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A NOTE ON PARABOLIC SUB-GROUPS

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Abstract: We investigate parabolic subgroups of connected semisimple Lie groups with finite center using the axiomatic approach of Tits to the theory of parabolic subgroups. The main result establishes an isomorphism between the group of Tits generators of the parabolic subgroups of a semisimple Lie group G and a certain subgroup of the Weyl group determined by a root space decomposition of the Lie algebra $\mathfrak{g}$ of G.

1. Tits systems and parabolic subgroups

In this section we present briefly the axiomatic approach of Tits to the theory of parabolic subgroups of groups (not necessarily endowed with a topology). The reader can find more details in [3].

1.1. Tits systems. Let G be a group and P, H subgroups of G. The triple (G, P, H) is called a Tits system with Weyl group W if the following conditions are satisfied:

(i) G is generated by P and H,
(ii) $N = P \cap H$ is a normal subgroup of H,
(iii) there exists a finite subset Δ of the group $W := H/N$ having the following properties:

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(1) Δ consists of elements of order 2 and generates W,
(2) $hPh^{-1} \neq P$ for any $hN \in \Delta$,
(3) $h'Ph \subseteq (PhP) \cup (Ph'hP)$ for any $h'N \in \Delta$ and all $hN \in W$.

1.2. Convention. Consider $w = hN \in W$. Since $N \subseteq P$ we shall write, by a slight abuse of notation, wP (Pw) for the set hP (Ph). In the same vein, we shall identify an element w of the Weyl group with any of its representatives $h \in H$, where $w = hN$.

1.3. Parabolic subgroups. Let (G, P, H) be a Tits system with Weyl group W. The conjugates of P in G are called the minimal parabolic subgroups of G. A parabolic subgroup of G is any subgroup of G containing a minimal parabolic subgroup. Let Δ be the (finite) set of generators of W mentioned in condition (iii) of 1.1. For every subset Θ of Δ denote by W_Θ the subgroup of W generated by Θ, and by P_Θ the set $PW_\Theta P$. Condition (3) of 1.1 implies that P_Θ is a subgroup of G, hence $P_\Delta = G$. Of course, $P_\emptyset = P$. According to the following theorem the subgroups P_Θ and their conjugates are the only parabolic subgroups of G. (For details see Sect. 1.2.1 of [3].)

1.4. Theorem. *Let (G, P, H) be a Tits system with Weyl group W, and let Δ be the finite set of generators of W mentioned in condition* (iii) *of* 1.1. *Then the following assertions hold:*

(i) *The map $w \mapsto PwP$ ($w \in W$) is a bijection between W and the set of double cosets PxP ($x \in G$) of G.*
(ii) *The subgroups P_Θ, $\Theta \subseteq \Delta$, are the only subgroups of G containing P.*
(iii) *If Θ_1 and Θ_2 are distinct subsets of Δ then P_{Θ_1} and P_{Θ_2} are not conjugate to one another.*
(iv) *Every parabolic subgroup of G is its own normalizer.*

1.5. Corollary. *With the notation of the preceding theorem, if $\Theta \subseteq \Delta$ then the map $h \in P_\Theta \cap H \mapsto hN \in W$ is a surjection onto W_Θ.*

Proof. We have only to show that $hN \in W_\Theta$ whenever $h \in P_\Theta \cap H$. Consider $h \in P_\Theta \cap H$. Since $P_\Theta = PW_\Theta P$, there exists $w \in W_\Theta$ such that $h \in PwP$. Thus $PhP = PwP$. In view of assertion (i) of Th. 1.4 we conclude that $hN = w \in W_\Theta$. $\Diamond$

1.6. Remarks. 1) The above corollary allows us to recover the subgroup of W that generates a given parabolic subgroup: Suppose that P' is a subgroup of G containing P. Then, due to assertion (ii) of Th. 1.4, there exists a subset Θ of Δ such that $P' = P_\Theta$. By Cor. 1.5 we know that $W_\Theta = \{hN \mid h \in P' \cap H\}$. The group W_Θ is called the group of Tits generators of P'.

2) We shall see that semisimple Lie groups admit Tits systems, and therefore we can consider parabolic subgroups in these groups. For this we have to recall some basic facts about orthogonal linear maps and root systems, and a few results on the structure of semisimple Lie algebras and Lie groups.

2. Basic facts about orthogonal linear maps and root systems

Throughout this section we are concerned with a fixed finite dimensional real vector space V endowed with the scalar product $\langle\cdot,\cdot\rangle$. Denote by V^* the vector space dual of V, by $\mathrm{End}(V)$ the space of endomorphisms (linear maps) $V \to V$, and by $\mathrm{Gl}(V)$ the group of automorphisms (bijective endomorphisms) of V. We recall that a map $f\colon V \to V$ is called orthogonal if $\langle f(v), f(w)\rangle = \langle v, w\rangle$ for every $v, w \in V$. It follows that every orthogonal endomorphism of V is an automorphism.

2.1. Example. Important examples of orthogonal linear maps are the reflections. Geometrically, a reflection is an automorphism of V leaving pointwise fixed some hyperplane (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative. Evidently a reflection is orthogonal. Any nonzero vector $v \in V$ determines a reflection σ_v with reflecting hyperplane $\{w \in V \mid \langle w, v\rangle = 0\}$. The explicit formula for σ_v is

$$\sigma_v(w) = w - \frac{2\,\langle w, v\rangle}{\langle v, v\rangle} v, \text{ for every } w \in V.$$

2.2. Definition. Define the maps $\varphi\colon V \to V^*$ and $\psi\colon \mathrm{End}(V) \to$ $\to \mathrm{End}(V^*)$ by

$$\varphi(v) = \langle\cdot, v\rangle\,, \text{ for all } v \in V, \quad \psi(f) = \varphi \circ f \circ \varphi^{-1}, \text{ for all } f \in \mathrm{End}(V).$$

These maps are isomorphisms of vector spaces. Moreover, when restricted to $\mathrm{Gl}(V)$, then ψ becomes a group isomorphism onto $\mathrm{Gl}(V^*)$. Of course the scalar product $\langle\cdot,\cdot\rangle$ can be transferred via φ to a scalar product on V^*.

2.3. Lemma. *If $f \in \mathrm{End}(V)$ is orthogonal then the equality $\varphi(f(v)) \circ$ $\circ f = \varphi(v)$ holds for every $v \in V$.*

Proof. Pick an arbitrary $w \in V$. Then

$$(\varphi(f(v)) \circ f)(w) = \langle f(w), f(v)\rangle = \langle w, v\rangle = \varphi(v)(w),$$

so the equality follows. ◊

2.4. Corollary. *If $g \in \mathrm{End}(V^*)$ is orthogonal and $\alpha \in V^*$ then*

$$g(\alpha) \circ \psi^{-1}(g) = \alpha.$$

Hence $g(\alpha) = \alpha \circ \psi^{-1}(g^{-1})$.

Proof. Consider $g \in \mathrm{End}(V^*)$ an orthogonal linear map and an arbitrary functional $\alpha \in V^*$. An easy computation yields that the map $\psi^{-1}(g) \in \mathrm{End}(V)$ is also orthogonal. Thus the assertion follows replacing in Lemma 2.3 the map f by $\psi^{-1}(g)$, and the vector v by $\varphi^{-1}(\alpha)$. ◊

2.5. Root systems. A root system in V is a finite set Φ of nonzero vectors in V (called roots) such that

(i) Φ spans V,

(ii) the reflections σ_v, $v \in \Phi$, leave Φ invariant,

(iii) $\frac{2\langle u,v\rangle}{\langle v,v\rangle}$ is an integer whenever u and v are in Φ.

2.6. The Weyl group of a root system. Let Φ be a root system in V. The subgroup $\mathcal{W}(\Phi)$ of $\mathrm{Gl}(V)$ generated by the reflections σ_v, $v \in \Phi$, is called the Weyl group of Φ. By condition (ii) of 2.5 the group $\mathcal{W}(\Phi)$ permutes the set Φ, which is finite and spans V. This allows us to identify $\mathcal{W}(\Phi)$ with a subgroup of the symmetric group on Φ; in particular, $\mathcal{W}(\Phi)$ is finite.

2.7. Base of a root system. Let Φ be a root system in V. A subset Δ of Φ is called a fundamental system of roots for Φ, or a base for Φ, if

(i) Δ is a vector space basis for V,

(ii) each root u can be written as $u = \sum k_v v$ $(v \in \Delta)$, where k_v, $v \in \Delta$, are integers all nonnegative or all nonpositive.

2.8. Positive and negative roots. If Δ is a base for Φ then a root $u = \sum k_v v$ $(v \in \Delta)$ is called positive (resp. negative) relative to Δ if all $k_v \geq 0$ (resp. all $k_v \leq 0$). The set of positive and negative roots (relative to Δ) will usually be denoted just by Φ^+ and Φ^-.

2.9. Theorem. *Let Φ be a root system in V. Then the following assertions hold:*

(i) *Φ admits at least one base.*

(ii) *If Δ and Δ' are two bases for Φ then there exists one and only one element $\omega \in \mathcal{W}(\Phi)$ such that $\omega(\Delta) = \Delta'$.*

(iii) *The Weyl group $\mathcal{W}(\Phi)$ is generated by the reflections σ_v for $v \in \Delta$.*

Proof. See 2.49, 2.62, and 2.63 of [2]. ◊

2.10. Lemma. *If Δ is a base for the root system Φ and if $\omega \in \mathcal{W}(\Phi)$ then $\omega(\Delta)$ is also a base for Φ.*

Proof. Since ω is an automorphism of V the set $\omega(\Delta)$ must be a vector space basis for V. Let $u \in \Phi$ be an arbitrary root, and write

$u = \sum t_v \omega(v)$ $(v \in \Delta,\ t_v \in \mathbb{R})$. Since $\mathcal{W}(\Phi)$ is a finite group there is a natural number n such that $\omega^n = 1_V$. Thus $\omega^{n-1}(u) = \sum t_v v$ $(v \in \Delta)$. Since Δ is a base for Φ and since $\omega^{n-1}(u) \in \Phi$, the reals t_v $(v \in \Delta)$ must be integers all nonnegative or all nonpositive (by condition (ii) of 2.7). $\Diamond$

2.11. Lemma. *Let Φ be a root system. If $v \in \Phi$ then the possible roots proportional to v are $\pm v, \pm\frac{1}{2}v$ and $\pm 2v$.*

Proof. Consider $v \in \Phi$ and $t \in \mathbb{R} \setminus \{0\}$ such that $tv \in \Phi$. Then, according to condition (iii) of 2.5, we have that

$$\frac{2\langle tv, v\rangle}{\langle v, v\rangle} = 2t \in \mathbb{Z}, \text{ and } \frac{2\langle v, tv\rangle}{\langle tv, tv\rangle} = \frac{2}{t} \in \mathbb{Z}.$$

It follows that $t \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. $\Diamond$

2.12. Reduced root systems. A root system Φ is called reduced if, for every $v \in \Phi$, v and $-v$ are the only roots in Φ proportional to v.

2.13. Lemma. *Let Φ be a root system. The subsets*

$$\Phi_1 = \{v \in \Phi \mid 2v \notin \Phi\} \text{ and } \Phi_2 = \{v \in \Phi \mid \frac{1}{2}v \notin \Phi\}$$

of Φ are reduced root systems. Moreover, $\mathcal{W}(\Phi) = \mathcal{W}(\Phi_1) = \mathcal{W}(\Phi_2)$.

Proof. We first verify that Φ_1 satisfies conditions (i)–(iii) of 2.5. Condition (i) is satisfied, since for a root $v \in \Phi$ we have $v \notin \Phi_1$ if and only if $2v \in \Phi_1$ (cf. Lemma 2.11). Consider now $u, v \in \Phi_1$. Then $\sigma_v(u) \in \Phi$ (because Φ is a root system). If $2\sigma_v(u) \in \Phi$ then $2u = \sigma_v(2\sigma_v(u)) \in \Phi$ (by condition (ii) of 2.5), contradicting $u \in \Phi_1$. So $2\sigma_v(u) \notin \Phi$, hence $\sigma_v(u) \in \Phi_1$. It follows that Φ_1 is a root system. It is clear that Φ_1 is reduced. An analogous argument holds for Φ_2. The last equality follows from the fact that $\sigma_v = \sigma_{tv}$ for every $v \in \Phi$ and every $t \in \mathbb{R} \setminus \{0\}$. $\Diamond$

2.14. Lemma. *If Δ is a base for the root system Φ then Δ is also a base for the reduced root system $\Phi_2 = \{v \in \Phi \mid \frac{1}{2}v \notin \Phi\}$.*

Proof. We have only to check that $\Delta \subseteq \Phi_2$ if Δ is a base for Φ. Consider $v \in \Delta$ and suppose that $\frac{1}{2}v \in \Phi$. Since $\frac{1}{2} \notin \mathbb{Z}$ we obtain a contradiction with condition (ii) of 2.7. Hence $\frac{1}{2}v \notin \Phi$, i.e., $v \in \Phi_2$. $\Diamond$

2.15. The length of an element of the Weyl group. For the remaining results of this section we assume that Φ is a reduced root system in V. Fix a base Δ for Φ, and let Φ^+ and Φ^- be the sets of positive, resp., negative roots (relative to Δ). By assertion (iii) of Th. 2.9 we know that for every $\omega \in \mathcal{W}(\Phi)$ there are roots $v_1, \dots, v_k \in \Delta$ such that $\omega = \sigma_{v_1} \dots \sigma_{v_k}$. So we can define the length $\ell(\omega)$ of ω in the following way: If $\omega = 1_V$ then $\ell(\omega) = 0$, if $\omega \neq 1_V$ then $\ell(\omega)$

is the smallest integer k such that ω can be written as a product of k reflections corresponding to roots from Δ.

2.16. Lemma. *Let $v \in \Delta$. Then σ_v permutes the positive roots other than v.*

Proof. See 2.61 of [2]. $\Diamond$

2.17. Lemma. *Let $v \in \Delta$ and $\omega \in \mathcal{W}(\Phi)$ be such that $\omega(v)$ is negative. Then $\ell(\omega\sigma_v) = \ell(\omega) - 1$.*

Proof. See 2.71 of [2]. $\Diamond$

2.18. Proposition. *For all $\omega \in \mathcal{W}(\Phi)$ we have that $\ell(\omega)$ is the number of positive roots $\alpha \in \Phi^+$ such that $\omega(\alpha) \in \Phi^-$.*

Proof. See 2.70 of [2]. $\Diamond$

3. Parabolic sets in root systems

Consider a root system Φ in a vector space V endowed with a scalar product, and let Δ be a base for Φ. As usual, Φ^+ and Φ^- denote the set of positive, resp. negative roots relative to Δ. For the sake of simplicity we shall denote the Weyl group $\mathcal{W}(\Phi)$ of Φ only with the letter $\mathcal{W}$.

3.1. Parabolic sets. A subset Γ of Φ is called parabolic if it satisfies the following conditions:

(i) $v_1, v_2 \in \Gamma$ and $v_1 + v_2 \in \Phi$ imply that $v_1 + v_2 \in \Gamma$.

(ii) $\Phi = \Gamma \cup (-\Gamma)$.

3.2. Examples. 1) It is clear that Φ^+ is a parabolic subset of Φ.

2) For every subset Θ of Δ we denote by

$$\mathcal{P}(\Theta) := \Phi^+ \cup \langle\Theta\rangle\,, \text{ where } \langle\Theta\rangle := \Phi \cap \operatorname{span}(\Theta).$$

(We recall that $\operatorname{span}(\Theta)$ stays for the intersection of all vector subspaces of V containing Θ.) $\mathcal{P}(\Theta)$ is a parabolic set, since Δ is a base for Φ, and since Φ^+ is parabolic. (Note that $\mathcal{P}(\emptyset) = \Phi^+$ and $\mathcal{P}(\Delta) = \Phi$.) Moreover, it can be shown that for every parabolic set $\Gamma \subseteq \Phi$ there exists a subset Θ of Δ and an element $\omega \in \mathcal{W}$ such that $\Gamma = \omega(\mathcal{P}(\Theta))$ (see 1.1.2.11 of [3]).

3.3. Notation. For a subset M of V we denote by $\operatorname{cone}(M)$ the set of all nonnegative linear combinations of elements of M.

3.4. Lemma. *If $\Theta \subseteq \Delta$ then* $\operatorname{cone}(\mathcal{P}(\Theta)) \cap \Phi \subseteq \mathcal{P}(\Theta)$.

Proof. First let us observe that $\mathcal{P}(\Theta) \subseteq \operatorname{cone}(\Delta \setminus \Theta) + \operatorname{span}(\Theta)$. Hence $\operatorname{cone}(\mathcal{P}(\Theta)) \subseteq \operatorname{cone}(\Delta \setminus \Theta) + \operatorname{span}(\Theta)$.

Consider now an element $x \in \text{cone}(\mathcal{P}(\Theta)) \cap \Phi$. If $x \in \Phi^+$ then clearly $x \in \mathcal{P}(\Theta)$. So let us suppose that $x \in \Phi^-$. We already know that $x \in \text{cone}(\Delta \setminus \Theta) + \text{span}(\Theta)$. Since Δ is a base for Φ, the negative root x must belong to $\text{span}(\Theta)$, hence to $\langle \Theta \rangle$. $\Diamond$

3.5. Notation. For any subset Θ of Δ denote by $\mathcal{W}_\Theta$ the subgroup of the Weyl group $\mathcal{W}$ generated by the reflections σ_v, $v \in \Theta$. Note that in view of assertion (iii) of Th. 2.9 the subgroup $\mathcal{W}_\Delta$ coincides with the whole Weyl group.

3.6. Proposition. *Let $\Theta \subseteq \Delta$. Then the following equality holds*

$$\mathcal{W}_\Theta = \{\omega \in \mathcal{W} \mid \omega(\mathcal{P}(\Theta)) = \mathcal{P}(\Theta)\}.$$

Proof. Denote by $\mathcal{W}' := \{\omega \in \mathcal{W} \mid \omega(\mathcal{P}(\Theta)) = \mathcal{P}(\Theta)\}$. We first prove the equality

$$\mathcal{W}' = \{\omega \in \mathcal{W} \mid \omega(\Delta) \subseteq \mathcal{P}(\Theta)\}. \tag{1}$$

Consider $\omega \in \mathcal{W}$ such that $\omega(\Delta) \subseteq \mathcal{P}(\Theta)$. Pick an arbitrary $v \in \in \mathcal{P}(\Theta)$. If v is positive, then $v \in \text{cone}(\Delta)$, so $\omega(v) \in \text{cone}(\omega(\Delta)) \subseteq \subseteq \text{cone}(\mathcal{P}(\Theta))$. It follows that $\omega(v) \in \text{cone}(\mathcal{P}(\Theta)) \cap \Phi$, thus $\omega(v) \in \mathcal{P}(\Theta)$ (by Lemma 3.4). If v is negative, then v must belong to $\langle \Theta \rangle$. Hence $\omega(v) \in \langle \Theta \rangle \subseteq \mathcal{P}(\Theta)$. This shows that (1) holds.

In view of (1) and the Lemmas 2.10, 2.13, and 2.14 we may assume without loosing generality that Φ is reduced. Observe that it follows right from its definition that $\mathcal{W}'$ is a subgroup of $\mathcal{W}$. If $v \in \Theta$ then $\sigma_v(\Delta \setminus \{v\}) \subseteq \Phi^+$ (by Lemma 2.16), and $\sigma_v(v) = -v$. Thus $\sigma_v(\Delta) \subseteq \subseteq \mathcal{P}(\Theta)$. According to (1), the reflection σ_v belongs to $\mathcal{W}'$. Since $\mathcal{W}'$ is a group we conclude that $\mathcal{W}_\Theta \subseteq \mathcal{W}'$. For the converse inclusion we prove by induction on $k \in \mathbb{N}$ the following assertion: *For every $\omega \in \in \mathcal{W}'$, such that the length $\ell(\omega)$ of ω equals k, we have that $\omega \in \mathcal{W}_\Theta$.* The assertion trivially holds for $k = 0$. (The only element of the Weyl group having length 0 is the identity 1_V.) So let us suppose that k is a positive integer and that the assertion holds for $k-1$. Consider an element $\omega \in \mathcal{W}'$ with $\ell(\omega) = k$. In view of Prop. 2.18 there exists an element $v \in \Delta$ such that $\omega(v)$ is negative. We prove that $\omega\sigma_v \in \mathcal{W}'$. For this let us observe that $(\omega\sigma_v)(\Delta \setminus \{v\}) \subseteq \omega(\Phi^+) \subseteq \mathcal{P}(\Theta)$ (the last inclusion follows from Lemma 3.4). On the other hand, $(\omega\sigma_v)(v) = = -\omega(v) \in \Phi^+$. We conclude that $\omega\sigma_v \in \mathcal{W}'$. We know from Lemma 2.17 that $\ell(\omega\sigma_v) = k-1$, so $\omega\sigma_v \in \mathcal{W}_\Theta$, by the induction hypothesis. It follows that $\sigma_v \in \mathcal{W}'$ (because $\mathcal{W}_\Theta \subseteq \mathcal{W}'$ and $\omega \in \mathcal{W}'$). Thus $-v = = \sigma_v(v)$ lies in $\mathcal{P}(\Theta)$. Since $-v$ is negative, it must belong to $\langle \Theta \rangle$.

So $v \in \Delta \cap \langle \Theta \rangle = \Theta$, and thus $\sigma_v \in \mathcal{W}_\Theta$. This yields finally that $\omega = (\omega\sigma_v)\sigma_v \in \mathcal{W}_\Theta$. ◊

4. On the structure of semisimple Lie algebras and semisimple Lie groups

As we have already mentioned in the second remark of 1.6, a semisimple Lie group admits Tits systems. The so-called Iwasawa decomposition plays an important role in the construction of these systems. We recall how this decomposition is obtained, but do not stress on details and proofs, since they can be found in every book treating the structure of semisimple real Lie algebras and semisimple Lie groups (for ex., [1], [2]).

Throughout this section $\mathfrak{g}$ will denote a semisimple real Lie algebra and G a connected Lie group having $\mathfrak{g}$ as Lie algebra. As usual, $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denotes the Killing form of $\mathfrak{g}$, $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$, and $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G. Let $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be a Cartan involution with the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, where $\mathfrak{k}$ and $\mathfrak{s}$ are the $+1$, resp., -1 eigenspaces of τ. Note that $\mathfrak{k}$ is a subalgebra and $\mathfrak{s}$ is a vector subspace of $\mathfrak{g}$. Denote by K the analytic subgroup of G determined by $\mathfrak{k}$, and by $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the bilinear map defined as

$$(*) \qquad \langle X, Y \rangle = -\kappa(X, \tau(Y)), \text{ for all } X, Y \in \mathfrak{g}.$$

4.1. Proposition. *With notation as above, the following assertions are true:*

(i) *$\langle\cdot,\cdot\rangle$ is a scalar product on $\mathfrak{g}$.*
(ii) *For every $k \in K$ the automorphism $\mathrm{Ad}(k)$ of $\mathfrak{g}$ is orthogonal (with respect to $\langle\cdot,\cdot\rangle$).*
(iii) *K is a closed subgroup of G which normalizes $\mathfrak{s}$ (i.e., $\mathrm{Ad}(k)(\mathfrak{s}) \subseteq \mathfrak{s}$ for every $k \in K$).*
(iv) *K is compact if and only if G has finite center.*

Proof. See III.6.2, III.6.3, III.6.4, and III.6.6 of [1]. ◊

4.2. The root space decomposition. In what follows $\mathfrak{g}$ is always assumed to be equipped with the scalar product $\langle\cdot,\cdot\rangle$ defined in $(*)$. Fix now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}$. The definition of $\langle\cdot,\cdot\rangle$ implies that the set $\{\mathrm{ad}(H) \mid H \in \mathfrak{a}\}$ is a commuting family of self-adjoint (hence diagonable) transformations of $\mathfrak{g}$. Thus $\mathfrak{g}$ can be written as the (orthogonal) direct sum of simultaneous eigenspaces

$$(**) \qquad \mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha,$$

where $\alpha \in \mathfrak{a}^*$,

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid \operatorname{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\},$$

and $\Phi = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\}$. Any $\alpha \in \Phi$ is called a root of $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{g}^\alpha$ is the corresponding root space, and Φ is the root system of the pair $(\mathfrak{g}, \mathfrak{a})$.

4.3. Proposition. *Let $\alpha, \beta \in \mathfrak{a}^*$. Then the following assertions hold:*

(i) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$.

(ii) $\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$, *and hence $\alpha \in \Phi$ implies $-\alpha \in \Phi$.*

(iii) $\mathfrak{g}^0 = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$.

(iv) *Φ is a root system in $\mathfrak{a}^*$ (when $\mathfrak{a}^*$ is equipped with the scalar product obtained by transferring to $\mathfrak{a}^*$ the restriction $\langle \cdot, \cdot \rangle |_{\mathfrak{a} \times \mathfrak{a}}$).*

Proof. See 6.40 and 6.53 of [2]. $\Diamond$

4.4. Notation. Choose a base Δ for Φ (we know from Th. 2.9 that such a base exists). Let Φ^+ be the set of positive roots relative to Δ, and define

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha.$$

4.5. Theorem (The Iwasawa decomposition of $\mathfrak{g}$). *With notation as above, $\mathfrak{n}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are subalgebras of $\mathfrak{g}$ with $\mathfrak{n}$ nilpotent and $\mathfrak{a} \oplus \mathfrak{n}$ solvable, and $\mathfrak{g}$ is a vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.*

Proof. See 6.43 of [2]. $\Diamond$

4.6. Theorem (The Iwasawa decomposition of G). *Let $\mathfrak{g} = \mathfrak{k} \oplus \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra of the connected semisimple Lie group G, and let K, A, and N be the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$. Then the following assertions hold:*

(i) $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$, *and the groups A and N are simply connected.*

(ii) *The multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism.*

Proof. See 6.46 of [2]. $\Diamond$

The adjoint action on the root spaces. Retain the notations of Th. 4.6. In what follows G is supposed to have finite center. Then K is compact (cf. assertion (iv) of Prop. 4.1). Next we look to the action of K on the root spaces by means of the adjoint representation Ad. For this we consider the closed (hence compact) subgroups $N_K(\mathfrak{a})$

$(= N_K(A))$ and $Z_K(\mathfrak{a})(= Z_K(A))$ of K. Evidently $Z_K(\mathfrak{a})$ is normal in $N_K(\mathfrak{a})$. Moreover, the two groups have the same Lie algebra, namely $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ (cf. 6.56 of [2]). Thus the group

$$W(G, A) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$$

is finite. This group shows all possible ways that members of K can act on $\mathfrak{a}$ by Ad. For the sake of simplicity, if $k \in N_K(\mathfrak{a})$, then we shall denote the restriction of $\mathrm{Ad}(k)$ to $\mathfrak{a}$ also by $\mathrm{Ad}(k)$.

We consider now the maps φ and ψ of Def. 2.2 defined here for the space $\mathfrak{a}$ (endowed with the scalar product $\langle\cdot,\cdot\rangle|_{\mathfrak{a}\times\mathfrak{a}}$).

4.8. Proposition. *Let Φ be the root system of $(\mathfrak{g}, \mathfrak{a})$. For $k \in N_K(\mathfrak{a})$ and $\alpha \in \mathfrak{a}^*$ the following equality holds*

$$\mathrm{Ad}(k)(\mathfrak{g}^\alpha) = \mathfrak{g}^{\psi(\mathrm{Ad}(k))(\alpha)}.$$

Thus, $\alpha \in \Phi$ implies $\psi(\mathrm{Ad}(k))(\alpha) \in \Phi$.

Proof. Consider arbitrary vectors $X \in \mathfrak{g}^\alpha$ and $H \in \mathfrak{a}$. Then $[H, X] = \alpha(H)X$, hence

$$[\mathrm{Ad}(k)(H), \mathrm{Ad}(k)(X)] = \alpha(H)\,\mathrm{Ad}(k)(X) = \\ = \alpha(\mathrm{Ad}(k^{-1})(\mathrm{Ad}(k)(H))\,\mathrm{Ad}(k)(X).$$

Since $\mathrm{Ad}(k)$ is orthogonal (by assertion (ii) of Prop. 4.1) we can apply Cor. 2.4 and obtain that

$$\alpha \circ \mathrm{Ad}(k^{-1}) = \psi(\mathrm{Ad}(k))(\alpha),$$

so $\mathrm{Ad}(k)(X) \in \mathfrak{g}^{\psi(\mathrm{Ad}(k))(\alpha)}$. Hence $\mathrm{Ad}(k)(\mathfrak{g}^\alpha) \subseteq \mathfrak{g}^{\psi(\mathrm{Ad}(k))(\alpha)}$. Replacing in this inclusion k by k^{-1}, and α by $\psi(\mathrm{Ad}(k))(\alpha)$, we obtain the converse inclusion. ◊

Theorem 4.9. *The map*

$$kZ_{\mathfrak{k}}(\mathfrak{a}) \in W(G, A) \longmapsto \psi(\mathrm{Ad}(k)) \in \mathrm{Gl}(\mathfrak{a}^*)$$

is a group isomorphism onto the Weyl group $\mathcal{W}(\Phi)$ of the root system Φ of $(\mathfrak{g}, \mathfrak{a})$.

Proof. The assertion follows from 6.57 of [2] and the above Prop. 4.8. ◊

5. Parabolic subgroups in semisimple Lie groups

We keep the notations from the previous section for a real semisimple Lie algebra $\mathfrak{g}$ with Cartan involution $\tau\colon \mathfrak{g} \to \mathfrak{g}$ and the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{s}$, Φ the root system of $(\mathfrak{g}, \mathfrak{a})$, and Δ a base for Φ. For a connected Lie group G with finite center and with Lie algebra $\mathfrak{g}$ let $G = KAN$ be the Iwasawa decomposition of Th. 4.6. Throughout this section we let $M := Z_K(\mathfrak{a})$ and $P := MAN$.

5.1. Lemma. *P is a closed subgroup of G.*

Proof. By Th. 4.6 we already know that AN is a closed subset of G. Since M is compact, P must be closed. In order to show that P is a subgroup of G it suffices to prove that both A and M normalize N in G. Since $\mathfrak{a}$ normalizes $\mathfrak{n}$ in $\mathfrak{g}$ (by assertion (i) of Prop. 4.3), it follows that $A = \exp \mathfrak{a}$ normalizes $N = \exp \mathfrak{n}$ in G. Pick now an arbitrary element $k \in M$, so $\mathrm{Ad}(k)|\mathfrak{a} = 1_{\mathfrak{a}}$. Prop. 4.8 shows then that $\mathrm{Ad}(k)(\mathfrak{n}) = \mathfrak{n}$. It follows that $kNk^{-1} = \exp(\mathrm{Ad}(k)(\mathfrak{n})) = \exp \mathfrak{n} = N$. Hence M normalizes N. $\diamondsuit$

5.2. Theorem. *The triple $(G, P, N_K(\mathfrak{a}))$ is a Tits system.*

Proof. This follows using the results 1.2.3.1, 1.2.3.17 of [3] and Th. 4.9. $\diamondsuit$

Remarks. 1) Since $P \cap N_K(\mathfrak{a}) = M$, Th. 4.9 just says that the Weyl group of the Tits system $(G, P, N_K(\mathfrak{a}))$ is isomorphic to the Weyl group $\mathcal{W}(\Phi)$ of the root system Φ of $(\mathfrak{g}, \mathfrak{a})$.

2) According to Th. 5.2 there are parabolic subgroups in G. In what follows we shall prove that these subgroups can be constructed with the aid of Lie theoretical methods. Also, we shall determine the group of Tits generators of the parabolic subgroups containing P. The parabolic sets introduced in Sect. 3 will play an important role for this purpose.

5.4. The sets $\mathfrak{p}_\Theta$ and $P(\Theta)$. Let Θ be a subset of Δ and consider the parabolic subsets $\mathcal{P}(\Theta)$ defined in Sect. 3. Define now

$$\mathfrak{p}_\Theta := \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \mathcal{P}(\Theta)} \mathfrak{g}^\alpha, \quad P_\Theta := N_G(\mathfrak{p}_\Theta).$$

Of course, $\mathfrak{p} := \mathfrak{p}_\emptyset = \mathfrak{g}^0 \oplus \mathfrak{n}$, $\mathfrak{p}_\Delta = \mathfrak{g}$, and $\mathfrak{p} \subseteq \mathfrak{p}_\Theta$ for every $\Theta \subseteq \Delta$.

5.5. Proposition. *For every $\Theta \subseteq \Delta$ the following assertions hold:*

(i) $\mathfrak{p}_\Theta$ *is a self-normalizing subalgebra of* $\mathfrak{g}$.

(ii) P_Θ *is a closed subgroup of G with Lie algebra* $\mathfrak{p}_\Theta$.

(iii) $MAN = P \subseteq P_\Theta$.

(iv) $\mathfrak{p} \cap \mathfrak{s} = \mathfrak{a}$.

(v) $P_\emptyset = P$.

Proof. (i) That $\mathfrak{p}_\Theta$ is a subalgebra follows from assertion (i) of Prop. 4.3 and the fact that $\mathcal{P}(\Theta)$ is a parabolic set.

Consider now an element $X \in \mathfrak{g}$ which normalizes $\mathfrak{p}_\Theta$. By the decomposition (**) of 4.2 we can write

$$X = X_0 + \sum_{\alpha\in\Phi} X_\alpha, \text{ where } X_0 \in \mathfrak{g}^0 \text{ and } X_\alpha \in \mathfrak{g}^\alpha,\ \alpha \in \Phi.$$

Since $\mathfrak{a} \subseteq \mathfrak{p}_\Theta$, we must have $[H, X] \in \mathfrak{p}_\Theta$ for every $H \in \mathfrak{a}$. Thus

$$\sum_{\alpha\in\Phi} \alpha(H)X_\alpha \in \mathfrak{p}_\Theta, \text{ for every } H \in \mathfrak{a}.$$

Now, for every $\alpha \in \Phi$, there exists an $H \in \mathfrak{a}$ such that $\alpha(H) \neq 0$, so we conclude that $X_\alpha = 0$ for every $\alpha \in \Phi \setminus \mathcal{P}(\Theta)$. Thus $X \in \mathfrak{p}_\Theta$ which finishes the proof of (i).

(ii) That P_Θ is closed follows from the continuity of the adjoint representation Ad. The assertion follows now from (i), since the Lie algebra of $N_G(\mathfrak{p}_\Theta) = P_\Theta$ is the normalizer of $\mathfrak{p}_\Theta$ in $\mathfrak{g}$.

(iii) Since $\mathfrak{a}$ and $\mathfrak{n}$ are subsets of $\mathfrak{p}_\Theta$ we have that $A = \exp\mathfrak{a}$ and $N = \exp\mathfrak{n}$ are subsets of $N_G(\mathfrak{p}_\Theta) = P_\Theta$. Consider now an arbitrary element $k \in M$. Then $\mathrm{Ad}(k)|_\mathfrak{a} = 1_\mathfrak{a}$, so, by Prop. 4.8, $\mathrm{Ad}(k)(\mathfrak{g}^\alpha) = \mathfrak{g}^\alpha$, for every $\alpha \in \mathfrak{a}^*$. It follows that $\mathrm{Ad}(k)(\mathfrak{p}_\Theta) \subseteq \mathfrak{p}_\Theta$, hence $k \in P_\Theta$. Thus $P = MAN \subseteq P_\Theta$.

(iv) Consider $X \in \mathfrak{g}^0$ and $Y \in \mathfrak{n}$ such that $X + Y \in \mathfrak{s}$. Then $\tau(X+Y) = -X - Y \in \mathfrak{g}^0 \oplus \mathfrak{n}$. On the other hand, $\tau(X+Y) \in \tau(\mathfrak{g}^0) \oplus \oplus\tau(\mathfrak{n}) = \mathfrak{g}^0 \oplus \tau(\mathfrak{n})$. (Note that $\tau(\mathfrak{g}^0) = \mathfrak{g}^0$ by assertion (ii) of Prop. 4.3.) Applying once again assertion (ii) of Prop. 4.3 and the fact that $\Phi^+ \cap \cap \Phi^- = \emptyset$, we see that $\mathfrak{n} \cap \tau(\mathfrak{n}) = \{0\}$, thus $\tau(X+Y) = -X - Y \in \mathfrak{g}^0$. It follows that $Y = 0$, so $X \in \mathfrak{g}^0 \cap \mathfrak{s} = \mathfrak{a}$.

(v) We already know from (iii) that $P \subseteq P_\emptyset$. For the converse inclusion, consider an arbitrary element $g \in P_\emptyset$. Due to the Iwasawa decomposition of G we find $k \in K$, $a \in A$, and $n \in N$ such that $g = = kan$. Since $A, N \subseteq P_\emptyset$ and since $P_\emptyset$ is a subgroup, the element k must belong to $P_\emptyset$. We prove that k lies in M. For this, observe first that $\mathrm{Ad}(k)(\mathfrak{a}) \subseteq \mathfrak{s}$ (this follows from assertion (iii) of Prop. 4.1). Thus $\mathrm{Ad}(k)(\mathfrak{a}) \subseteq \mathfrak{p}_\Theta \cap \mathfrak{s} = \mathfrak{a}$ (by (iv)), showing that $k \in N_K(\mathfrak{a})$. Assume that $k \notin M$, i.e., $\mathrm{Ad}(k)|_\mathfrak{a} \neq 1_\mathfrak{a}$. According to Prop. 2.18 and Th. 4.9, we find a root $\alpha \in \Phi^+$ such that $\psi(\mathrm{Ad}(k))(\alpha) \in \Phi^-$. Hence $\mathrm{Ad}(k)(\mathfrak{g}^\alpha) \not\subseteq \mathfrak{p}_\Theta$ (by Prop. 4.8), a contradiction. Thus $k \in M$. This finishes the proof. ◊

5.6. Theorem. *Let $\Theta \subseteq \Delta$. Then P_Θ is a parabolic subgroup of G, and the group of its Tits generators is isomorphic to the subgroup $\mathcal{W}_\Theta$ of the Weyl group $\mathcal{W}(\Phi)$. (We recall that $\mathcal{W}_\Theta$ is the subgroup generated by the reflections σ_α with $\alpha \in \Theta$.)*

Proof. We already know from assertion (iii) of Prop. 5.5 that $P \subseteq$ $\subseteq P_\Theta$. Thus, by 1.1 and Th. 5.2, P_Θ is a parabolic subgroup. According to Cor. 1.5, $\{kM \mid\ k \in P_\Theta \cap N_K(\mathfrak{a})\}$ is the set of Tits generators of P_Θ. Let $k \in N_K(\mathfrak{a})$. Then $k \in P_\Theta$ if and only if $\mathrm{Ad}(k)(\mathfrak{p}_\Theta) \subseteq$ $\subseteq \mathfrak{p}_\Theta$. Since $\mathrm{Ad}(k)(\mathfrak{g}^0) = \mathfrak{g}^0$ (by Prop. 4.8), this is equivalent to $\mathrm{Ad}(k)(\mathfrak{g}^\alpha) \subseteq \mathfrak{p}_\Theta$ for every $\alpha \in \mathcal{P}(\Theta)$. Applying Prop. 4.8 once again, this means that $\psi(\mathrm{Ad}(k))(\alpha) \in \mathcal{P}(\Theta)$, for every $\alpha \in \mathcal{P}(\Theta)$, or, equivalently, $\psi(\mathrm{Ad}(k))(\mathcal{P}(\Theta)) = \mathcal{P}(\Theta)$ (note that $\mathcal{P}(\Theta)$ is finite). By Prop. 3.6 this is tantamount to say that $\psi(\mathrm{Ad}(k)) \in \mathcal{W}_\Theta$. The assertion follows now from Th. 4.9. $\Diamond$

5.7. Corollary. *If G' is a subgroup of G containing P then there exists a subset Θ of Δ such that $G' = P_\Theta$.*

Proof. This follows from assertion (ii) of Th. 1.4 and from Th. 5.6. $\Diamond$

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