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ON A SECOND-ORDER NONLINEAR DIFFERENTIAL SUBORDINATION

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Abstract: We find conditions on the complex-valued functions A, B, C, D, E defined in the unit disc U such that the differential inequality
$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$
implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc
$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$
For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let
$$\mathcal{H}[a,n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$
and
$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$
with $\mathcal{A}_1 = \mathcal{A}$.

In [1, Chap. IV], the authors have analyzed a second-order linear differential subordination

$$(1) \qquad A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where A, B, C, D and h are complex-valued functions in the unit disc, and $p \in \mathcal{H}[0, n]$. A more general version of (1) is given by:

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega,$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions A, B, C in the unit disc U and the positive numbers M and N such that

$$|A(z)zp'(z) + B(z)p^2(z) + C(z)p(z) + D(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$(2) \quad |A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M,$$

where $p \in \mathcal{H}[0, n]$.

We find conditions on complex-valued functions A, B, C, D, E and the positive numbers M and N such that (2) implies

$$|p(z)| < N,$$

where $p \in \mathcal{H}[0, n]$.

In order to prove the new results we shall use the following lemma:

Lemma A. [1, p. 34] *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $M > 0$, $N > 0$, n a positive integer, satisfy*

$$(3) \qquad |\psi(Ne^{i\theta}, Ke^{i\theta}, L; z)| \geq M$$

whenever

$$\mathrm{Re}\,[Le^{-i\theta}] \geq (n-1)K, \quad K \geq nN,$$

$z \in U$ and $\theta \in \mathbb{R}$.

If $p \in \mathcal{H}[0, n]$ and

$$|\psi(p(z), zp'(z), z^2p''(z); z)| < M$$

then $|p(z)| < N$.

2. Main results

Theorem 1. *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $A, B, C, D, E : U \to \mathbb{C}$ satisfy $A(z) \neq 0$,*

$$\text{(4)}\qquad \begin{cases} \operatorname{Re}\dfrac{nB(z)+D(z)}{A(z)} \geq \dfrac{M+N^2|C(z)|+|E(z)|}{N|A(z)|} \\ \operatorname{Re}\dfrac{B(z)}{A(z)} \geq 0, \ z \in U. \end{cases}$$

If $p \in \mathcal{H}[0,n]$ *and*

$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

then

$$|p(z)| < N.$$

Proof. Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be defined by

$$\text{(5)}\qquad \begin{aligned} \psi(p(z), zp'(z), z^2p''(z); z) = A(z)z^2p''(z) + B(z)zp'(z) + \\ + C(z)p^2(z) + D(z)p(z) + E(z). \end{aligned}$$

From (2) we have

$$\text{(6)}\qquad |\psi(p(z), zp'(z), z^2p''(z); z)| < M, \text{ for } z \in U.$$

Using (4) in (5) we have

$$\begin{aligned}
&|\psi(Ne^{i\theta}, Ke^{i\theta}, L; z)| = \\
&= |A(z)L + B(z)Ke^{i\theta} + C(z)N^2e^{2i\theta} + D(z)Ne^{i\theta} + E(z)| = \\
&= |A(z)| \left| Le^{-i\theta} + K\frac{B(z)}{A(z)} + N^2\frac{C(z)}{A(z)}e^{i\theta} + \frac{D(z)}{A(z)}N + \frac{E(z)}{A(z)}e^{-i\theta} \right| \geq \\
&\geq |A(z)| \left[\left| Le^{-i\theta} + K\frac{B(z)}{A(z)} + N^2\frac{C(z)}{A(z)}e^{i\theta} + \frac{D(z)}{A(z)}N \right| - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[\left| Le^{-i\theta} + K\frac{B(z)}{A(z)} + \frac{D(z)}{A(z)}N \right| - N^2\left| \frac{C(z)}{A(z)} \right| - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[\operatorname{Re} Le^{-i\theta} + K\operatorname{Re}\frac{B(z)}{A(z)} + N\operatorname{Re}\frac{D(z)}{A(z)} - N^2\left| \frac{C(z)}{A(z)} \right| - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[(n-1)K + K\operatorname{Re}\frac{B(z)}{A(z)} + N\operatorname{Re}\frac{D(z)}{A(z)} - N^2\left| \frac{C(z)}{A(z)} \right| - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[n(n-1)N + nN\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left| \frac{C(z)}{A(z)} \right| + N\operatorname{Re}\frac{D(z)}{A(z)} - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[(n-1)nN + N\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} - \left| \frac{C(z)}{A(z)} \right| N^2 - \left| \frac{E(z)}{A(z)} \right| \right] \geq \\
&\geq |A(z)| \left[N\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} - N^2\left| \frac{C(z)}{A(z)} \right| - \left| \frac{E(z)}{A(z)} \right| \right] \geq M.
\end{aligned}$$

Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)| < N$. ◊

Instead of prescribing the constant N in Th. 1, in some cases we can use (4) to determine an appropriate $N = N(M, n, A, B, C, D, E)$ so that (2) implies $|p(z)| < N$. This can be accomplished by solving (4) for N and by taking the supremum of the resulting function over U.

Condition (4) is equivalent to:

$$N^2|C(z)| - N|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} + M + |E(z)| \leq 0. \tag{7}$$

If we suppose $C(z) \neq 0$, then the inequality (7) holds if

$$|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} \geq 2\sqrt{|C(z)|(M+|E(z)|)}. \tag{8}$$

If (8) holds, then the roots of the trinomial in (7) are

$$N_{1,2}(z) = \\ = \frac{|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)}}{2|C(z)|} \pm \frac{\sqrt{\left[|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)}\right]^2 - 4|C(z)|[M+|E(z)|]}}{2|C(z)|}.$$

We let

$$N(z) = \frac{2[M+|E(z)|]}{|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} + \sqrt{\left[|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)}\right]^2 - 4|C(z)|[M+|E(z)|]}}.$$

If the supremum of $N(z)$ is finite, Th. 1 can be rewritten as follows:

Corollary 1. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and that the functions $A, B, C, D, E : U \to \mathbb{C}$, with $A(z) \neq 0$, satisfy* (4).

If

$$N = \\ = \sup_{|z|<1} \frac{2[M+|E(z)|]}{|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)} + \sqrt{\left[|A(z)|\operatorname{Re}\frac{nB(z)+D(z)}{A(z)}\right]^2 - 4|C(z)|[M+|E(z)|]}} < \infty$$

then

$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

implies

$$|p(z)| < N.$$

Let $n = 2$, $A(z) = 4 + 3i$, $B(z) = 4 - z$, $C(z) = 1 - i\sqrt{3}$, $D(z) = 1 + 2z$, $E(z) = 1 + i\sqrt{3}$, $M = 0{,}16$, $N = 0{,}32$.

In this case from Cor. 1 we deduce:

Example 1. If $p \in \mathcal{H}[0,2]$ and

$$|(4+3i)z^2p''(z)+(4-z)zp'(z)+(1-i\sqrt{3})p^2(z)+(1+2z)p(z)+1+\sqrt{3}|<0{,}16$$

then $|p(z)| < 0{,}32$.

Let $n = 3$, $A(z) = 1+i$, $B(z) = 5+z$, $C(z) = \sqrt{3}-i$, $D(z) = 1-3z$, $E(z) = i$, $M = 9$, $N = 1{,}1$.

In this case from Cor. 1 we deduce

Example 2. If $p \in \mathcal{H}[0,3]$ and

$$|(1+i)zp''(z) + (5+z)zp'(z) + (\sqrt{3}-i)p^2(z) + (1-3z)p(z) + i| < 9$$

then $|p(z)| < 1{,}1$.

Theorem 2. *Let $M > 0$, $N > 0$ and let n be a positive integer. Suppose that the functions $A, B, C, D, E : U \to \mathbb{C}$ satisfy $A(z) \neq 0$,*

$$(9)\qquad \begin{cases} \operatorname{Re}\dfrac{nB(z)+D(z)}{A(z)} \geq \dfrac{M-(n-1)nN|A(z)|+N^2|C(z)|+|E(z)|}{N|A(z)|} \\ \operatorname{Re}\dfrac{B(z)}{A(z)} \geq 0,\ z \in U. \end{cases}$$

If $p \in \mathcal{H}[0,n]$ and

$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

then

$$|p(z)| < N.$$

Proof. Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be defined by (5). Using (9) in (5) we have as in proof of Th. 1 that $|\psi(Ne^{i\theta}, Ke^{i\theta}, L; z)| \geq M$. Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)| < N$. ◊

If $A(z) = A > 0$ then Th. 2 can be rewritten as follows:

Corollary 2. *Let $M > 0$, $N > 0$ and let n be a positive integer. Suppose that the functions $B, C, D, E : U \to \mathbb{C}$ satisfy*

$$\begin{cases} \operatorname{Re}[nB(z)+C(z)] \geq \dfrac{M-(n-1)nNA+N^2|C(z)|+|E(z)|}{N} \\ \operatorname{Re}B(z) \geq 0 \end{cases}$$

If $p \in \mathcal{H}[0,n]$ and

$$|Azp''(z) + B(z)zp'(z) + C(z)p'(z) + D(z)p(z) + E(z)| < M$$

then

$$|p(z)| < N.$$

Let $n = 1$, $A = 6$, $B(z) = 12+z$, $C(z) = 4+3i$, $D(z) = 14-z$, $E(z) = \sqrt{2}-\sqrt{7}i$, $M = 10$, $N = 4$.

In this case from Cor. 2 we deduce

Example 3. If $p \in \mathcal{H}[0,1]$ and

$|6z^2p''(z)+(12+z)zp'(z)+(4+3i)p^2(z)+(14-z)p(z)+\sqrt{2}-\sqrt{7}i| < 10$

then $|p(z)| < 4$.

Let $n=2$, $A=4$, $B(z)=6-2z$, $C(z)=\sqrt{3}+i$, $D(z)=8+4z$, $E(z)=1-\sqrt{3}i$, $M=8$, $N=1$.

In this case from Cor. 2 we deduce

Example 4. If $p \in \mathcal{H}[0,2]$ and

$|4z^2p''(z)+(6-2z)zp'(z)+(\sqrt{3}+i)p^2(z)+(8+4z)p(z)+1-\sqrt{3}i| < 8$

then $|p(z) < 1$.

Instead of prescribing the constant N in Th. 2, in some cases we can use (9) to determine an appropriate $N = N(M,n,A,B,C,D,E)$ so that (2) implies $|p(z)| < N$. This can be accomplished by solving (9) for N by taking the supremum of the resulting function over U.

Condition (9) is equivalent to:

$$N^2|C(z)|-N\left[|A(z)|\mathrm{Re}\,\frac{nB(z)+D(z)}{A(z)}+(n-1)n|A(z)|\right]+ \\ +M+|E(z)| \le 0. \tag{10}$$

If we suppose $C(z) \neq 0$, then inequality (10) holds if

$$|A(z)|\mathrm{Re}\,\frac{nB(z)+D(z)}{A(z)}+(n-1)n|A(z)| \ge 2|C(z)|[M+|E(z)|]. \tag{11}$$

If (11) holds, the roots of the trinomial in (10) are

$$N_{1,2}(z) = \frac{|A(z)|\mathrm{Re}\,\frac{nB(z)+D(z)}{A(z)}+(n-1)n|A(z)| \pm \sqrt{\Delta}}{2|C(z)|}$$

where

$$\Delta = |A(z)|^2\left[\mathrm{Re}\,\frac{nB(z)+D(z)}{A(z)}+(n-1)n\right]^2-4|C(z)|[M+|E(z)|]. \tag{12}$$

We let

$$N = \sup_{|z|<1} \frac{2[M+|E(z)|]}{|A(z)|\mathrm{Re}\,\frac{nB(z)+D(z)}{A(z)}+(n-1)n|A(z)|+\sqrt{\Delta}}.$$

If this supremum is finite, Th. 2 can be rewritten as follows:

Corollary 3. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and that the functions $A, B, C, D : U \to \mathbb{C}$, with $A(z) \neq 0$ satisfy* (9).

If

$$N = \sup_{|z|<1} \frac{2[M + |E(z)|]}{|A(z)| \mathrm{Re}\, \dfrac{nB(z) + D(z)}{A(z)} + (n-1)n|A(z)| + \sqrt{\Delta}}$$

then

$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

implies

$$|p(z)| < N.$$

References

[1] MILLER, S. S. and MOCANU, P. T.: Differential Subordinations. Theory and Applications, Marcel Dekker Inc., New York, Basel, 2000.

[2] OROS, Gh. and OROS, G. I.: On a first-order nonlinear differential subordination I, *Analele Univ. Oradea Fasc. Matematica* **9** 65–70.