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ON THE PROJECTION NORM FOR A WEIGHTED INTERPOLATION USING CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

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Abstract: In a 1995 paper, J.C. Mason and G.H. Elliott studied polynomial interpolation weighted by $(1-x^2)^{1/2}$ at the zeros of Chebyshev polynomials of the second kind. In particular, they obtained an asymptotic result for the norm of the resulting projection. However, this result was based on an (unproven) conjecture about the points of attainment of the supremum of a function which defines the norm. In this paper the validity of the asymptotic result for the projection norm is established by a method that does not depend on the conjecture.

1. Introduction

Let $x_i = \cos[(i+1)\pi/(n+2)]$, $0 \le i \le n$, denote the zeros of the Chebyshev polynomial of the second kind $U_{n+1}(x) = [\sin(n+2)\theta]/\sin\theta$, where $x = \cos\theta$ and $0 \le \theta \le \pi$. Also let $w(x) = (1-x^2)^{1/2}$ and denote the set of all polynomials of degree no greater than n by P_n.

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In the paper [5], J.C. Mason and G.H. Elliott introduced the projection L_n of $C[-1,1]$ on wP_n that is defined by

$$(L_n f)(x) = w(x) \sum_{i=0}^{n} \ell_i(x) \frac{f(x_i)}{w(x_i)},$$

where $\ell_i(x)$ is the fundamental Lagrange polynomial

$$\ell_i(x) = \prod_{\substack{k=0 \\ k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}.$$

They showed that if $\|\cdot\|_\infty$ denotes the uniform norm $\|g\|_\infty = \sup_{-1\le x\le 1} |g(x)|$, the projection norm

$$\|L_n\| = \sup_{\|f\|_\infty \le 1} \|L_n f\|_\infty$$

satisfies $\|L_n\| = \sup_{0\le\theta\le\pi} F_n(\theta)$, where

$$F_n(\theta) = \frac{|\sin(n+2)\theta|}{n+2} \sum_{i=0}^{n} \left| \frac{\sin\theta_i}{\cos\theta - \cos\theta_i} \right| \tag{1}$$

and $\theta_i = (i+1)\pi/(n+2)$.

On the basis of numerical computations, the authors conjectured that the supremum of $F_n(\theta)$ occurs at $\theta = \pi/2$ when n is odd and at a value of θ that is asymptotic to $[\pi(n+1)]/[2(n+2)]$ as $n \to \infty$ when n is even. Furthermore, they proved that $F_n(\pi/2)$ (for odd n) and $F_n([\pi(n+1)]/[2(n+2)])$ (for even n) both have an asymptotic expansion

$$\frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + o(1),$$

where $\gamma = 0.577\ldots$ denotes Euler's constant. Therefore, assuming the conjecture about the points of attainment of the supremum of $F_n(\theta)$ to be correct,

$$\|L_n\| = \frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right) + o(1). \tag{2}$$

As pointed out by Mason and Elliott, the projection norm for the much-studied (unweighted) Lagrange interpolation method based on the zeros of the Chebyshev polynomial of the first kind $T_{n+1}(x) = \cos(n+1)\theta$, where $x = \cos\theta$ and $0 \le \theta \le \pi$, is

$$\frac{2}{\pi}\log n + \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma\right) + o(1)$$

(see Luttmann and Rivlin [4] for a short proof of this based on a conjecture that was subsequently established by Ehlich and Zeller [2]). Therefore, if (2) is valid, the norm of the weighted interpolation method is smaller by a quantity asymptotic to $2\pi^{-1}\log 2$.

In this paper it will be shown that the asymptotic formula (2) is indeed correct. The result is a consequence of the following theorem which will be proved in Sect. 2.

Theorem. *There exists a positive constant C so that for all $\theta \in [0, \pi]$ and $n \geq 2$,*

$$F_n(\theta) \leq \begin{cases} F_n\left(\frac{\pi}{2}\right) + \frac{C}{\log n} & \text{if } n \text{ is odd,} \\ F_n\left(\frac{\pi(n+1)}{2(n+2)}\right) + \frac{C}{\log n} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

Observe that (2) follows immediately from (3) and the asymptotic analysis of $F_n(\pi/2)$ and $F_n([\pi(n+1)]/[2(n+2)])$ that was carried out by Mason and Elliott. It should be noted, however, that although the validity of (2) is established in this paper, the conjecture about the points of attainment of the supremum of $F_n(\theta)$ remains an open question.

2. Proof of the theorem

The methods that will be employed to prove (3) are modelled on those used by Brutman [1] and Günttner [3] in their studies of the Lebesgue function for Lagrange interpolation at the zeros of Chebyshev polynomials of the first kind.

The proof will be presented via a sequence of lemmas. We begin by noting that the θ_i $(0 \leq i \leq n)$ are symmetrically arranged about $\pi/2$, so $F_n(\pi/2 - \theta) = F_n(\pi/2 + \theta)$ and hence $\|L_n\| = \sup_{0\leq\theta\leq\pi/2} F_n(\theta)$.

In the first lemma an alternative representation of $F_n(\theta)$ to that in (1) is obtained. In the statement of the lemma and subsequently, $[\cdot]$ denotes the integer part of a number.

Lemma 1. *For $j = 0, 1, \ldots, [(n+1)/2]$, let I_j denote the interval $[\theta_{j-1}, \theta_j]$. Then for $\theta \in I_j$ and with $\eta = \theta - \theta_{j-1}$,*

(4)

$$F_n(\theta) \equiv F_n(I_j, \eta) =$$

$$= \frac{\sin(n+2)\eta}{n+2} \left[\sum_{k=0}^{j-1} \cot(\eta+\theta_{k-1}) - \sum_{k=j}^{2j-1} \cot(\eta+\theta_k) + \sum_{k=2j}^{n} \csc(\eta + \theta_k) \right].$$

Proof. By using the trigonometric identity

$$\frac{\sin \theta_i}{\cos \theta - \cos \theta_i} = \frac{1}{2} \left(\cot \frac{\theta + \theta_i}{2} - \cot \frac{\theta - \theta_i}{2} \right),$$

the expression (1) becomes

$$F_n(\theta) = \frac{|\sin(n+2)\theta|}{2(n+2)} \sum_{i=0}^{n} \left| \cot \frac{\theta + \theta_i}{2} - \cot \frac{\theta - \theta_i}{2} \right|.$$

Suppose $\theta = \theta_{j-1} + \eta$ where $0 \le \eta \le \pi/(n+2)$. Then $|\sin(n+2)\theta| = = \sin(n+2)\eta$ and

$$\sum_{i=0}^{n} \left| \cot \frac{\theta + \theta_i}{2} - \cot \frac{\theta - \theta_i}{2} \right| =$$

$$= \sum_{i=0}^{n} \left| \cot \frac{\eta + \theta_{i+j}}{2} - \cot \frac{\eta + \theta_{j-i-2}}{2} \right| =$$

$$= \sum_{i=0}^{j-1} \left| \cot \frac{\eta + \theta_{i+j}}{2} - \cot \frac{\eta + \theta_{j-i-2}}{2} \right| +$$

$$+ \sum_{i=j}^{n-j} \left| \cot \frac{\eta + \theta_{i+j}}{2} - \cot \frac{\eta + \theta_{j-i-2}}{2} \right| +$$

$$+ \sum_{i=n-j+1}^{n} \left| \cot \frac{\eta + \theta_{i+j}}{2} - \cot \frac{\eta + \theta_{j-i-2}}{2} \right| =$$

$$= \sum_{k=j}^{2j-1} \left| \cot \frac{\eta + \theta_k}{2} - \cot \frac{\eta + \theta_{2j-2-k}}{2} \right| +$$

$$+ \sum_{k=2j}^{n} \left(\cot \frac{\eta + \theta_k}{2} + \tan \frac{\eta + \theta_k}{2} \right) +$$

$$+ \sum_{k=1}^{j} \left| - \tan \frac{\eta + \theta_{k-2}}{2} + \tan \frac{\eta + \theta_{2j-k}}{2} \right|.$$

On noting that

$$\cot\frac{\eta+\theta_k}{2} < \cot\frac{\eta+\theta_{2j-2-k}}{2} \quad \text{for } j \le k \le 2j-1$$

and

$$\tan\frac{\eta+\theta_{k-2}}{2} < \tan\frac{\eta+\theta_{2j-k}}{2} \quad \text{for } 1 \le k \le j,$$

and using the trigonometric identities $\cot\alpha + \tan\alpha = 2\csc(2\alpha)$ and $\cot\alpha - \tan\alpha = 2\cot(2\alpha)$, the representation (4) is obtained. ◊

Observe that (4) defines F_n as a function of $\eta \in [0, \pi/(n+2)]$ in each interval I_j for $0 \le j \le [(n+1)/2]$. For convenience we write

$$\eta = \frac{\pi}{2(n+2)} + \frac{\delta\pi}{n+2}$$

where $-1/2 \le \delta \le 1/2$. Therefore for $\theta \in I_j$, (4) can be written as

$$F_n(\theta) \equiv F_n^j(\delta) = \tag{5}$$

$$= \frac{\cos\delta\pi}{n+2}\left[\sum_{k=0}^{j-1}\cot\theta_{k-1/2+\delta} - \sum_{k=j}^{2j-1}\cot\theta_{k+1/2+\delta} + \sum_{k=2j}^{n}\csc\theta_{k+1/2+\delta}\right]$$

where the definition $\theta_i = (i+1)\pi/(n+2)$ has been extended to non-integer values of i.

In the next lemma the values of $F_n(\theta)$ in the left- and right-halves of each interval I_j are compared.

Lemma 2. *Suppose* $0 \le j \le [(n+1)/2]$ *and* $0 \le \delta \le 1/2$. *Then* $F_n^j(\delta) \ge F_n^j(-\delta)$.

Proof. From (5) it follows that for $0 \le j \le [(n-1)/2]$,

$$F_n^{j+1}(\delta) - F_n^j(\delta) = \tag{6}$$

$$= \frac{\cos\delta\pi}{n+2}\big[\cot\theta_{j-1/2+\delta} + \cot\theta_{j+1/2+\delta} - \cot\theta_{2j+1/2+\delta} - \cot\theta_{2j+3/2+\delta} -$$

$$-\csc\theta_{2j+1/2+\delta} - \csc\theta_{2j+3/2+\delta}\big] =$$

$$= \frac{\cos\delta\pi}{n+2}\big[\cot\theta_{j-1/2+\delta} + \cot\theta_{j+1/2+\delta} - \cot\theta_{j-1/4+\delta/2} - \cot\theta_{j+1/4+\delta/2}\big]$$

where the identity $\cot\alpha + \csc\alpha = \cot\alpha/2$ has been employed. Therefore, if $0 \le \delta \le 1/2$ and $G_n^j(\delta) = F_n^j(\delta) - F_n^j(-\delta)$, it follows that

$$\Delta G_n^j(\delta) = G_n^{j+1}(\delta) - G_n^j(\delta) =$$
$$= \frac{\cos\delta\pi}{n+2}\big[\cot\theta_{j-1/2+\delta} - \cot\theta_{j-1/2-\delta} + \cot\theta_{j+1/2+\delta} - \cot\theta_{j+1/2-\delta} -$$
$$- \cot\theta_{j-1/4+\delta/2} + \cot\theta_{j-1/4-\delta/2} - \cot\theta_{j+1/4+\delta/2} + \cot\theta_{j+1/4-\delta/2}\big].$$

We show that $\Delta G_n^j(\delta) \le 0$. If $A = (1/2+\delta)\pi/(n+2)$, $B = (1/2-\delta)\pi/$ $/(n+2)$ and

$$H(x) = \cot(\theta_j - xB) - \cot(\theta_j - xA) + \cot(\theta_j + xA) - \cot(\theta_j + xB),$$

then $\Delta G_n^j(\delta)$ can be written as

$$\Delta G_n^j(\delta) = \frac{\cos\delta\pi}{n+2}\left[H(1) - H(1/2)\right]. \tag{7}$$

Now, H is a decreasing function of x on $[0,1]$. This follows from

$$H'(x) = B\csc^2(\theta_j - xB) - A\csc^2(\theta_j - xA) - A\csc^2(\theta_j + xA) +$$
$$+ B\csc^2(\theta_j + xB) \le$$
$$\le B\left[\left(\csc^2(\theta_j - xB) - \csc^2(\theta_j - xA)\right) +\right.$$
$$\left.+ \left(\csc^2(\theta_j + xB) - \csc^2(\theta_j + xA)\right)\right] =$$
$$= -2xB(A-B)\left(\cot\theta_1\csc^2\theta_1 - \cot\theta_2\csc^2\theta_2\right)$$

for some $\theta_1 \in (\theta_j - xA, \theta_j - xB)$, $\theta_2 \in (\theta_j + xB, \theta_j + xA)$ by the Mean-Value Theorem, so $H'(x) < 0$ on $(0,1]$. From (7) it now follows that for each j and δ,

$$G_n^{j+1}(\delta) \le G_n^j(\delta). \tag{8}$$

We next show that $G_n^{[(n+1)/2]}(\delta) \ge 0$. If n is odd, then $G_n^{[(n+1)/2]}(\delta) = 0$ because $F_n^{[(n+1)/2]}(\delta)$ is an even function of δ by (5). If n is even, say $n = 2m$, then by (5),

$$G_n^{[(n+1)/2]}(\delta) = F_{2m}^m(\delta) - F_{2m}^m(-\delta) =$$
$$= \frac{\cos\delta\pi}{2m+2}\left[\sum_{k=0}^{m-1}\left(\cot\theta_{k-1/2+\delta} + \cot\theta_{k+1/2-\delta} - \cot\theta_{k-1/2-\delta} - \cot\theta_{k+1/2+\delta}\right) +\right.$$
$$\left.+ \left(\csc\theta_{-1/2-\delta} - \csc\theta_{-1/2+\delta}\right)\right] =$$
$$= \frac{\cos\delta\pi}{2m+2}\left(\cot\theta_{-1/2+\delta} - \cot\theta_{m-1/2+\delta} - \cot\theta_{-1/2-\delta} + \cot\theta_{m-1/2-\delta} +\right.$$
$$\left.+ \csc\theta_{-1/2-\delta} - \csc\theta_{-1/2+\delta}\right).$$

On using the identity $\csc\alpha - \cot\alpha = \tan\alpha/2$ this can be written as

$$G_n^{[(n+1)/2]}(\delta) = \frac{\cos\delta\pi}{2m+2}\left[\left(\tan\theta_{-1/2+\delta} - \tan\theta_{-3/4+\delta/2}\right) - \right.$$
$$\left. - \left(\tan\theta_{-1/2-\delta} - \tan\theta_{-3/4-\delta/2}\right)\right],$$

and since $\theta_{-3/4\pm\delta/2} = (\theta_{-1/2\pm\delta})/2$ and $\tan x - \tan(x/2)$ is increasing on $[0,\pi/2)$, it follows that $G_n^{[(n+1)/2]}(\delta) \geq 0$. Thus, whatever the parity of n, $G_n^{[(n+1)/2]}(\delta) \geq 0$. Hence by (8), $G_n^j(\delta) \geq 0$ for $0 \leq j \leq [(n+1)/2]$ and $0 \leq \delta \leq 1/2$, which establishes the lemma. ◊

Note that by Lemma 2, the supremum of $F_n(\theta)$ in each interval I_j for $0 \leq j \leq [(n+1)/2]$ is attained in the right half of the interval. Also observe that by (6), $F_n^{j+1}(0) - F_n^j(0) > 0$ for $0 \leq j \leq [(n-1)/2]$, so

$$F_n^j(0) < F_n^{[(n+1)/2]}(0), \qquad 0 \leq j \leq [(n-1)/2]. \tag{9}$$

One final lemma is needed. This lemma provides an upper bound for the difference between the supremum of $F_n(\theta)$ on I_0 and its value at the midpoint of the interval.

Lemma 3. *If* $-1/2 \leq \delta \leq 1/2$ *and* $n \geq 1$, *then*

$$F_n^0(\delta) \leq F_n^0(0) + \frac{\pi^3}{64\left(\log(n+1) + \gamma + \log 4 - \pi^2/4\right)}. \tag{10}$$

Proof. By Lemma 2 and continuity it can be assumed that $0 \leq \delta <$ $< 1/2$. From (5),

$$(n+2)\left(F_n^0(\delta) - F_n^0(0)\right) = \sum_{k=0}^{n}\left(\cos\delta\pi\csc\theta_{k+1/2+\delta} - \csc\theta_{k+1/2}\right) =$$
$$= \sum_{k=0}^{n}\left(\cos\delta\pi\csc\theta_{k-1/2-\delta} - \csc\theta_{k-1/2}\right).$$

We employ the inequality $\cos\delta\pi \leq 1 - 4\delta^2$ $(0 \leq \delta \leq 1/2)$. Since

$$\csc\theta = \frac{1}{\theta} + \sum_{r=1}^{\infty}\frac{|B_{2r}|(2^{2r}-2)}{(2r)!}\,\theta^{2r-1}, \qquad 0 < |\theta| < \pi,$$

where the B_{2r} are the Bernoulli numbers, then

$$\sum_{k=0}^{n}\left(\cos\delta\pi\csc\theta_{k-1/2-\delta}-\csc\theta_{k-1/2}\right)\le\sum_{k=0}^{n}\left(\frac{1-4\delta^2}{\theta_{k-1/2-\delta}}-\frac{1}{\theta_{k-1/2}}\right)=$$

$$=\frac{2(n+2)}{\pi}\sum_{k=0}^{n}\left(\frac{1-4\delta^2}{2k+1-2\delta}-\frac{1}{2k+1}\right)=$$

$$=\frac{2(n+2)}{\pi}\sum_{k=0}^{n}\left[\frac{1-4\delta^2}{2k+1}\left(1+\sum_{m=1}^{\infty}\left(\frac{2\delta}{2k+1}\right)^m\right)-\frac{1}{2k+1}\right]\le$$

$$\le\frac{2(n+2)}{\pi}\left[-4\delta^2\sum_{k=0}^{n}\frac{1}{2k+1}+(1-4\delta^2)\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{(2\delta)^m}{(2k+1)^{m+1}}\right].$$

Write $\sum_{k=0}^{n}\frac{1}{2k+1}=S_n$ and note that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{m+1}}\le\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}.$$

Therefore

$$F_n^0(\delta)-F_n^0(0)\le\frac{2}{\pi}\left[-4\delta^2S_n+\frac{\pi^2}{8}(1-4\delta^2)\sum_{m=1}^{\infty}(2\delta)^m\right]=$$

$$=\frac{\pi\delta}{2}\left[1-2\delta\left(\frac{8}{\pi^2}S_n-1\right)\right].$$

This latter expression is a quadratic in δ which has a maximum value of $\pi/[16(8\pi^{-2}S_n-1)]$. The result (10) then follows from the inequality of Güntter [3, eq. 3.3],

$$S_n>\frac{1}{2}(\log(n+1)+\gamma)+\log 2.\ \Diamond$$

We are now in a position **to prove the theorem**. For $0\le j\le$ $\le[(n-1)/2]$ and $0\le\delta\le 1/2$ let

$$P_j(\delta)=\cot\theta_{j-1/2+\delta}+\cot\theta_{j+1/2+\delta}-\cot\theta_{j-1/4+\delta/2}-\cot\theta_{j+1/4+\delta/2},$$

so that

$$P_j'(\delta) = \frac{\pi}{n+2}\Big[-\csc^2\theta_{j-1/2+\delta} - \csc^2\theta_{j+1/2+\delta} + \frac{1}{2}\csc^2\theta_{j-1/4+\delta/2} + \\ + \frac{1}{2}\csc^2\theta_{j+1/4+\delta/2}\Big] < \\ < \frac{\pi}{n+2}\left[-\csc^2\theta_{j+1/2+\delta}\right] < 0.$$

Since $P_j(0) > 0$ it follows from (6) that for each j with $0 \leq j \leq \leq [(n-1)/2]$,

$$\max_{0\leq\delta\leq 1/2}\left(F_n^{j+1}(\delta) - F_n^j(\delta)\right) = F_n^{j+1}(0) - F_n^j(0).$$

Therefore

$$\max_{0\leq\delta\leq 1/2}\left(F_n^{j+1}(\delta) - F_n^0(\delta)\right) = F_n^{j+1}(0) - F_n^0(0). \tag{11}$$

Now choose any $\theta \in [0, \pi/2]$ and write $\theta = \theta_{m-1/2+\delta}$ where $0 \leq \leq m \leq [(n+1)/2]$ and $-1/2 \leq \delta \leq 1/2$. Then, by (9), (11) and Lemmas 2 and 3,

$$F_n(\theta) = F_n^m(\delta) \leq F_n^m(|\delta|) = \\ = \left(F_n^m(|\delta|) - F_n^0(|\delta|)\right) + F_n^0(|\delta|) \leq \\ \leq \left(F_n^m(0) - F_n^0(0)\right) + F_n^0(0) + \frac{\pi^3}{64\left(\log(n+1) + \gamma + \log 4 - \pi^2/4\right)} \leq \\ \leq F_n^{[(n+1)/2]}(0) + \frac{\pi^3}{64\left(\log(n+1) + \gamma + \log 4 - \pi^2/4\right)},$$

from which (3) follows. Therefore the proof of the theorem is completed. ◊

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