

DISCRETE ORTHOGONALITY OF ZERNIKE FUNCTIONS

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Abstract: In this paper we introduce a set of points within the unit circle and a discrete measure and we prove that the Zernike polynomials of degree less than $2N$ are discrete orthogonal regarding to the discrete scalar product induced by this discrete measure. We prove that the limit of our discrete measure when $N \rightarrow \infty$ is the continuous measure over the unit circle. Using the discrete orthogonality property of Zernike polynomials we can compute the exact values of Zernike moments of T_N , where T_N is an arbitrary linear combination of Zernike polynomials of degree less than $2N$.

1. Introduction. The circle polynomials of Zernike were introduced by Zernike in 1934 (see [10]). A short summary regarding to these polynomials we can find for example in [1], [10] and [11]. First of all we summarize the most important properties of these functions.

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There exists an infinity of complete sets of polynomials in two real variables x, y which are orthogonal in the interior of the unit circle, i.e. which satisfy the orthogonality condition

$$(1.1) \quad \int \int_{x^2+y^2 \leq 1} V_{(\alpha)}(x, y) \overline{V_{(\beta)}(x, y)} dx dy = A_{\alpha\beta} \delta_{\alpha\beta}.$$

The circle polynomials of Zernike are distinguished from the other sets by certain simple invariance properties which can be explained from group theoretical considerations. The circle polynomials of Zernike are invariant in form with respect to rotations of axes about origin, which means that when any rotation

$$(1.2) \quad \begin{cases} x' = x \cos \theta + y \sin \theta, \\ y' = -x \sin \theta + y \cos \theta, \end{cases}$$

is applied, each polynomial $V_{(\alpha)}(x, y)$ is transformed into a polynomial of the same form, i.e.

$$(1.3) \quad V_{(\alpha)}(x, y) = G(\theta) V_{(\alpha)}(x', y'),$$

where $G(\theta)$ is a continuous function with period 2π of the angle of rotation θ and $G(0) = 1$. The application of two successive rotations through angles θ_1 and θ_2 is equivalent to a single rotation through an angle $\theta_1 + \theta_2$. Hence it follows from (1.3) that G must satisfy the functional equation

$$(1.4) \quad G(\theta_1)G(\theta_2) = G(\theta_1 + \theta_2).$$

The general solution with period 2π of this equation is

$$(1.5) \quad G(\theta) = e^{i\ell\theta}, \quad \ell \in \mathbb{Z}.$$

Substituting (1.5) in (1.3), setting $x' = \rho$, $y' = 0$ and using (1.2) we obtain that

$$(1.6) \quad V_{\alpha}(\rho \cos \theta, \rho \sin \theta) = R_{\alpha}(\rho) e^{i\ell\theta},$$

where $R_{\alpha}(\rho) = V(\rho, 0)$ is a function of ρ . If $V(x, y)$ is a polynomial of degree n in variables $x = \rho \cos \theta$, $y = \rho \sin \theta$, then from (1.6) it follows that $R_{\alpha}(\rho) := R_n^{\ell}(\rho)$ is a polynomial in ρ of degree n and contains no power lower than $|\ell|$ and these are called the Zernike radial polynomials. Moreover R_{α} is an even or an odd polynomial according as ℓ is even or odd. Consequently the index α can be replaced by two indices n and ℓ . For any given value ℓ the index n can only take the

values $|\ell|, |\ell| + 2, |\ell| + 4, \dots$, and the corresponding radial polynomials $R_{|\ell|}^\ell, R_{|\ell|+2}^\ell, R_{|\ell|+4}^\ell, \dots$ may be obtained by orthogonalizing the powers

$$(1.7) \quad \rho^{|\ell|}, \rho^{|\ell|+2}, \rho^{|\ell|+4}, \dots$$

with the weighting factor ρ over interval $0 \leq \rho \leq 1$. Since in (1.7) occurs only the absolute value of ℓ ,

$$R_n^{-\ell} = R_n^\ell.$$

We will denote by

$$(1.8) \quad \{Y_n^\ell(\rho, \theta) := \sqrt{2n + |\ell| + 1} R_{|\ell|+2n}^{|\ell|} e^{i\ell\theta}, \ell \in \mathbb{Z}, n \in \mathbb{N}, |\ell| + 2n < 2N\}$$

the set of Zernike polynomials of degree less than $2N$.

This set contains $N(2N + 1)$ linearly independent two variables polynomials of degree less than $2N$. Hence every monomial $x^i y^j$, ($i, j \in \mathbb{N}$) and consequently every polynomial in x, y may be expressed as a linear combination of finite number of circle polynomials Y_n^ℓ , consequently the product of two such polynomials $Y_n^\ell Y_{n'}^{\ell'}$ can be expressed also as a finite linear combination of circle Zernike polynomials.

The orthogonality relation for radial Zernike polynomials is

$$(1.9) \quad \int_0^1 R_{|\ell|+2n}^{|\ell|}(\rho) R_{|\ell|+2n'}^{|\ell|}(\rho) \rho d\rho = \frac{1}{2(|\ell| + 2n + 1)} \delta_{nn'},$$

and the orthogonality relation in polar coordinates for Zernike polynomials is the following:

$$(1.10) \quad \frac{1}{\pi} \int_0^{2\pi} \int_0^1 Y_n^\ell(\rho, \phi) Y_{n'}^{\ell'}(\rho, \phi) \rho d\rho d\phi = \delta_{nn'} \delta_{\ell\ell'}.$$

The radial terms $R_{|\ell|+2n}^{|\ell|}(\rho)$ are related to the Jacobi polynomials in the following way:

$$(1.11) \quad R_{|\ell|+2n}^{|\ell|}(\rho) = \rho^{|\ell|} P_n^{(0, |\ell|)}(2\rho^2 - 1).$$

Zernike polynomials are often used to express wavefront data on optical tests, since they are made up of terms that are of the same form as the types of aberrations often observed in optical tests. In [2] it is presented that first order wavefront aberrations coefficients can be obtained from Zernike polynomials expansion coefficients, which are often called Zernike moments of the wavefront. In order to find the Zernike coefficients for a wavefront there are used well known approximation

processes as the minimum square fit method, finite element method applied on the set of points which form an equidistant division along the radial interval and angular interval (see [3], [4], [5]). In [6] the approximation of Zernike moments is made on a set of points which corresponds to the equidistant divisions along the Ox and Oy of the $[-1, 1] \times [-1, 1]$ which belong to the unit circle. Because the Zernike polynomials are orthogonal in a continuous fashion over the interior of the unit circle and they are not orthogonal over the discrete points considered in previously mentioned papers, the computations give only an approximation of Zernike moments.

An open question, as in [2] is mentioned, is to find a set X of points within the unit circle so that the Zernike polynomials on this set to have the discrete orthogonality property. In what follows we will introduce this set of points and a discrete measure so that the set of functions given by (1.8) will be discrete orthogonal on this set of points regarding to the discrete scalar product induced by discrete measure.

In construction we will use similar technique as in our earlier paper connected to spherical functions (see [7]). For this purpose we will need the following quadrature formula which can be found in [8].

Theorem A. *Let denote by $\lambda_k^N \in (-1, 1)$, $k \in \{1, \dots, N\}$ the roots of Legendre polynomials P_N of order N , and for $j = 1, \dots, N$, let*

$$(1.12) \quad \ell_j^N(x) := \frac{(x - \lambda_1^N) \dots (x - \lambda_{j-1}^N)(x - \lambda_{j+1}^N) \dots (x - \lambda_N^N)}{(\lambda_j^N - \lambda_1^N) \dots (\lambda_j^N - \lambda_{j-1}^N)(\lambda_j^N - \lambda_{j+1}^N) \dots (\lambda_j^N - \lambda_N^N)},$$

be the corresponding fundamental polynomials of Lagrange interpolation. Denote by

$$(1.13) \quad A_k^N := \int_{-1}^1 \ell_k^N(x) dx, \quad (1 \leq k \leq N),$$

the corresponding Cristoffel-numbers. Then for every polynomial f of order less then $2N$,

$$(1.14) \quad \int_{-1}^1 f(x) dx = \sum_{k=1}^N f(\lambda_k^N) A_k^N.$$

2. Discretisation. Let define the following numbers with the help of the roots of Legendre polynomials of order N

$$(2.1) \quad \rho_k^N := \sqrt{\frac{1 + \lambda_k^N}{2}}, \quad k = \overline{1, N}$$

and the set of nodal points

$$(2.2) \quad X := \left\{ z_{jk} := \left(\rho_k^N, \frac{2\pi j}{4N + 1} \right), \quad k = \overline{1, N}, \quad j = \overline{0, 4N} \right\}.$$

We define

$$\nu(z_{jk}) := \frac{\mathcal{A}_k^N}{2(4N + 1)}$$

and we introduce the following discrete integral

$$(2.3) \quad \int_X f(\rho, \phi) d\nu_N := \sum_{k=1}^N \sum_{j=0}^{4N} f\left(\rho_k^N, \frac{2\pi j}{4N + 1}\right) \frac{\mathcal{A}_k^N}{2(4N + 1)}.$$

Theorem 2.1. *If $n + n' + |m| \leq 2N - 1, n + n' + |m'| \leq 2N - 1, n, n' \in \mathbb{N}, m, m' \in \mathbb{Z}$, then*

$$(2.4) \quad \int_X Y_n^m(\rho, \phi) Y_{n'}^{m'}(\rho, \phi) d\nu_N = \delta_{nn'} \delta_{mm'}.$$

Proof. Due to (1.9) and (1.11)

$$(2.5) \quad \begin{aligned} \frac{1}{2(2n + |m| + 1)} \delta_{nn'} &= \int_0^1 R_{2n+|m|}^{|m|}(\rho) R_{2n'+|m|}^{|m|}(\rho) \rho d\rho = \\ &= \int_0^1 \rho^{2|m|} P_n^{(0, |m|)}(2\rho^2 - 1) P_{n'}^{(0, |m|)}(2\rho^2 - 1) \rho d\rho. \end{aligned}$$

If in this last integral we make the change of variable $u := 2\rho^2 - 1$, then we obtain the following:

$$(2.6) \quad \frac{1}{2(2n + |m| + 1)} \delta_{nn'} = \frac{1}{4} \int_0^1 \left(\frac{1+u}{2}\right)^{|m|} P_n^{(0, |m|)}(u) P_{n'}^{(0, |m|)}(u) du.$$

Let denote by $f(\rho) := \left(\frac{1+u}{2}\right)^{|m|} P_n^{(0, |m|)}(u) P_{n'}^{(0, |m|)}(u)$ and $\rho_k^N := \sqrt{\frac{1+\lambda_k^N}{2}}$ $k = \overline{1, N}$. Then the order of f is $n + n' + |m|$. From (1.1) follows that $Y_N^0(\rho_k^N, \phi) = P_N^{(0, 0)}(2(\rho_k^N)^2 - 1) = P_N(\lambda_k^N) = 0$. If $n + n' + |m| \leq 2N - 1$ then we can apply Th. A and we obtain

$$\begin{aligned}
 (2.7) \quad & \frac{1}{2(2n + |m| + 1)} \delta_{nn'} = \int_0^1 R_{2n+|m|}^{|m|}(\rho) R_{2n'+|m|}^{|m|}(\rho) \rho d\rho = \\
 & = \frac{1}{4} \sum_{k=1}^N f(\lambda_k^N) \mathcal{A}_k^N = \frac{1}{4} \sum_{k=1}^n \mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N) R_{2n'+|m|}^{|m|}(\rho_k^N)
 \end{aligned}$$

$$\begin{aligned}
 (2.8) \quad & \int_X Y_n^m(\rho, \phi) \overline{Y_{n'}^{m'}(\rho, \phi)} d\nu_N = \\
 & = \sum_{k=1}^N \sum_{j=0}^{4N} Y_n^m \left(\rho_k^N, \frac{2\pi j}{4N+1} \right) \overline{Y_{n'}^{m'} \left(\rho_k^N, \frac{2\pi j}{4N+1} \right)} \frac{\mathcal{A}_k^N}{2(4N+1)} = \\
 & = \sum_{j=0}^{4N} e^{i(m-m') \frac{2\pi j}{4N+1}} \frac{\sqrt{2n + |m| + 1} \sqrt{2n' + |m'| + 1}}{2(4N+1)} \times \\
 & \quad \times \sum_{k=1}^N \mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N) R_{2n'+|m'|}^{|m'|}(\rho_k^N).
 \end{aligned}$$

If $m \neq m'$ the first sum it is equal to 0, and if $m = m'$ then it is equal to $4N + 1$. Taking into account this and (2.7) we obtain

$$\begin{aligned}
 & \int_X Y_n^m(\rho, \phi) Y_{n'}^{m'}(\rho, \phi) d\nu_N = \\
 & = \delta_{mm'} \frac{\sqrt{2n + |m| + 1} \sqrt{2n' + |m'| + 1}}{2} \sum_{k=1}^N \mathcal{A}_k^N R_{2n+|m|}^{|m|}(\rho_k^N) R_{2n'+|m'|}^{|m'|}(\rho_k^N) = \\
 & = \delta_{mm'} 2 \sqrt{2n + |m| + 1} \sqrt{2n' + |m'| + 1} \int_0^1 R_{2n+|m|}^{|m|}(\rho) R_{2n'+|m'|}^{|m'|}(\rho) \rho d\rho = \\
 & \quad = \delta_{mm'} \delta_{nn'}.
 \end{aligned}$$

Theorem 2.2. For all $f \in C(\overline{D})$

$$\lim_{N \rightarrow \infty} \int_X f d\nu_N = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(\rho, \phi) \rho d\rho d\phi.$$

Proof. Let denote by $U = C(\overline{D})$ and introduce the bounded linear functionals $A_N(f) = \int_X f d\nu_N$, $A(f) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(\rho, \phi) \rho d\rho d\phi$. Th. 2.2 is a consequence of the Banach-Steinhaus theorem. We will check that all conditions of this theorem are satisfied for the functionals $A_N : U \rightarrow \mathbb{C}$ and $A : U \rightarrow \mathbb{C}$. Let denote by Z the set of all Zernike circle polynomials. It can be proved that Z is a dense subset of $C(\overline{D})$ on

the base of Stone–Weierstrass theorem, because of the points of $C(\overline{D})$ are separated by the functions in Z . Namely, if $(\rho, \phi) \neq (\rho', \phi')$, $\rho, \rho' \in [0, 1]$, $\phi, \phi' \in [0, 2\phi]$, then $Y_0^1(\rho, \phi) \neq Y_0^1(\rho', \phi')$. As in introduction we have mentioned, the product of two Zernike functions can be expressed as a finite linear combination of Zernike functions. From [Szegő, p. 48 (3.4.5)] it follows that A_N is a bounded linear operator, namely

$$\|A_N\| = \sum_{k=1}^N \sum_{j=0}^{4N} \frac{|A_k^N|}{2(4N+1)} = \sum_{k=1}^N \frac{|A_k^N|}{2} = 1 < \infty.$$

From the orthonormality property it follows that for all $z = Y_n^m \in Z$ and for all N so that $2n + |m| < 2N - 1$ we have $A_N(z) - A(z) = 0$, consequently $\lim_{N \rightarrow \infty} |A_N(z) - A(z)| = 0, z \in Z$. Applying the Banach–Steinhaus theorem we get that

$$|A_N(f) - A(f)| \rightarrow 0, \quad \text{for all } f \in C(\overline{D}), N \rightarrow \infty. \quad \diamond$$

In fact, Th. 2.2 means that the limit of the $(0, 0)$ -th discrete Zernike coefficient is equal by the $(0, 0)$ -th continuous Zernike coefficient. In an analogous way can be proved that in general the discrete Zernike coefficients of the function from $C(S^2)$ tend to the corresponding continuous Zernike coefficients of $f \in C(\overline{D})$.

3. Zernike moments. Let

$$T_N(\rho, \phi) = \sum_{2n+|m| \leq 2N-1} A_{mn} Y_n^m(\rho, \phi)$$

be an arbitrary linear combination of Zernike polynomials of degree less than $2N$. Using the discrete orthogonality (2.4) and the continuous orthogonality property (1.10) we obtain that the coefficients A_{mn} can be expressed in the following two ways:

$$(3.1) \quad A_{mn} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T_N(\rho', \phi') \overline{Y_n^m(\rho', \phi')} \rho' d\rho' d\phi',$$

$$(3.2) \quad A_{mn} = \int_X T_N(\rho', \phi') \overline{Y_n^m(\rho', \phi')} d\nu_N(\rho', \phi').$$

With formula (3.2) we can determine the exact value of the Zernike coefficients (moments) of T_N if we can measure the values of T_N on the points of the set X . This means that with the construction of the set X we give answer to the question where the Placido ring system is worth situated.

From (3.1) and (3.2) it follows that

$$\begin{aligned}
 (3.3) \quad T_N(\rho, \phi) &= \\
 &= \sum_{2n+|m| \leq 2N-1} \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T_N(\rho', \phi') \overline{Y_n^m(\rho', \phi')} \rho' d\rho' d\phi' Y_n^m(\rho, \phi) = \\
 &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T_N(\rho', \phi') \sum_{2n+|m| \leq 2N-1} \overline{Y_n^m(\rho', \phi')} Y_n^m(\rho, \phi) \rho' d\rho' d\phi'
 \end{aligned}$$

and

$$\begin{aligned}
 (3.4) \quad T_N(\rho, \phi) &= \sum_{2n+|m| \leq 2N-1} \int_X T_N(\rho', \phi') \overline{Y_n^m(\rho', \phi')} d\nu_N(\rho', \phi') Y_n^m(\rho, \phi) = \\
 &= \int_X T_N(\rho', \phi') \sum_{2n+|m| \leq 2N-1} \overline{Y_n^m(\rho', \phi')} Y_n^m(\rho, \phi) d\nu_N(\rho', \phi').
 \end{aligned}$$

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