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PERIODIC SOLUTIONS OF A HIGH ORDER EQUATION WITH DEVIATING ARGUMENTS

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Abstract: In this paper, by using the coincidence degree theory, sufficient conditions are given for the existence of periodic solutions of the high order Rayleigh equation with deviating arguments

$$x^{(n)}(t)+f(x'(t-\sigma))+\beta(t)g[x(t-\tau(t))]=p(t).$$

In this paper, we will be concerned with a high order Rayleigh equation with deviating arguments

$$x^{(n)}(t)+f(x'(t-\sigma))+\beta(t)g[x(t-\tau(t))]=p(t), \tag{1}$$

where $n \geq 2$ is a positive integer; σ is a constant; f, g, β, p and τ are real continuous functions defined on $\mathbb{R}$ such that $f(0)=0$, β, τ and p are periodic with period 2π, $\min_{t\in\mathbb{R}}\beta(t)>0$ and $\int_0^{2\pi}p(t)dt=0$. Using coincidence degree theory, we establish a theorem for the existence of 2π-periodic solutions of Eq. (1). When $n=2$, $\sigma=0$ and $\beta(t)=1$, Eq. (1) has been studied in [2]. We also hope to extend the results in [2].

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Theorem 1. *Suppose there are positive constants K, D and M such that*

(i) $|f(x)| \leq K$ *for* $x \in \mathbb{R}$;
(ii) $xg(x) > 0$ *and* $\beta(t)|g(x)| > K$ *for* $t \in \mathbb{R}$ *and* $|x| \geq D$,
(iii) $g(x) \geq -M$ *for* $x \leq -D$.

Then there exists a 2π*-periodic solution of Eq.* (1).

Example. Consider the equation

$$x^{(5)}(t) + \exp\{-(x'(t-1))^2\} - 1 + e^{\sin t+1} \arctan(x(t-\pi)) = \sin t.$$

Take $f(x) = \exp\{-u^2\} - 1$, $g(t,x) = \arctan x$, $\beta(t) = e^{\sin t+1}$, $\sigma = 1$ and $p(t) = \sin t$, $\tau(t) = \pi$. It is then easy to verify that all the assumptions in Th. 1 are satisfied with $K = 1$, $D > \pi/4$, and $M = \pi/2$. Thus this equation has a 2π-periodic solution.

For the proof we want some preliminaries. Set

$$X := \{x \in C^{n-1}(\mathbb{R}, \mathbb{R}) \mid x(t + 2\pi) = x(t)\},$$

and $x^{(0)} = x$; define the norm on X as follows

$$\|x\| = \max_{0 \leq j \leq n-1} \max_{0 \leq t \leq 2\pi} |x^{(j)}(t)|;$$

and set

$$Y := \{y \in C^{n-1}(\mathbb{R}, \mathbb{R}) \mid y(t + 2\pi) = y(t)\}.$$

Define the norm on Y by $\|y\|_0 = \max_{0 \leq t \leq 2\pi} |y(t)|$. Thus both $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_0)$ are Banach spaces. Define the operators L and N by

$$L : X \cap C^n(\mathbb{R}, \mathbb{R}) \to Y, \quad x(t) \mapsto x^{(n)}(t), \quad t \in \mathbb{R},$$

and

$$N : X \to Y, \quad x(t) \mapsto -f(x'(t-\sigma)) - \beta(t) g[x(t - \tau(t))] + p(t), \quad t \in \mathbb{R},$$

respectively.

Let $\operatorname{Im} L$ and $\operatorname{Ker} L$ be, respectively, the image and kernel of the operator L. Clearly, $\operatorname{Ker} L = \mathbb{R}$. Define the projections $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y/\operatorname{Im} L$ by

$$(Px)(t) = \frac{1}{2\pi} \int_0^{2\pi} x(t)dt, \quad t \in \mathbb{R},$$

and

$$(Qy)(t) = \frac{1}{2\pi} \int_0^{2\pi} y(t)dt, \quad t \in \mathbb{R},$$

respectively, then $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Consider the equation

$$x^{(n)}(t) + \lambda f(x'(t-\sigma)) + \lambda\beta(t)g[x(t-\tau(t))] = \lambda p(t), \tag{2}$$

where $\lambda \in (0,1)$. We have

Lemma. *Suppose that all conditions of* Th. 1 *are satisfied, then there exist positive constants D_j $(1 \le j \le n-1)$, which are independent of λ, if $x(t)$ is any 2π-periodic solution of Eq.* (2), *such that*

$$|x^{(j)}(t)| \le D_j \text{ and } |x(t)| \le D + 2\pi D_1 \ (t \in [0,2\pi],\ 1 \le j \le n-1). \tag{3}$$

Proof. Let $x = x(t)$ be a 2π-periodic solution of Eq. (2). Since $x^{(n-2)}(0) = x^{(n-2)}(2\pi)$, there exists $t_1 \in [0,2\pi]$ such that $x^{(n-1)}(t_1) = 0$. In view of (1), we see that any $t \in [0,2\pi]$,

$$\begin{aligned} |x^{(n-1)}(t)| &= \left|\int_{t_1}^{t} x^{(n)}(s)ds\right| \le \int_0^{2\pi} |x^{(n)}(s)|ds \le \\ &\le \lambda \int_0^{2\pi} |f(x'(s-\sigma))|ds + \\ &+ \lambda \int_0^{2\pi} \beta(s)|g[x(s-\tau(s))]|ds + \lambda \int_0^{2\pi} |p(s)|ds \le \\ &\le 2\pi K + \int_0^{2\pi} \beta(s)|g[x(s-\tau(s))]|ds + 2\pi \max_{0\le s\le 2\pi} |p(s)|. \end{aligned} \tag{4}$$

We assert that

$$\int_0^{2\pi} \beta(t)|g[x(t-\tau(t))]|dt \le 2\pi K + 4\pi\beta_1 M_1 \tag{5}$$

for some positive number M_1, where $\beta_1 = \max_{0\le t\le 2\pi} \beta(t)$. Indeed, integrating Eq. (2) from 0 to 2π, and noting condition (i), we see that

$$\begin{aligned} &\int_0^{2\pi} \{\beta(t)g[x(t-\tau(t))] - K\}dt \le \\ &\le \int_0^{2\pi} \{\beta(t)g[x(t-\tau(t))] - |f(x'(t-\sigma))|\}dt \le \\ &\le \int_0^{2\pi} \{f(x'(t-\sigma)) + \beta(t)g[x(t-\tau(t))]\}dt = 0. \end{aligned} \tag{6}$$

Thus letting

$$E_1 = \{t \in [0,2\pi] \mid x(t-\tau(t)) > D\}, \quad E_2 = [0,2\pi] \setminus E_1,$$

we have

$$\int_{E_2} \beta(t)|g[x(t-\tau(t))]|dt \le 2\pi\beta_1 \max\Big\{M, \sup_{x\in[-D,D]} |g(x)|\Big\},$$

and

$$\int_{E_1} \{\beta(t)|g[x(t-\tau(t))| - K\}dt \le \int_{E_1} |\beta(t)g[x(t-\tau(t)] - K|dt =$$
$$= \int_{E_1} \{\beta(t)g[x(t-\tau(t))] - K\}dt \le$$
$$\le -\int_{E_2} \{\beta(t)g[x(t-\tau(t))] - K\}dt \le$$
$$\le \int_{E_2} \beta(t)|g[x(t-\tau(t))]|dt + \int_{E_2} K dt.$$

Therefore,

$$\int_0^{2\pi} \beta(t)|g[x(t-\tau(t))]|dt \le 2\pi K + 4\pi\beta_1 \max\Big\{M, \sup_{x\in[-D,D]} |g(x)|\Big\},$$

as required. Combining (4) and (5), we see that

$$|x^{(n-1)}(t)| \le D_{n-1}, \quad t \in [0, 2\pi] \tag{7}$$

for some positive number D_{n-1}. Next, since $x^{(n-3)}(0) = x^{(n-3)}(2\pi)$, there is some $t_2 \in [0, 2\pi]$, such that $x^{(n-2)}(t_2) = 0$. In view of (1), we see that for any $t \in [0, 2\pi]$,

$$|x^{(n-2)}(t)| = \left| \int_{t_0}^{t} x^{(n-1)}(s)ds \right| \le \int_0^{2\pi} |x^{(n-1)}(t)|dt \le 2\pi D_{n-1}. \tag{8}$$

Similarly, we conclude for any $t \in [0, 2\pi]$,

$$|x^{(n-1-i)}(t)| \le (2\pi)^i D_{n-1} \qquad (i = 1, 2, \ldots, n-2). \tag{9}$$

Let $D_{n-1-i} = (2\pi)^i D_{n-1}$ $(i = 1, 2, \ldots, n-2)$. By (7) and (9), implies that

$$|x^{(j)}(t)| \le D_j, \qquad t \in [0, 2\pi] \qquad (j = 1, 2, \ldots, n-1). \tag{10}$$

Further, note that last equality in (6) implies

$$f(x'(t_0)) + \beta(t_0)g(x(t_0 - \tau(t_0))) = 0$$

for some t_0 in $[0, 2\pi]$. Thus in view of condition (i),

$$|\beta(t_0)g[x(t_0 - \tau(t_0))]| = |f(x'(t_0))| \le K,$$

and in view of (ii),

$$|x(t_0 - \tau(t_0))| < D.$$

Since $x(t)$ is 2π-periodic, we may infer that $|x(t^*)| < D$ for some t^* in $[0, 2\pi]$. Finally, we see that

$$\begin{aligned} |x(t)| = \left| x(t^*) + \int_{t^*}^{t} x'(s)ds \right| \leq D + \int_0^{2\pi} |x'(t)|dt \leq \\ \leq D + 2\pi D_1, \ t \in [0, 2\pi]. \end{aligned} \tag{11}$$

The proof is complete. ◊

Proof of Theorem 1. Suppose that $x(t)$ is any 2π-periodic solution of Eq. (2). By Lemma, there exist positive constants D_j $(1 \leq j \leq n-1)$, which are independent of λ such that (4) is true. For any fixed positive constant $\overline{D} > \max\{D_1, D_2, \dots, D_{n-1}, D + 2\pi D_1\}$, set

$$\Omega = \{x \in X \mid \|x\| < \overline{D}\}.$$

We know that the operator L is a Fredholm operator with index zero, and the operator N is L-compact on the closure $\overline{\Omega}$ of Ω (see, e.g., [1, p. 176]).

For any $\lambda \in (0, 1)$ and any $x = x(t)$ in the domain of L is which also belongs to $\partial\Omega$, we must have $Lx \neq \lambda Nx$. For otherwise in view of (4), we see that x belongs to the interior of Ω, which is contrary to the assumption that $x \in \partial\Omega$. Next, note that a function $x = x(t)$ in the intersection of $\operatorname{Ker} L$ and $\partial\Omega$ must be the constant functions $x(t) \equiv \overline{D}$ or $x(t) \equiv -\overline{D}$. Hence

$$\begin{aligned} (QN)(x) &= \frac{1}{2\pi} \int_0^{2\pi} (-f(x'(t)) - \beta(t)g[x(t - \tau(t))] + p(t))dt = \\ &= -\frac{1}{2\pi} g(\pm\overline{D}) \int_0^{2\pi} \beta(t)dt \neq 0. \end{aligned}$$

Finally, consider the mapping

$$H(x, s) = sx + \frac{1}{2\pi}(1 - s)g(x) \int_0^{2\pi} \beta(t)dt, \quad 0 \leq s \leq 1.$$

Since for every $s \in [0, 1]$ and x in the intersection of $\operatorname{Ker} L$ and $\partial\Omega$, we have

$$xH(x, s) = sx^2 + \frac{1}{2\pi}(1 - s)xg(x) \int_0^{2\pi} \beta(t)dt > 0,$$

thus $H(x, s)$ is a homotopy. This shows that

$$\begin{aligned}&\deg\{QNx, \Omega\cap \operatorname{Ker} L, 0\} = \\&= \deg\left\{ -\frac{1}{2\pi}g(x)\int_0^{2\pi}\beta(t)dt, \Omega\cap \operatorname{Ker} L, 0\right\} = \\&= \deg\{-x, \Omega\cap \operatorname{Ker} L, 0\} = \deg\{-x, \Omega\cap \mathbb{R}, 0\} \neq 0.\end{aligned}$$

By Mawhin continuing theorem [1, p. 40], we find that equation $Lx = = Nx$ has a solution in $\operatorname{dom} L \cap \overline{\Omega}$, that is to say that there exists a 2π-periodic solution of Eq. (1). The proof is thus complete. ◊

References

[1] GAINES, R. E. and MAWHIN, J. L.: Coincidence degree and nonlinear differential equations, Lecture Notes in Math. No. 568, Springer-Verlag, 1977.

[2] WANG, G. Q. and CHENG, S. S.: A priori bounds for periodic solutions of a delay Rayleigh equation, *Applied Mathematics Letters* **12** (1999), 41–44.