CONGRUENCES AND ISOTONE MAPS ON PARTIALLY ORDERED SETS

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Abstract: An equivalence ρ is called an order-congruence of a poset P if it is the kernel of an isotone mapping defined on P. The order-congruences are exactly those equivalence relations on P whose classes are intervals of some linear extension of $\leq$. If P is a finite poset then its order-congruences form a relatively complemented lattice which satisfies the Jordan–Hölder condition.

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1. Introduction

In the last decade several papers dealing with the generalization of the notion of semilattice (or lattice) congruence to partially ordered sets appeared. A common property of the congruence notions developed in these papers was that all these "congruences" were defined as the kernels of some particular isotone maps (see e.g. [6], [2], [3], [7] and [14]). From here naturally arises the idea to define the kernels of all isotone maps of a poset $(P, \leq)$ as congruences of it. This idea appeared already in the seventies and was developed in the papers of T. Sturm in a more general context (see e.g. [21], [22] and [23]). The same notion, in a different formulation, was also introduced by W. T. Trotter (see [26]), who called the partitions corresponding to the above congruences order-preserving partitions. Notice that in the case of a semilattice or lattice L the kernels of isotone maps of $(L, \leq)$ are not semilattice or lattice congruences in general. An other related general notion in the framework of ordered algebras was introduced by G. Czédli and A. Lenkehegyi ([4], [5]). Their approach was related in some aspects to a paper of S. L. Bloom[1]. We note that both W. T. Trotter's and our congruence notion can be deduced from their definition.

In this paper we give several characterizations of these congruences, showing that they are related with the interval decompositions in partially ordered sets, investigated in several papers. (See e.g. [9], [17] or [18].) We prove that the order-congruences of a finite poset $(P, \leq)$ form a relatively complemented lattice which satisfies the Jordan–Hölder condition, however it is not semimodular in general. We show that this lattice is 0-distributive if and only if $(P, \leq)$ is either a chain or a two-element antichain.

2. Preliminaries

The kernel $\operatorname{Ker} f$ of a function $f : P \to Q$ is the equivalence relation $\{(x, y) \in P^2 \mid f(x) = f(y)\}$. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are partially ordered sets, f is called isotone (or order-preserving) if for any $x, y \in P$, $x \leq_P y$ implies $f(x) \leq_Q f(y)$. If $\rho \subseteq P^2$ is an equivalence relation, then $\rho[x]$ denotes the equivalence class of an element $x \in P$ and P/ρ the set of all equivalence classes of ρ. The partition induced by ρ on P is denoted by π_ρ. Conversely, if π is a partition of the set P, then ρ_π

stands for the equivalence relation induced by π. We denote by ω the identity relation and by ι the total relation (on a set P). The following definition is inspired by [4].

Definition 2.1. Let $(P, \leqq)$ be a poset and $\rho \subseteq P^2$ an equivalence relation on it.

(i) A sequence $x_0, x_1, \ldots, x_n \in P$ (with $n \geqq 1$) is called a ρ-*sequence* if for each $i \in \{1, \ldots n\}$ either $(x_{i-1}, x_i) \in \rho$ or $x_{i-1} < x_i$ holds. If in addition $x_0 = x_n$, then $x_0, x_1, \ldots, x_n$ is called a ρ-*circle*.

(ii) ρ is called an *order-congruence* of $(P, \leqq)$ if for any ρ-circle $x_0, x_1, \ldots, x_n \in P$ $\rho[x_0] = \rho[x_1] = \cdots = \rho[x_n]$ is satisfied.

(iii) A partition π of P is called an *order-preserving partition* of $(P, \leqq)$ if ρ_π is an order-congruence of $(P, \leqq)$.

Remark 2.2. Clearly, if the partial order $\mathcal{R}$ extends $\leqq$ then any ρ-sequence and ρ-circle of $(P, \leqq)$ is also a ρ-sequence and ρ-circle of $(P, \mathcal{R})$. Hence any order-congruence of $(P, \mathcal{R})$ is also an order-congruence of $(P, \leqq)$.

Lemma 2.3. *If ρ is an order-congruence of the poset $(P, \leqq)$, then it induces a partial order $\leqq_\rho$ defined on the set $P/\rho = \{\rho[x] \mid x \in P\}$ as follows:*

$\rho[x]\leqq_\rho\rho[y]$ if there exists a ρ-sequence $x_0, x_1, \ldots, x_n \in P$, with $x_0 = x$ and $x_n = y$.

Proof. Clearly, $\leqq_\rho$ is reflexive and transitive by its definition. In order to prove that $\leqq_\rho$ is antisymmetric, assume $\rho[x]\leqq_\rho\rho[y]$ and $\rho[y]\leqq_\rho\rho[x]$, for some $x, y \in P$. Then there exists a ρ-sequence $x_0, x_1, \ldots, x_n \in P$, with $x_0 = x$ and $x_n = y$ and a ρ-sequence $y_0, y_1, \ldots, y_m \in P$ with $y_0 = y$ and $y_m = x$. Clearly, $x = x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m = x$ is a ρ-circle, hence we obtain $\rho[x] = \rho[x_0] = \rho[x_n] = \rho[y]$. $\vee$

Remark 2.4. Consequently, any order-congruence ρ determines a factor-poset $(P/\rho, \leqq_\rho)$. We note that the canonical projection $f_\rho : P \to$ $\to P/\rho$, $f_\rho(x) = \rho[x]$ maps isotonly the poset $(P, \leqq)$ into $(P/\rho, \leqq_\rho)$. Indeed, take any $x, y \in P$ with $x \leqq y$. Then x, y is a ρ-sequence, and hence we get $\rho[x]\leqq_\rho\rho[y]$, i.e. $f_\rho(x) \leqq f_\rho(y)$.

By a (partially) ordered algebra we mean a triple $(A, F, \leqq)$, where (A, F) is an algebra, $(A, \leqq)$ is a poset and all the operations $f \in F$ are isotone with respect to the partial order $\leqq$ (see [4]). By a homomorphism of (partially) ordered algebras we mean an operation preserving isotone map from an algebra $(A, F, \leqq_A)$ to an algebra $(B, F, \leqq_B)$.

Definition 2.5.([4]) A relation $\theta \subseteq A^2$ is called an *order-congruence* of

an ordered algebra $(A, F, \leq)$ if it is a congruence of the algebra (A, F) and in the same time an order-congruence of the poset $(A, \leq)$.

Clearly, if the set F of the (term) operations of the above algebra is empty, we remain with the notion of an order-congruence of a poset. This observation makes possible to valorize the results of [4] and [5] in our paper. For instance, [4, Prop. 2.1] asserts that $\theta \subseteq A^2$ is an order-congruence of the partially ordered algebra $(A, F, \leq_A)$ iff there exists a partially ordered algebra $(B, F, \leq_B)$ and a homomorphism $f : A \to B$ such that $\theta = \mathrm{Ker} f$. As a consequence we obtain

Corollary 2.6. $\theta \subseteq A^2$ *is an order-congruence of a poset* $(A, \leq_A)$ *iff there exists a poset* $(B, \leq_B)$ *and an isotone mapping* $f : A \to B$ *with* $\theta = \mathrm{Ker} f$.

Definition 2.7. (i) A nonempty set $I \subseteq P$ is called an *interval* (or *modul*, or *autonomous set*) of a poset $(P, \leq)$ if for any elements $a, b \in I$ and $x, y \in P \setminus I$ $x < a$ implies $x < b$ and $a < y$ implies $b < y$.

(ii) A partition $\pi = \{A_i \mid i \in I\}$ of the set P is called an *interval decomposition* of $(P, \leq)$ if each block A_i of it is an interval of $(P, \leq)$.

For instance, in the case of a linearly ordered set $(P, \leq)$ its intervals are the usual intervals, all the singletons $\{a\}$, $a \in P$, P and $\varnothing$. We note, that this interval notion goes back to Hausdorff [15] and it can be generalized for an arbitrary n-ary relation $\mathcal{R}$ (see [11]), even more, the system of all intervals can be defined as a particular closure system on P (see e.g. [9], [10] or [17]).

Lemma 2.8. (i) *If* $\pi = \{A_i \mid i \in I\}$ *is an interval decomposition of the poset* $(P, \leq)$ *and* $x_0, x_1, \ldots, x_n \in P$ *is a* ρ_π-sequence, then for all $i \in \{1, \ldots, n\}$ either $(x_0, x_i) \in \rho_\pi$ or $x_0 < x_i$ holds.

(ii) *Any interval-decomposition* $\pi = \{A_i \mid i \in I\}$ *of the poset* $(P, \leq)$ *is also an order-preserving partition of* $(P, \leq)$.

Proof. (i) Take a ρ_π-sequence $x_0, x_1, \ldots, x_n \in P$, $n \geq 1$. We prove (i) by induction. Clearly, the assertion is satisfied for $i = 1$. Now suppose that either $(x_0, x_i) \in \rho_\pi$ or $x_0 < x_i$ holds for some $i \in \{1, \ldots, n-1\}$. As we have either $(x_i, x_{i+1}) \in \rho_\pi$ or $x_i < x_{i+1}$, we get four possible cases: $(x_0, x_i) \in \rho_\pi, (x_i, x_{i+1}) \in \rho_\pi$; $x_0 < x_i, x_i < x_{i+1}$; $(x_0, x_i) \in \rho_\pi, x_i < x_{i+1}$ and $x_0 < x_i, (x_i, x_{i+1}) \in \rho_\pi$. In the first two cases $(x_0, x_{i+1}) \in \rho_\pi$ or $x_0 < x_{i+1}$. As each $\rho_\pi[x_i]$ is an interval of $(P, \leq)$, in the last two cases we obtain $x_0 < x_{i+1}$, completing the proof.

(ii) Assume by contradiction that there is a ρ_π-circle $x_0, x_1, \ldots, x_n \in P$, $x_0 = x_n$ whose elements do not belong to a single class of ρ_π. Then there exists an x_i with $(x_0, x_i) \notin \rho_\pi$. Hence

$x_0 < x_i$, according to (i). As $x_i, x_{i+1}, \ldots, x_n$ is also a ρ_π-sequence, (i) and $(x_i, x_n) \notin \rho_\pi$ imply $x_i < x_n = x_0$, a contradiction. ◊

3. Main properties of the order-congruences

The following lemma will be used in several proofs of our paper:

Lemma 3.1. *Let* $(P, \leqq_P)$, $(Q, \leqq_Q)$ *be two posets,* $f : P \to Q$ *an isotone mapping and define a relation* $\mathcal{R}^f \subseteq P^2$ *as follows:*

(1) $$x\mathcal{R}^f y \Leftrightarrow x \leqq_P y \text{ or } f(x) <_Q f(y).$$

Then the following assertions are true:

(i) $\mathcal{R}^f$ *is a partial order on* P *which extends* $\leqq_P$, f *is an isotone mapping of the poset* $(P, \mathcal{R}^f)$ *into* $(Q, \leqq_Q)$, *and the partition induced by* $\mathrm{Ker} f$ *is an interval decomposition of* $(P, \mathcal{R}^f)$.

(ii) *If* $\leqq_Q$ *is a linear order and* $\mathcal{L}$ *is a linear extension of* $\mathcal{R}^f$, *then* f *is an isotone mapping of* $(P, \mathcal{L})$ *into* $(Q, \leqq_Q)$ *and the partition induced by* $\mathrm{Ker} f$ *on* P *is an interval decomposition of* $(P, \mathcal{L})$.

Proof. (i) $\mathcal{R}^f$ is reflexive and extends $\leqq_P$ by its definition. As $f(x) <_Q f(y)$ excludes $y \leqq_P x$ and $f(y) <_Q f(x)$, the relations $x\mathcal{R}^f y$ and $y\mathcal{R}^f x$ together imply $x \leqq_P y$ and $y \leqq_P x$, i.e. $x = y$. Hence $\mathcal{R}^f$ is antisymmetric. Take $a, b, c \in P$ with $a\mathcal{R}^f b$ and $b\mathcal{R}^f c$. Then using (1) we obtain $f(a) \leqq_Q f(b) \leqq_Q f(c)$. Now $f(a) <_Q f(c)$ implies $a\mathcal{R}^f c$, while $f(a) = f(c)$ gives $f(a) = f(b) = f(c)$ and $a \leqq_P b \leqq_P c$, hence we get $a\mathcal{R}^f c$ again. Thus $\mathcal{R}^f$ is transitive as well, hence it is a partial order. Clearly, if $\leqq_P$ is a linear order then $\mathcal{R}^f$ is equal to $\leqq_P$.

As by (1) $x\mathcal{R}^f y$ implies $f(x) \leqq_Q f(y)$, f maps isotonly $(P, \mathcal{R}^f)$ into $(Q, \leqq_Q)$. Let A be an equivalence class of $\mathrm{Ker} f$ and take $a, b \in A$, $x, y \in P \setminus A$ with $x\mathcal{R}^f a$ and $a\mathcal{R}^f y$. Then $f(x) \neq f(a) \neq f(y)$ implies $f(x) <_Q f(a) <_Q f(y)$. As $f(a) = f(b)$, we get

$f(x) <_Q f(b) <_Q f(y)$, and this implies $x\mathcal{R}^f b$ and $b\mathcal{R}^f y$. Thus A is an interval and the partition induced by $\mathrm{Ker} f$ is an interval decomposition of $(P, \mathcal{R}^f)$.

(ii) Take $u, v \in P$ with $(u, v) \in \mathcal{L}$ and $u \neq v$. We prove that $f(u) \leqq_Q f(v)$. Indeed, assume by contradiction $f(v) <_Q f(u)$. Then using (1) we get $v\mathcal{R}^f u$, whence $(v, u) \in \mathcal{L}$. Now $(u, v), (v, u) \in \mathcal{L}$ implies $u = v$, a contradiction. Hence f maps isotonly $(P, \mathcal{L})$ into $(Q, \leqq_Q)$. Define now the relation $\mathcal{R}^f_{\mathcal{L}}$ for the posets $(P, \mathcal{L})$, $(Q, \leqq_Q)$ and for the map f, using (1). Then $\mathcal{R}^f_{\mathcal{L}} = \mathcal{L}$ and (i) imply that the partition

induced by $\mathrm{Ker} f$ is an interval decomposition of $(P, \mathcal{L})$. $\Diamond$

A reflexive transitive binary relation $\theta \subseteq P^2$ is called a quasiorder on the set P. Clearly, if θ is a quasiorder, then its inverse θ^{-1} is also a quasiorder and $\theta \cap \theta^{-1}$ is an equivalence on P.

Theorem 3.2. *Let $(P, \leqq_P)$ be a poset and $\rho \subseteq P^2$ an equivalence on P. Then the following are equivalent:*

(i) *ρ is an order-congruence of $(P, \leqq_P)$.*

(ii) *There exists a poset $(Q, \leqq_Q)$ and an isotone map $f : P \to Q$ such that $\rho = \mathrm{Ker} f$.*

(iii) *$\leqq$ can be extended to a quasiorder θ such that $\rho = \theta \cap \theta^{-1}$.*

(iv) *$\leqq$ can be extended to a linear order $\mathcal{L} \subseteq P^2$ such that π_ρ is an interval decomposition of $(P, \mathcal{L})$.*

(v) *$\leqq$ can be extended to a partial order $\mathcal{R} \subseteq P^2$ such that π_ρ is an interval decomposition of $(P, \mathcal{R})$.*

Proof. The equivalence of (i), (ii) and (iii) was established in [5]. (Take $F = \varnothing$ in theorems 1.1, 1.2 and Prop. 1.4. from [5].) We note that (i)$\Leftrightarrow$(ii) also follows from [23] (see sections 45 and 47).

(ii)$\Rightarrow$(iv). Let $(Q, \leqq_Q)$ be a poset, $f : P \to Q$ an isotone mapping with $\rho = \mathrm{Ker} f$ and $\mathcal{L}_Q$ a linear extension of $\leqq_Q$ on Q. Then f is an isotone mapping of $(P, \leqq_P)$ into $(Q, \mathcal{L}_Q)$, too. Now, by Lemma 3.2(ii), there exists a linear extension $\mathcal{L}$ of $\leqq_P$ such that π_ρ is an interval decomposition of $(P, \mathcal{L})$.

(iv)$\Rightarrow$(v) is obvious. (v)$\Rightarrow$(i). Let π_ρ an interval decomposition of $(P, \mathcal{R})$. Then in virtue of Lemma 2.8(ii) ρ is an order-congruence of $(P, \mathcal{R})$. Since $\mathcal{R}$ is an extension of $\leqq$, in view of Remark 2.2, ρ is an order-congruence of $(P, \leqq_P)$, too. $\Diamond$

Corollary 3.3. *Let $(P, \mathcal{L})$ be a linearly ordered set. Then the interval decompositions of $(P, \mathcal{L})$ and the order-preserving partitions of $(P, \mathcal{L})$ are the same.*

Proof. In view of Th. 3.2(iv) any order-preserving partition of $(P, \mathcal{L})$ is an interval decomposition of $(P, \mathcal{L})$. Conversely, according to Lemma 2.8(ii), any interval decomposition of $(P, \mathcal{L})$ is also an order-preserving partition of $(P, \mathcal{L})$. $\Diamond$

Let $(P, \leqq)$ be a poset and $\varepsilon \subseteq P^2$ an equivalence relation. We define two binary relations q_ε and $\bar{\varepsilon}$ on P as follows:

(2) $(x, y) \in q_\varepsilon \Leftrightarrow$ there exists an ε-sequence $x_0, x_1, \ldots, x_n \in P$ with

$$x_0 = x \quad \text{and} \quad x_n = y.$$

(3) $\bar{\varepsilon} = q_\varepsilon \cap q_\varepsilon^{-1}$.

Clearly, q_ε is a quasiorder which extends $\leqq$ by its definition. Denote the lattice of all quasiorders of P ordered by set inclusion by $(\mathrm{Quord}(P), \cap, \vee)$. Then, according to (2), $q_\varepsilon = \varepsilon \vee \leqq$, i.e. q_ε is the smallest quasiorder containing both ε and $\leqq$. In view of Th. 3.2(iii) $\overline{\varepsilon}$ is an order-congruence of $(P, \leqq)$. Now let L be a lattice and $a, b, c \in L$. (a, b, c) is called a **dually distributive triple** if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ (see e.g. [20]).

Proposition 3.4. *Let $(P, \leqq)$ be a poset and ε an equivalence relation on P. Then:*

(i) *$\overline{\varepsilon}$ is the smallest order-congruence of $(P, \leqq)$ which contains ε, and q_ε is the smallest element $q \in \mathrm{Quord}(P)$ which contains $\leqq$ and satisfies $\overline{\varepsilon} = q \cap q^{-1}$.*

(ii) *ε is an order-congruence of $(P, \leqq)$ if and only if the relations $\leqq, \geqq$ and ε form a dually distributive triple in the lattice $\mathrm{Quord}(P)$.*

Proof. (i) Clearly, $\varepsilon \subseteq q_\varepsilon \cap q_\varepsilon^{-1} = \overline{\varepsilon}$. Take any order-congruence ν of $(P, \leqq)$ with $\varepsilon \subseteq \nu$. In view of Th. 3.2(iii) ν has the form $\nu = q \cap q^{-1}$, where q is a quasiorder on P containing $\leqq$. Now, $\varepsilon \subseteq q$ implies $q_\varepsilon \subseteq q$ and so $q_\varepsilon^{-1} \subseteq q^{-1}$. Hence $\overline{\varepsilon} = q_\varepsilon \cap q_\varepsilon^{-1} \subseteq q \cap q^{-1} = \nu$. Further, take a $q \in \mathrm{Quord}(P)$ such that q contains $\leqq$ and $\overline{\varepsilon} = q \cap q^{-1}$. Then $\varepsilon \subseteq q$ implies that $q_\varepsilon = \varepsilon \vee \leqq$ is included in q.

(ii) If $(\leqq, \geqq, \varepsilon)$ is a dually distributive triple, then $\varepsilon = \triangle \vee \varepsilon = (\leqq \cap \geqq) \vee \varepsilon = (\leqq \vee \varepsilon) \cap (\geqq \vee \varepsilon) = q_\varepsilon \cap q_\varepsilon^{-1} = \overline{\varepsilon}$. Hence ε is an order-congruence of $(P, \leqq)$. Conversely, let ε be an order-congruence of $(P, \leqq)$. Then the relations $\varepsilon = \overline{\varepsilon} = q_\varepsilon \cap q_\varepsilon^{-1} = (\leqq \vee \varepsilon) \cap (\geqq \vee \varepsilon) \geqq (\leqq \cap \geqq) \vee \varepsilon = \varepsilon$ imply $(\leqq \vee \varepsilon) \cap (\geqq \vee \varepsilon) = (\leqq \cap \geqq) \vee \varepsilon$, hence $(\leqq, \geqq, \varepsilon)$ is a dually distributive triple in $\mathrm{Quord}(P)$. $\Diamond$

A family $\mathfrak{F} = \{\mathcal{L}_i \mid i \in I\}$ of linear orders on P is called a *realizer* of the partial order $\leqq$ on P if $\cap\{\mathcal{L}_i \mid i \in I\}$ is equal to $\leqq$. (Such a family always exists – see [25]. Our terminology is similar to [24].)

Theorem 3.5. *Let $(P, \leqq)$ be a poset and ρ an equivalence relation on P. Then the following assertions are equivalent:*

(i) *π_ρ is an interval decomposition of $(P, \leqq)$.*

(ii) *$\rho \cup \leqq$ is a quasiorder on P.*

(iii) *ρ is an order-congruence of $(P, \leqq)$ and the partial order $\mathcal{R}^{f_\rho}$ corresponding to the canonical projection $f_\rho : P \to P/\rho$ by relation (1) is the same as $\leqq$.*

(iv) *There exists a realizer $\mathfrak{F}$ of $\leqq$ on P such that ρ is an order-congruence of each linearly ordered set $(P, \mathcal{L})$, $\mathcal{L} \in \mathfrak{F}$.*

Proof. (i)$\Rightarrow$(ii). If π_ρ is an interval decomposition of $(P, \leqq)$ then, in virtue of Lemma 2.8(i), we have $q_\rho = \rho \cup \leqq$. Hence $\rho \cup \leqq$ is a quasiorder on P.

(ii)$\Rightarrow$(iii). If $q = \rho \cup \leqq$ is a quasiorder, then we have $q \cap q^{-1} =$
$= (\rho \cup \leqq) \cap (\rho \cup \geqq) = \rho \cup (\leqq \cap \geqq) = \rho$. Hence, by Th. 3.2(iii), ρ is an order-congruence of $(P, \leqq)$. Clearly, the quasiorder $\rho \cup \leqq$ is equal to $q_\rho = \rho \vee \leqq$. As $\mathcal{R}^{f_\rho}$ is an extension of $\leqq$, to prove (iii) we have to show that $\mathcal{R}^{f_\rho}$ is included in $\leqq$. Take $(a,b) \in \mathcal{R}^{f_\rho}$ and assume by contradiction $a \not\leqq b$. Then, relation (1) gives $f_\rho(a) <_\rho f_\rho(b)$, i.e. $\rho[a] <_\rho \rho[b]$. Thus there exists a ρ-sequence $x_0, x_1, \ldots, x_n \in P$ with $x_0 = a$ and $x_n = b$. Then $(a,b) \in q_\rho = \rho \cup \leqq$. As $(a,b) \notin \rho$, we obtain $a \leqq b$, a contradiction.

(iii)$\Rightarrow$(iv). Let us consider the factor-poset $(P/\rho, \leqq_\rho)$ and a realizer $\mathfrak{F} = \{\mathcal{L}_i \mid i \in I\}$ for $\leqq_\rho$. Clearly, f_ρ is an isotone mapping of $(P, \leqq)$ into each $(P/\rho, \mathcal{L}_i)$, $i \in I$. Further, let $\mathcal{R}_i^{f_\rho}$, $i \in I$ be the partial orders on P corresponding to f_ρ and to the posets $(P/\rho, \mathcal{L}_i)$, $i \in I$ by relation (1). We claim that $\cap\{\mathcal{R}_i^{f_\rho} \mid i \in I\}$ is equal to $\leqq$. Indeed, as $\leqq$ is extended by each $\mathcal{R}_i^{f_\rho}$, $i \in I$, $\leqq$ is included in $\cap\{\mathcal{R}_i^{f_\rho} \mid i \in I\}$. To prove the converse inclusion, take any $(a,b) \in \cap\{\mathcal{R}_i^{f_\rho} \mid i \in I\}$ and assume by contradiction $a \not\leqq b$. Since $(a,b) \in \mathcal{R}_i^{f_\rho}$, for all $i \in I$ and $a \not\leqq b$, by using relation (1) for f_ρ and $(P/\rho, \mathcal{L}_i)$, we get $(f_\rho(a), f_\rho(b)) \in \mathcal{L}_i$, for all $i \in I$ and $f_\rho(a) \neq f_\rho(b)$. As $\{\mathcal{L}_i \mid i \in I\}$ is a realizer for $\leqq_\rho$, we get $f_\rho(a) <_\rho f_\rho(b)$. Now, applying relation (1) for f_ρ and $(P/\rho, \leqq_\rho)$, we deduce $(a,b) \in \mathcal{R}^{f_\rho}$. As by assumption $\mathcal{R}^{f_\rho}$ coincides with $\leqq$, we obtain $a \leqq b$, a contradiction. Further, take a realizer family $\mathfrak{F}_i$ for each $\mathcal{R}_i^{f_\rho}$, $i \in I$. In view of Lemma 3.1(ii), π_ρ is an interval decomposition of each $(P, \mathcal{L})$, $\mathcal{L} \in \mathfrak{F}_i$, $i \in I$. Let $\mathbb{F} = \cup\{\mathfrak{F}_i \mid i \in I\}$. In view of Cor. 3.3, ρ is an order-congruence of each linearly ordered set $(P, \mathcal{L})$, $\mathcal{L} \in \mathbb{F}$. As $\cap\{\mathcal{L} \mid \mathcal{L} \in \mathbb{F}\} = \cap\{\mathcal{R}_i^{f_\rho} \mid i \in I\}$ is equal to $\leqq$, the family $\mathbb{F}$ is a realizer of $\leqq$ on P.

(iv)$\Rightarrow$(i) Assume (iv) and take any block A of π_ρ and arbitrary $a, b \in A$, $x, y \in P \setminus A$ with $x \leqq a$ and $a \leqq y$. As any member of $\mathfrak{F}$ is an extension of $\leqq$, we get $x \mathcal{L} a$ and $a \mathcal{L} y$, for each $\mathcal{L} \in \mathfrak{F}$. As by assumption A is an interval in each ordered set $(P, \mathcal{L})$, we obtain that $x \mathcal{L} b$ and $b \mathcal{L} y$ hold for all $\mathcal{L} \in \mathfrak{F}$. Since $\mathfrak{F}$ is a realizer of $\leqq$ on P, these relations imply $x \leqq b$ and $b \leqq y$, proving that A is an interval in $(P, \leqq)$. Hence π_ρ is an interval decomposition of $(P, \leqq)$ $\Diamond$

The following proposition, essentially also present in [22], will be useful in the next section.

Proposition 3.6.(i) *If $\pi = \{A_i \mid i \in I\}$ is an order-preserving partition of the poset $(P, \leqq)$, then each block A_i of it is a convex set in $(P, \leqq)$.*

(ii) *If A is a convex set in the poset $(P, \leqq)$ then $\pi_A = \{A\} \cup \{\{x\} \mid x \in P \setminus A\}$ is an order-preserving partition of $(P, \leqq)$.*

Proof. (i). In view of Th. 3.2(iv) there exists a linear extension $\mathcal{L}$ of $\leqq$ such that each A_i is an interval in $(P, \mathcal{L})$. As now A_i is a convex set in $(P, \mathcal{L})$, it is also a convex set in $(P, \leqq)$.

(ii). Let ρ_A be the equivalence relation induced by π_A on P. Assume by contradiction that $(P, \leqq)$ contains a ρ_A-circle $x_0, x_1, \ldots, x_n$ such that not every element of it is in A. If none of these elements belong to A, then we get $x_0 < x_1 < \cdots < x_n = x_0$, a contradiction. Clearly, without loss of generality we can assume $n \geq 2$ and $x_0 = x_n \in A$. Then obviously there exist $i, j \in \{1, \ldots, n\}$ with $i < j$ such that $x_{i-1} \in A$, $x_i, \ldots, x_{j-1} \notin A$ and $x_j \in A$. As all the blocs of π_A different from A are singletons, we get $x_{i-1} < x_i \leqq \ldots \leqq x_{j-1} < x_j$. Since $x_{i-1}, x_j \in A$ and since A is convex, we obtain $x_i, \ldots, x_{j-1} \in A$, a contradiction. Hence ρ_A is an order-congruence and π_A is an order-preserving partition of $(P, \leqq)$. ◊

4. On the lattice of the order-congruences of a poset

Denote by $\mathcal{O}(P)$ the set of all order-congruences of a poset $(P, \leqq)$ and by op(P) the set of all order-preserving partitions of it. Clearly, $\mathcal{O}(P)$ with the set-theoretical inclusion of the order-congruences and op(P) with the partitions' ordering are isomorphic partially ordered sets. From [4] one can also deduce that $(\mathcal{O}(P), \subseteq)$ is a complete lattice, moreover, [22], Sect. 30 gives the following

Proposition 4.1. *$(\mathcal{O}(P), \subseteq)$ is an algebraic lattice with the greatest element ι, where $\inf\{\theta_i \mid i \in I\} = \bigcap\{\theta_i \mid i \in I\}$ for any system $\theta_i \in \mathcal{O}(P), i \in I$.*

Consequently, $(\mathrm{op}(P), \leqslant)$ is also an algebraic lattice, where the meet operation is the intersection. As $(\mathcal{O}(P), \subseteq) \cong (\mathrm{op}(P), \leqslant)$, we denote the meet and join operations in these lattices with the same symbols $\sqcap$ and $\sqcup$, respectively. According to [10] and [18] the interval decompositions of a poset $(P, \leqq)$ form a complete semimodular sublattice of the partition lattice $(\mathrm{Part}(P), \sqcap, \vee)$ which will be denoted by

$(\mathcal{D}(P), \cap, \vee)$ (shortly $\mathcal{D}(P)$). We note that $\mathrm{op}(P)$ in general is neither a sublattice of $\mathrm{Part}(P)$ nor semimodular. In fact we will prove (in Cor. 4.7) that whenever one of these conditions hold $(P, \leqq)$ is a so-called interval order.

In [17] and [18] it is proved that for a finite chain $(P, \leqq)$ $\mathcal{D}(P)$ is a Boolean sublattice of $\mathrm{Part}(P)$. (An equivalent formulation of this result can be also found in [21].) It was also shown, that any atom of $\mathcal{D}(P)$ has the form $\nu_{a,b} = \{a, b\} \cup \{\{x\} \mid x \in P \setminus \{a, b\}\}$, where $a, b \in P$ and $a \prec b$. Here we add:

Proposition 4.2. (i) *If* $(P, \leqq)$ *is a chain, then the lattices* $\mathrm{op}(P)$ *and* $\mathcal{D}(P)$ *are the same. If in addition* P *is finite and* $\varphi \prec \theta$ *holds in* $\mathcal{O}(P)$ *then* $\pi_\varphi \prec \pi_\theta$ *holds in* $\mathrm{Part}(P)$, *too.*

(ii) *If* $(P, \leqq)$ *is a poset and* $\mathcal{L}$ *is a linear extension of* $\leqq$ *then* $\mathcal{O}(P, \mathcal{L})$ *is a sublattice of* $\mathcal{O}(P, \leqq)$.

Proof. (i) In view of Cor. 3.3, the underlying sets $\mathrm{op}(P)$ and $\mathcal{D}(P)$ are the same. As the partial order is also the same in these lattices, they are identical. Thus we obtain $\mathcal{O}(P) \cong \mathcal{D}(P)$, as well. Hence, to prove the second assertion of (i), it suffices to show that $\pi_1 \prec \pi_2$ in $\mathcal{D}(P)$ implies $\pi_1 \prec \pi_2$ in $\mathrm{Part}(P)$. As now $\mathcal{D}(P)$ is a finite Boolean lattice, it is atomistic, too. Hence, $\pi_2 = \pi_1 \vee \nu_{a,b}$, for an atom $\nu_{a,b} \in \in \mathcal{D}(P)$, where $a, b \in P$, $a \prec b$. Clearly, $\omega \prec \nu_{a,b}$ holds in $\mathrm{Part}(P)$, too. As $\pi_1 \cap \nu_{a,b} = \omega$ and since $\mathrm{Part}(P)$ is a semimodular lattice, we obtain $\pi_1 \prec \pi_1 \vee \nu_{a,b}$, i.e. $\pi_1 \prec \pi_2$ in $\mathrm{Part}(P)$.

(ii) Clearly, it is enough to show that $\mathrm{op}(P, \mathcal{L})$ is a sublattice of $\mathrm{op}(P, \leqq)$. Since, in view Remark 2.2, $\mathrm{op}(P, \mathcal{L}) \subseteq \mathrm{op}(P, \leqq)$ and since the meet operations in these lattices coincide with the intersection, $\mathrm{op}(P, \mathcal{L})$ is a subsemilattice of $\mathrm{op}(P, \leqq)$. Now take $\pi_1, \pi_2 \in \mathrm{op}(P, \mathcal{L})$. As $\mathrm{op}(P, \mathcal{L}) = \mathcal{D}(P, \mathcal{L})$ and since $\mathcal{D}(P, \mathcal{L})$ is a sublattice of $\mathrm{Part}(P)$, we get $\pi_1 \vee \pi_2 \in \mathrm{op}(P, \mathcal{L}) \subsetneq \mathrm{op}(P, \leqq)$. As for any $\nu \in \mathrm{op}(P, \leqq)$ $\nu \geqslant \pi_1, \pi_2$ implies $\nu \geqslant \pi_1 \vee \pi_2$, the join $\pi_1 \sqcup \pi_2$ of π_1 and π_2 in $\mathcal{O}(P, \leqq)$ is the same as $\pi_1 \vee \pi_2$. Thus $\pi_1 \sqcup \pi_2 \in \mathrm{op}(P, \mathcal{L})$, and hence $\mathrm{op}(P, \mathcal{L})$ is a sublattice of $\mathrm{op}(P, \leqq)$. $\Diamond$

Let φ and θ be two equivalence relations on the set $P \neq \varnothing$ and $\varphi \subseteq \theta$. As for any $x, y \in P$ $\varphi[x] = \varphi[y]$ implies $\theta[x] = \theta[y]$, the map $\widetilde{f} : P/\varphi \to P/\theta$, $\widetilde{f}(\varphi[x]) = \theta[x]$ is well-defined and surjective.

Now the equivalence θ/φ on P/φ is defined as

$$\theta/\varphi = \mathrm{Ker}\,\widetilde{f} = \{(\varphi[a], \varphi[b]) \in (P/\varphi)^2 \mid (a, b) \in \theta\}.$$

If φ and θ are order-congruences of the poset $(P, \leqq)$ with $\varphi \subseteq \theta$ then

it is easy to see that $\widetilde{f}$ is an isotone mapping of $(P/\varphi, \leqq_\varphi)$ into $(P/\theta, \leqq_\theta)$. Indeed, take $a, b \in P$ with $\varphi[a] \leqq_\varphi \varphi[b]$. Then there is a φ-sequence $x_0, x_1, \ldots, x_n \in P$ with $x_0 = a$ and $x_n = b$. As $\varphi \subseteq \theta$, $x_0, x_1, \ldots, x_n$ is also a θ-sequence in $(P, \leqq)$, hence we get $\theta[a] \leqq_\theta \theta[b]$ i.e. $\widetilde{f}(\varphi[a]) \leqq_\theta \widetilde{f}(\varphi[b])$. Since $\theta/\varphi =$ Ker $\widetilde{f}$, as a conclusion we obtain that θ/φ is an order-congruence of the factor-poset $(P/\varphi, \leqq_\varphi)$.

Using the above notion and the results of [5] (Th. 1.6 and the note after Prop. 1.9) it follows:

Proposition 4.3. *If $(P, \leqq)$ is a poset and φ is an order-congruence of it then the principal filter $[\varphi)$ of $\mathcal{O}(P)$ is isomorphic to the lattice $\mathcal{O}(P/\varphi)$.*

The following lemma will be used in the proof of Th. 4.5.

Lemma 4.4. *Let φ and θ two order-congruences of the poset $(P, \leqq)$ with $\varphi \subseteq \theta$. Then there exists a linear extension $\mathcal{L}$ of $\leqq$ on P such that φ and θ are both order congruences of $(P, \mathcal{L})$.*

Proof. As θ/φ is an order-congruence of the poset $(P/\varphi, \leqq_\varphi)$, in virtue of Th. 3.2(iv) there exists a linear extension $\mathcal{L}_\varphi$ of $\leqq_\varphi$ on P/φ such that θ/φ is an interval decomposition of $(P/\varphi, \mathcal{L}_\varphi)$. Let $f_\varphi : P \to P/\varphi$ stand for the canonical projection of P in P/φ. Since f_φ is an isotone mapping of the poset $(P, \leqq)$ into $(P/\varphi, \leqq_\varphi)$, it is also an isotone mapping of $(P, \leqq)$ into $(P/\varphi, \mathcal{L}_\varphi)$. Now, denote by $\mathcal{R}^{f_\varphi}$ the partial order defined by f_φ and the latter two posets via relation (1) and take a linear extension $\mathcal{L}$ of $\mathcal{R}^{f_\varphi}$ on P. Then f_φ is an isotone mapping of $(P, \mathcal{L})$ into $(P/\varphi, \mathcal{L}_\varphi)$, according to Lemma 3.1(ii). As $\varphi = $ Ker f_φ, φ is an order-congruence of $(P, \mathcal{L})$, too. Further, to show that θ is an order congruence of $(P, \mathcal{L})$ it is enough to prove that the partition $\pi_\theta = \{B_j \mid j \in J\}$ induced by θ on P is an interval decomposition of $(P, \mathcal{L})$. For this purpose take any $B_j \in \pi_\theta$ and any elements $a, b \in B_j$, $x, y \in P \setminus B_j$ with $x\mathcal{L}a$ and $a\mathcal{L}y$. Then we get $f_\varphi(x)\mathcal{L}_\varphi f_\varphi(a)$ and $f_\varphi(a)\mathcal{L}_\varphi f_\varphi(y)$, i.e. $\varphi[x]\mathcal{L}_\varphi\varphi[a]$ and $\varphi[a]\mathcal{L}_\varphi\varphi[y]$. Since $(a, b) \in \theta$, we have $(\varphi[a], \varphi[b]) \in \theta/\varphi$. As the classes of θ/φ are intervals in $(P/\varphi, \mathcal{L}_\varphi)$, we get $\varphi[x]\mathcal{L}_\varphi\varphi[b]$ and $\varphi[b]\mathcal{L}_\varphi\varphi[y]$, i.e. $f_\varphi(x)\mathcal{L}_\varphi f_\varphi(b)$ and $f_\varphi(b)\mathcal{L}_\varphi f_\varphi(y)$. Moreover, $f_\varphi(x) \neq f_\varphi(b) \neq f_\varphi(y)$ since $\varphi[b] \subseteq \theta[b]$ and x and b, as well as b and y, are in different classes of θ. These relations imply $x\mathcal{R}^{f_\varphi}b$ and $b\mathcal{R}^{f_\varphi}y$, according to (1). As $\mathcal{R}^{f_\varphi}$ is extended by $\mathcal{L}$, we get $x\mathcal{L}b$ and $b\mathcal{L}y$, proving that B_j is an interval in $(P, \mathcal{L})$. Hence π_θ is an interval decomposition of $(P, \mathcal{L})$. $\Diamond$

We say that a lattice $(L, \leq)$ satisfies the Jordan–Hölder (chain) condition if for any elements $a, b \in L$, $a < b$, all the maximal chains

between a and b have the same length.

Theorem 4.5. *Let $(P, \leq)$ be a finite poset. Then the lattice $\mathcal{O}(P)$ is relatively complemented and satisfies the Jordan–Hölder condition. If the relation $\nu \prec \theta$ holds in $\mathcal{O}(P)$ for some $\nu, \theta \in \mathcal{O}(P)$, then $\pi_\nu \prec \pi_\theta$ holds in $\mathrm{Part}(P)$, too.*

Proof. As any interval $[\varphi, \theta]$ in $\mathcal{O}(P)$ is isomorphic to the principal ideal $(\theta/\varphi]$ in the lattice $\mathcal{O}(P/\varphi)$ corresponding to the factor-poset $(P/\varphi, \leq_\varphi)$, it is enough to prove that for any poset $(P, \leq)$ every principal ideal in $\mathcal{O}(P)$ is complemented.

Take any $\nu, \theta \in \mathcal{O}(P, \leq)$ with $\nu \subseteq \theta$. Then, in view of Lemma 4.4, there is a linear extension $\mathcal{L}$ of $\leq$ on P such that $\varphi, \theta \in \mathcal{O}(P, \mathcal{L})$. As in view of Prop. 4.2 and [18] $\mathcal{O}(P, \mathcal{L})$ is a Boolean sublattice of $\mathcal{O}(P, \leq)$, there exists an order-congruence $\nu^* \in \mathcal{O}(P, \leq)$ such that $\nu \cap \nu^* = \omega$ and $\nu \sqcup \nu^* = \theta$. Thus the principal ideal $(\theta]$ is complemented. Now, suppose that $\nu \prec \theta$ holds in $\mathcal{O}(P, \leq)$. Then ν is covered by θ in $\mathcal{O}(P, \mathcal{L})$ as well, and hence using Prop. 4.2 again, we obtain that $\pi_\nu \prec \pi_\theta$ holds in $\mathrm{Part}(P)$.

Now take any $\varphi, \psi \in \mathcal{O}(P)$ with $\varphi < \psi$. As $\mathcal{O}(P)$ is a finite lattice, all the maximal chains between φ and ψ are finite. Let $\varphi = \theta_0 \prec \theta_1 \prec \ldots \prec \theta_n = \psi$ and $\varphi = \nu_0 \prec \nu_1 \prec \ldots \prec \nu_m$ be two such maximal chains (where $m, n \geq 1$). Then, in view of the previous result, $\pi_\varphi = \pi_{\theta_0} \prec \pi_{\theta_1} \prec \ldots \prec \pi_{\theta_n} = \pi_\psi$ and $\pi_\varphi = \pi_{\nu_0} \prec \pi_{\nu_1} \prec \ldots \prec \pi_{\nu_m} = \pi_\psi$ are also two maximal chains connecting π_φ and π_ψ in $\mathrm{Part}(P)$. Since $\mathrm{Part}(P)$, as a semimodular lattice, satisfies the Jordan–Hölder condition, we obtain $n = m$. Hence $\mathcal{O}(P)$ satisfies the same condition. $\Diamond$

Corollary 4.6. *Let $(P, \leq)$ be a finite poset. Then $\mathcal{O}(P)$ and $\mathrm{op}(P)$ are atomistic and dually atomistic lattices. All the atoms of $\mathrm{op}(P)$ have the form $\nu_{a,b} = \{a, b\} \cup \{\{x\} \mid x \in P \setminus \{a, b\}\}$, where $a, b \in P$, and either $a \prec b$ or a and b are incomparable in $(P, \leq)$.*

Proof. Since any finite relatively complemented lattice is in the same time atomistic and dually atomistic and since $\mathcal{O}(P) \cong \mathrm{op}(P)$, the first assertion is obvious. Further, if $a, b \in P$ with $a \prec b$ or a and b incomparable then $\{a, b\}$ is a convex set. Hence $\nu_{a,b} = \{a, b\} \cup \{\{x\} \mid x \in P \setminus \{a, b\}\} \in \mathrm{op}(P)$, according to Prop. 3.6(ii). As $\nu_{a,b}$ is an atom in $\mathrm{Part}(P)$, it is also an atom in the lattice $\mathrm{op}(P)$. Conversely, let ν be an atom in $\mathrm{op}(P)$. Then $\omega \prec \rho_\nu$ in $\mathcal{O}(P)$ implies $0 \prec \nu$ in $\mathrm{Part}(P)$, i.e. we get that ν is an atom in $\mathrm{Part}(P)$. Hence ν has the form $\nu = \{a, b\} \cup \{\{x\} \mid x \in P \setminus \{a, b\}$, where $a, b \in P$, $a \neq b$. As by Prop. 3.6(i) $\{a, b\}$

is a convex set in $(P, \leqq)$, we obtain that either $a \prec b$ or a and b are incomparable in $(P, \leqq)$. $\Diamond$

A poset $(P, \leqq)$ is called an *interval order* if there is a function F assigning to each point $x \in P$ a nondegenerate closed interval $F(x) = = [a_x, b_x]$ on the real line $\mathbb{R}$ such that $x < y$ in $P \Leftrightarrow b_x < a_y$ in $\mathbb{R}$. In view of the result of Fishburn[8], a poset is an interval order iff it does not contain S_2 (see Figure 1) as a subposet.

Corollary 4.7. (i) *If* op(P) *is a sublattice of* Part(P)*, then it is a semimodular lattice.*

(ii) *If* op(P) *is semimodular then* $(P, \leqq)$ *is an interval order.*

Proof. (i) Assume that op(P) is a sublattice of Part(P) and take $\pi_1, \pi_2 \in$op(P) with $\pi_1 \cap \pi_2 \prec \pi_2$ in op(P). Then, in view of Th. 4.5, $\pi_1 \cap \pi_2 \prec \pi_2$ holds in Part(P). As Part(P) is a semimodular lattice, $\pi_1 \prec \pi_1 \vee \pi_2 = \pi_1 \sqcup \pi_2$ is true in Part(P). Since π_1 is covered by $\pi_1 \sqcup \pi_2$ in op(P) as well, op(P) is a semimodular lattice.

(ii) Assume by contradiction that $(P, \leqq)$ is not an interval order. Then $(P, \leqq)$ contains as a subposet S_2 (shown in Figure 1) and, in view of Cor. 4.6, $\nu_{\{a,d\}}$ and $\nu_{\{c,b\}}$ are atoms in op(P). As op(P) is semimodular, $\nu_{\{a,d\}} \cap \nu_{\{c,b\}} = \omega$ and $\omega \prec \nu_{\{c,b\}}$ imply $\nu_{\{a,d\}} \prec \nu_{\{a,d\}} \sqcup \sqcup \nu_{\{c,b\}}$ in op(P). Hence $\nu_{\{a,d\}} \prec \nu_{\{a,d\}} \sqcup \nu_{\{c,b\}}$ holds in Part(P), too. On the other hand, we have $\nu_{\{a,d\}} \vee \nu_{\{c,b\}} = \{a, d\} \cup \{c, b\} \cup \{\{x\} \mid x \in \in P \setminus \{a, b, c, d\}\}$. Let ρ be the equivalence induced by $\nu_{\{a,d\}} \vee \nu_{\{c,b\}}$. As now a, b, c, d, a is a ρ-circle in $(P, \leqq)$ with $\rho[a] \neq \rho[b]$, $\nu_{\{a,d\}} \vee \nu_{\{c,b\}}$ is not an order-preserving partition of $(P, \leqq)$. Thus we get $\nu_{\{a,d\}} < < \nu_{\{a,d\}} \vee \nu_{\{c,b\}} < \nu_{\{a,d\}} \sqcup \nu_{\{c,b\}}$, contrary to $\nu_{\{a,d\}} \prec \nu_{\{a,d\}} \sqcup \nu_{\{c,b\}}$. $\Diamond$

Remark 4.8. We note that even for an interval order, op(P) is not a sublattice of Part(P) in general. Indeed, assume that $(P, \leqq)$ contains as a subposet N_2 in Figure 1.
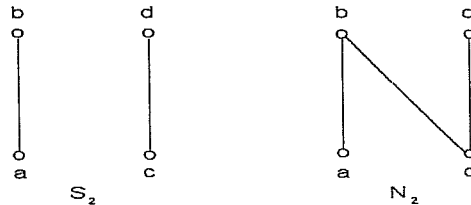

Figure 1.

As any usual interval $[u, v] = \{x \mid u \leqq x \leqq v\}$ is a convex set in $(P, \leqq)$, by Prop. 3.7(ii), $\pi_{[c,b]} \in$op(P) and $\{a, d\} \cap [c, b] = \varnothing$ implies $\nu_{\{a,d\}} \vee \pi_{[c,b]} = \{a, d\} \cup [c, b] \cup \{\{x\} \mid x \in P \setminus (\{a, d\} \cup [c, b])\}$. Let ρ^* stand for the equivalence induced by $\nu_{\{a,d\}} \vee \pi_{[c,b]}$. Then a, b, c, d, a is

a ρ^*-circle in $(P, \leqq)$ with $\rho^*[a] \neq \rho^*[b]$. Hence, $\nu_{\{a,d\}} \vee \pi_{[c,b]}$ is not an order-preserving partition of $(P, \leqq)$.

We say that a lattice L with 0 is weakly 0-distributive if, for any different atoms $a, b, c \in L$ $(a \vee b) \wedge c = 0$ holds.

Theorem 4.9. *For a finite poset $(P, \leqq)$ the following conditions are equivalent*

We note that (ii)$\Leftrightarrow$(iii) *is essentially contained in* [21], *Sect. 19:*
(i) $\mathcal{O}(P)$ *is a weakly 0-distributive lattice,*
(ii) $\mathcal{O}(P)$ *is a Boolean lattice,*
(iii) $(P, \leqq)$ *is either a chain or an antichain with two elements.*

Proof. (iii)$\Rightarrow$(ii). If $(P, \leqq)$ is a finite chain then by [18] $\mathcal{O}(P)$ is a Boolean lattice. If $(P, \leqq)$ is a two-element antichain then $\mathcal{O}(P) = \{\omega, \iota\}$, hence $\mathcal{O}(P)$ is a Boolean lattice again. (ii)$\Rightarrow$(i) is obvious.

(i)$\Rightarrow$(iii). Let $\mathcal{O}(P)$ be weakly 0-distributive. Then $op(P)$ satisfies the same property. If $| P | \leqq 2$, then $(P, \leqq)$ is either a chain or a two-element antichain, thus (iii) is trivially satisfied. Now assume by contradiction that $| P | \geqq 3$ and $(P, \leqq)$ is not a chain. Then $(P, \leqq)$ contains at least two incomparable elements $a, b \in P$. Take any element $c \in P \setminus \{a, b\}$. If $\{a, b, c\}$ would be an antichain, then, according to Cor. 4.9, $\nu_{\{a,b\}}, \nu_{\{b,c\}}, \nu_{\{a,c\}}$ would be atoms in $op(P)$ with $(\nu_{\{a,b\}} \sqcup \sqcup \nu_{\{b,c\}}) \cap \nu_{\{a,c\}} = \nu_{\{a,c\}} \neq 0$ and hence $op(P)$ would not be weakly 0-distributive. Hence, c is comparable either with a or with b. Without loss of generality we can assume $a \leqq c$. As P is finite, there is an element $d \in P$ with $a \prec d \leqq c$. If b and d would be incomparable, then $\nu_{\{a,b\}}, \nu_{\{b,d\}}$ and $\nu_{\{a,d\}}$ would be atoms in $op(P)$ which satisfy $(\nu_{\{a,b\}} \sqcup \sqcup \nu_{\{b,d\}}) \cap \nu_{\{a,d\}} = \nu_{\{a,d\}} \neq 0$, in contradiction to (i). Hence b and d are comparable. Clearly, we have $b \leqq d$, otherwise $a \leqq d$ and $d \leqq b$ imply $a \leqq b$, a contradiction. As P is finite, there is an $x \in P$ with $b \leqq x \prec d$. Obviously, a and x are incomparable. Hence $\nu_{\{a,x\}}, \nu_{\{x,d\}}$ and $\nu_{\{a,d\}}$ are atoms in the lattice $op(P)$ and satisfy $(\nu_{\{a,x\}} \sqcup \nu_{\{x,d\}}) \cap \cap \nu_{\{a,d\}} = \nu_{\{a,d\}} \neq 0$, contradicting our assumption again. Therefore, if $| P | \geqq 3$ then $(P, \leqq)$ is a chain. $\Diamond$

Order-congruences with a linearly ordered factor-poset have important applications in Queuing theory (see e.g. [16]). Using Th. 4.9 we give a characterization of these order-congruences. A poset $(P, \leqq)$ is called connected if for any elements $a, b \in P$ there exist elements $c_1, c_2, \ldots, c_n \in P$ such that $a \leqq c_1 \geqq c_2 \leqq \ldots \geqq c_n \leqq b$. It is not hard to see that in this case any factor-poset $\mathcal{O}(P/\rho, \leqq_\rho)$ of $(P, \leqq)$ (where $\rho \in \mathcal{O}(P)$) is connected, as well.

Corollary 4.10. *Let $(P, \leqq)$ be a finite connected poset and ρ an order-congruence on it. Then the factor-poset $(P/\rho, \leqq_\rho)$ is linearly ordered if and only if the principal filter $[\rho)$ of $\mathcal{O}(P)$ is a Boolean lattice.*

Proof. The "only if" part is clear. To prove the "if part" suppose $[\rho)$ a Boolean lattice. As now, in view of Prop. 4.6, the lattice $\mathcal{O}(P/\rho, \leqq_\rho)$ is also Boolean, Th. 4.9 gives that $(P/\rho, \leqq_\rho)$ is either a chain or a two-element antichain. The latter case can be excluded since $\mathcal{O}(P/\rho, \leqq_\rho)$, as a factor-poset of $(P, \leqq)$, is connected. $\Diamond$

5. Closing remarks

Finally, we note that the order-congruences of a poset can be studied by the methods of Formal Concept Analysis, as well. A formal context is triple $\mathcal{K} = (G, M, I)$ where G and M are sets and $I \subseteq G \times M$ is a binary relation. For any $A \subseteq G$ and $B \subseteq M$ define

$A' = \{m \in M \mid g\,Im, \text{ for all } g \in A\}$,
$B' = \{g \in G \mid g\,Im, \text{ for all } m \in B\}$.

A concept of the context $\mathcal{K}$ is defined as a pair (A, B) with $A' = = B$ and $B' = A$. The concepts of $\mathcal{K}$ together with the partial order defined by $(A_1, B_1) \leqslant (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ (or equivalently $B_1 \supseteq B_2$) form a complete lattice $\mathcal{L}(G, M, I)$, which is called the *concept lattice* of the context (G, M, I). For instance, the concept lattice $\mathcal{L}(P, P, \leqq)$ is identical to the Dedekind-McNeil completion of the poset $(P, \leqq)$.

Now let $(P, \leqq)$ be a poset. Denote by $\mathrm{Equ}(P)$ the set of the equivalence relations defined on the set P and by $\mathrm{Q}^{\leqq}(P)$ the set of those quasiorders on P which contain as a subrelation the ordering $\leqq$. We define a relation $I \subseteq \mathrm{Equ}(P) \times \mathrm{Q}^{\leqq}(P)$ as follows:

$\varepsilon I q \Leftrightarrow \varepsilon \subseteq q$, for $\varepsilon \in \mathrm{Equ}(P), q \in \mathrm{Q}^{\leqq}(P)$.

We remark that the concept lattice $\mathcal{L}(\mathrm{Equ}(P), \mathrm{Q}^{\leqq}(P), I)$ is isomorphic to the lattice $\mathcal{O}(P)$. Recall that q_ε stands for the smallest element of $\mathrm{Q}^{\leqq}(P)$ containing $\varepsilon \in \mathrm{Equ}(P)$, $(\varepsilon]$ denotes the principal ideal of ε in $(\mathrm{Equ}(P), \subseteq)$ and $[q)$ the principal filter of a $q \in \mathrm{Q}^{\leqq}(P)$. Now it is not hard to see that any concept $(A, B) \in \mathcal{L}(\mathrm{Equ}(P), \mathrm{Q}^{\leqq}(P), I)$ has the form $((\rho], [q_\rho))$, where $\rho = \vee\{\varepsilon \mid \varepsilon \in A\}$ and $\rho = \overline{\rho} = q_\rho \cap q_\rho^{-1}$, i.e. ρ is an order-congruence of $(P, \leqq)$ (according to Prop. 3.4). Conversely, it is easy to check that for any $\rho \in \mathcal{O}(P)$, the pair $((\rho], [q_\rho))$ is a concept of the above context.

From here it follows also that the mapping $\Psi : \mathcal{O}(P) \to Q^{\leqq}(P)$, $\rho \mapsto q_\rho$ is an order-embedding. Indeed, for any $\rho_1, \rho_2 \in \mathcal{O}(P)$ we have: $\rho_1 \subseteq \rho_2 \Leftrightarrow (\rho_1] \subseteq (\rho_2] \Leftrightarrow [q_{\rho_1}) \supseteq [q_{\rho_2}) \Leftrightarrow q_{\rho_1} \subseteq q_{\rho_2}$.

We note that the context $(\mathrm{Equ}(P), Q^{\leqq}(P), I)$ has remarkable properties. For instance, it is a so called *quasiordered* context, defined in [12]. Let $\mathrm{OI}(P)$ denote the order ideals of $(P, \leqq)$ and $\mathrm{Sub}(\mathrm{OI}(P))$ the complete sublattices of the lattice $\mathrm{OI}(P)$. In view of [12, Cor. 2] $Q^{\leqq}(P)$ is dually isomorphic to $\mathrm{Sub}(\mathrm{OI}(P))$.

Further, a binary relation $\mathcal{R} \subseteq G \times M$ is called a *Ferrers relation* if for any elements $a, c \in G$ and $b, d \in M$, $(a, b), (c, d) \in \mathcal{R}$ and $(a, d) \notin \mathcal{R}$ imply $(c, b) \in \mathcal{R}$ (see e.g. [19]). In view of [13, Prop. 103] $\mathcal{R}$ is a Ferrers relation iff the concept lattice $\mathcal{L}(G, M, \mathcal{R})$ is a chain. Now consider a poset $(P, \leqq)$. Then, using the argument of Cor. 4.7(ii), we can deduce that $<$ is a Ferrers relation, whenever the lattice $\mathrm{op}(P)$ is semimodular. Therefore, if $\mathcal{O}(P)$ is a semimodular lattice then the concept lattice $\mathcal{L}(P, P, <)$ is a chain.

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