|\cdot\|_U$	$\|\cdot\|_V$	$\|F_{,y}^{-1}F_{,xx}\|$	$\|F_{,y}^{-1}F_{,xy}\|$	$\|F_{,y}^{-1}F_{,yy}\|$	$\|G_{,x}\|$	$C_{\max}/s$	C'/s
∞	∞	0.00	0.00	4.43	2.98	7.42	22.15
∞	1	0.00	0.00	1.85	8.04	6.58	52.86
∞	2	0.00	0.00	2.85	4.53	7.58	34.38
1	∞	0.00	0.00	4.43	1.39	15.97	22.15
1	1	0.00	0.00	1.85	1.56	33.82	52.86
1	2	0.00	0.00	2.85	1.39	24.70	34.38
2	∞	0.00	0.00	4.43	1.97	11.27	22.15
2	1	0.00	0.00	1.85	3.64	14.53	52.86
2	2	0.00	0.00	2.85	1.97	17.44	34.38

Table 8. Experimental values for various norms and the values $C_{\max}$ and C' according for the constraint problem of Sec. 5.5 ($s = 0.0051$).

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