Mathematica Pannonica
16/2 (2005), 205–209

# ON THE NUMBER OF PRIME DIVISORS OF THE ITERATES OF THE CARMICHAEL FUNCTION

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**Dedicated to Professor László Leindler on his 70th anniversary**

*Received*: May 2005

*MSC 2000*: 11 N 60

*Keywords*: Carmichael function, limit distribution, iterates.

**Abstract**: Let $\lambda(n)$ be the Carmichael function, $\lambda_k(n)$ be its $k$-fold iterate, $\omega(n)$ be the number of prime factors of $n$. Let
$\mu_k(n) := \dfrac{\omega(\lambda_k(n)) - a_k(\log\log n)^{k+1}}{b_k \cdot (\log\log n)^{k+1/2}}$, $a_k = \dfrac{1}{(k+1)!}$, $b_k = \dfrac{1}{\sqrt{2k+1}} \cdot \dfrac{1}{k!}$.

It is proved that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x \mid \mu_k(n) < y\} = \Phi(y),$$

and that

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x} \#\{p \le x \mid \mu_k(p+a) < y\} = \Phi(y),$$

where $p$ runs over the set of primes, $a \ne 0$, $a$ integer, $\Phi$ is the Gaussian law.

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Research supported by the Applied Number Theory Research Group of the Hungarian Academy of Sciences and by a grant from OTKA T46993.

## § 1. Introduction

Let $\mathcal{P}$ be the set of primes, $p, q$ with and without suffixes denote primes.

The so called Carmichael function $\lambda$ is defined for prime powers $p^\alpha$ according to

$$\lambda(p^\alpha) = \begin{cases} p^{\nu-1}(p-1) & \text{if } p \geq 3, \text{ or } \nu \leq 2, \\ 2^{\nu-2} & \text{if } p = 2 \text{ and } \nu \geq 3, \end{cases}$$

and for $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$,

$$\lambda(n) = LCM\left[\lambda(p_1^{\alpha_1}), \dots, \lambda(p_r^{\alpha_r})\right],$$

if $p_1, \dots, p_r$ are distinct primes. Here $LCM$ = least common multiple.

Let $\omega(n)$ be the number of prime factors of $n$, and $\varphi(n)$ be Euler's totient function.

Let $\lambda_k(n) = \lambda(\lambda_{k-1}(n))$, $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ $(k = 2, 3, \dots)$ be the $k$-fold iterate of $\lambda$ and $\varphi$.

Let $x_1 = \log x$, $x_2 = \log x_1$, $x_3 = \log x_2, \dots$ . Let $P(n)$ be the largest prime divisor of $n$.

In [1] it was proved

**Theorem A.** *Let $k \geq 1$ be a fixed integer, $a_k = \frac{1}{(k+1)!}$, $b_k = \frac{1}{\sqrt{2k+1}} \cdot \frac{1}{k!}$, and*

$$\nu_k(n) := \frac{\omega(\varphi_k(n)) - a_k(\log\log n)^{k+1}}{b_k(\log\log n)^{k+1/2}}. \tag{1.1}$$

*Then $\nu_k(n)$ is distributed according to the Gaussian law, i.e.*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x \mid \nu_k(n) < y\} = \Phi(y). \tag{1.2}$$

*Furthermore, if $a$ is a nonzero integer, then*

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x}\#\{p \leq x \mid \nu_k(p+a) < y\} = \Phi(y). \tag{1.3}$$

In this short paper hence we deduce

**Theorem 1.** *Let $k \geq 1$ be a fixed integer, $a_k = \frac{1}{(k+1)!}$, $b_k = \frac{1}{\sqrt{2k+1}} \cdot \frac{1}{k!}$ and*

$$\mu_k(n) := \frac{\omega(\lambda_k(n)) - a_k(\log\log n)^{k+1}}{b_k(\log\log n)^{k+1/2}} \tag{1.4}$$

*Then*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \le x \mid \mu_k(n) < y\} = \Phi(y), \tag{1.5}$$

*and for every nonzero integer* $a$,

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x}\#\{\mu_k(p+a) < y\} = \Phi(y). \tag{1.6}$$

## § 2. Lemmata

**Lemma 1.** (Brun–Titchmarsh inequality.) *Let* $\pi(x,k,l) = \#\{p \le x,\ p \equiv l \pmod k\}$. *Then, for* $k < x$, $(l,k) = 1$,

$$\pi(x,k,l) < c\frac{x}{\varphi(k)\log\frac{x}{k}},$$

*where* $c$ *is an absolute constant.*

**Lemma 2.** *Let* $a \ne 0$ *be a fixed integer. Then for* $0 < \delta < 1/2$

$$\#\{p < x \mid P(p+a) > x^{1-\delta}\} < c\delta \operatorname{li} x,$$

*where* $c$ *may depend only on* $a$.

The proof of Lemma 1 can be found in [2], and Lemma 2 can be deduced from Cor. 2.4.1 in [2].

**Lemma 3.** *Let* $q$ *be an arbitrary prime,* $q < x$. *Then*

$$\sum_{\substack{p \le x \\ p \equiv 1 \pmod q}} \frac{1}{p} < c\frac{x_2}{q},$$

*where* $c$ *is an absolute constant.*

This is known. A proof is given in [1].

## § 3. Proof of Theorem 1

From the definition we have:

a. if $d|n$, then $\varphi(d)|\varphi(n)$, $\lambda(d)|\lambda(n)$, and

b. $\lambda(n)|\varphi(n)$.

Hence, by induction on $k$,

$$\lambda_k(n) \mid \varphi_k(n) \quad (k = 1, 2, \dots), \tag{3.1}$$

and so

$$\Delta_k(n) := \omega(\varphi_k(n)) - \omega(\lambda_k(n)) \quad (k = 1, 2, \dots) \tag{3.2}$$

is nonnegative.

If $q_0 | \varphi_k(n)$, then either there exists $q_1 \equiv 1 \pmod{q_0}$, $q_1 | \varphi_{k-1}(n)$, or $q_0^2 | \varphi_{k-1}(n)$, whence especially $q_0 | \varphi_{k-1}(n)$.

Continuing this argument, we obtain that $q_0 | \varphi_k(n)$ implies the existence of a "chain of primes" (defined in [1]): $q_0, q_1, \dots, q_h$ such that $q_j - 1 \equiv 0 \pmod{q_{j-1}}$ $(j = 1, \dots, h)$, and $q_h | n$, and the length $h \le k$.

Let us observe that if the chain is of maximal length, i.e. $h = k$, then $q_0 | \lambda_k(n)$ holds as well.

Thus

$$\sum_{n \le x} \Delta_k(n) \le \sum_{h=0}^{k-1} \sum_{q_0 \to \dots \to q_h} \frac{x}{q_h}. \tag{3.3}$$

We observe that

$$\sum_{q_0 \to \dots \to q_h} \frac{1}{q_h} \le c_1 x_2 \sum \frac{1}{q_{h-1}} \le \dots \le c_1^h x_2^{h+1},$$

whence (3.3) is less than $O(x x_2^k)$. Hence we obtain that

$$\frac{1}{x} \# \left\{ n \le x \;\middle|\; |\Delta_k(n)| > x_2^{k+1/4} \right\} = O\left( \frac{x}{x_2^{1/4}} \right),$$

and so

$$\nu_k(n) - \mu_k(n) = O\left( \frac{1}{x_2^{1/4}} \right) \quad \text{for all but} \quad O\left( \frac{x}{x_2^{1/4}} \right)$$

integers $n \le x$. Hence (1.5) is straightforward (since $\phi$ is a continuous function).

To prove (1.6) we argue similarly. First we choose a small $\delta > 0$ and drop all the primes $p \le x$ for which $P(p + a) > x^{1-\delta}$, the size of which is $O(\delta \operatorname{li} x)$. Let $\mathcal{B}_\delta$ be the set of primes $p \le x$ which remain.

As earlier, we have

$$\sum_{p \in \mathcal{B}_\delta} \Delta_k(p + a) \le \sum_{h=0}^{k-1} \sum_{\substack{q_0 \to \dots \to q_h \\ q_h \le x^{1-\delta}}} \pi(x, q_h, -a).$$

Applying the Burn–Titchmarsh inequality, the right-hand side is less than

$$\ll \frac{1}{\delta}\operatorname{li} x \sum_{h=0}^{k-1} \sum_{\substack{q_0 \to \dots \to q_h \\ q_h \le x}} \frac{1}{q_h} \ll \frac{\operatorname{li} x}{\delta} x_2^k,$$

and so

$$\#\left\{p \le x \mid \; p \in \mathcal{B}_\delta, \; |\Delta_k(p+a)| > \frac{x_2^{1/4}}{\delta}\right\} = O\left(\operatorname{li} x \cdot \frac{1}{x_2^{1/4}}\right).$$

Since the density of the primes which were dropped is $O(\delta)$, therefore

$$\limsup \frac{1}{\operatorname{li} x}\#\{\mu_k(p+a) < y\} \le \Phi(y+\delta),$$

and

$$\liminf \frac{1}{\operatorname{li} x}\#\{\mu_k(p+a) < y\} > \Phi(y-\delta).$$

The inequalities hold for every $\delta > 0$, therefore (1.6) is true.

The proof of the theorem is completed. $\Diamond$

## References

[1] BASSILY, N. L., WIJSMULLER, M. and KÁTAI, I.: Number of prime divisors $\varphi_k(n)$ where $\varphi_k$ is the $k$-fold iterate of $\varphi$, *J. Number Theory* **65** (1997), 226–239.

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