Mathematica Pannonica
15/2 (2004), 231–239

TENSOR PRODUCTS OF NEAR-RING MODULES, 2

Suraiya J. **Mahmood**

Department of Mathematics, King Saud University, P.O. Box 22452, Riyadh 11495, Kingdom of Saudi Arabia

Received: June 2004

MSC 2000: 51 M 10, 51 F 15, 03 B 30

Keywords: Ordered Euclidean spaces, vector lattices.

Abstract: In [4] a tensor product of near-ring modules has been defined. But it turned out to be an abelian group. In order to avoid this special situation, this concept is generalised further in this paper. Now there are two tensor products for the same pair of modules. This situation fits better in the theory of near-rings and their modules. It has been seen in [2] that there may be two duals for a pair of modules.

1. Introduction

We write maps on the right and hence use left near-rings and the traditional near-ring modules are right modules. Let $(R, +, \cdot)$ be a left near-ring. A group $(G, +)$ is called an R-module (traditional one) if there is a near-ring homomorphism θ from R to $\mathrm{Map}(G)$. As usual, we write gr to mean $g(r\theta)$ for $g \in G$ and $r \in R$. In this case the group elements distribute over the near-ring elements. G is called a complementary R-module or R-comodule, for short, if there is a semigroup homomorphism θ from $(R, +, \cdot)$ to $(\mathrm{End}(G), \circ)$. In this case the near-ring elements distribute over the group elements and the action of R is usually written on the left of the elements of G.

Let R and S be two left near-rings. A group G is called an (R, S)-bimodule if:

E-mail address: suraiya@ksu.edu.sa

i) G is an R-comodule,
ii) G is an S-module,
iii) $(rg)s = r(gs), \forall g \in G, r \in R, s \in S.$

G is called a left strong R-module if the action of R is defined on the left of G satisfying the following conditions $\forall r, r' \in R$ and $g, g' \in G$:

i) $(rr')g = r(r'g)$,
ii) $r(g + g') = rg + rg'$,
iii) $(r + r')g = rg + r'g$.

A right strong R-module is defined similarly. $(R, +)$ is an $(R-R)$-bimodule for a left as well as a right near-ring R. If R is a distributive near-ring then $(R, +)$ is a left as well as a right strong R-module. Many more examples of these structures are given in Grainger [2].

Let G and H be two R-modules (R-comodule. A group homomorphism θ from G to H is called an R-homomorphism if $\forall g \in G$, and $r \in R$, $(gr)\theta = (g\theta)r$, $((rg)\theta = r(g\theta))$. An $(R - S)$-homomorphism for $(R - S)$-bimodule are defined in a similar way.

We refer to Clay [1] for definitions and results about near-rings and to Hungerford [3] for groups and tensor products of ring modules.

2. Tensor product

Let R be a left near-ring, A an R-module and B an R-comodule. Let F be the free group on $A \times B$. Let L and K be the normal subgroups of F generated by:

$$\{(a + a', b) - (a', b) - (a, b), (ar, b) - (a, rb) | a, a' \in A, b \in B, r \in R\}$$

and

$$\{(a, b + b') - (a, b') - (a, b), (ar, b) - (a, rb) | a \in A, b, b' \in B, r \in R\}$$

respectively.

We call F/L the left tensor product of A and B and denote it by $A_R \otimes B$ and call F/K the right tensor product of A and B and denote it by $A \otimes {}_R B$. The coset $(a, b) + L$ is denoted by $a_l \otimes b$ and $(a, b) + K$ by $a \otimes {}_r b$. The coset L is denoted by 0 in both cases. Since F is generated by $A \times B$, $F/L = A_R \otimes B$ and $F/K = A \otimes {}_R B$ are generated by $\{a_l \otimes b | a \in A, b \in B\}$ and $\{a \otimes {}_r b | a \in A, b \in B\}$ respectively. An element of $A_R \otimes B$ $(A \otimes {}_R B)$ is a finite sum of the form $\sum \varepsilon_i(a_i {}_l \otimes$ $\otimes b_i)$ $(\sum \varepsilon_i(a_i \otimes {}_r b_i))$, where each $\varepsilon_i = \pm 1$.

The following result is a direct consequence of the definition of tensor products.

Theorem 2.1. **1.** *$\forall a, a' \in A, b \in B, r \in R$, the following are satisfied in $A_R \otimes B$:*

i) $(a + a')_l \otimes b = a_l \otimes b + a'_l \otimes b$,

ii) $ar_l \otimes b = a_l \otimes rb$,

iii) $0_{Al} \otimes b = 0$,

iv) $(-a)_l \otimes b = -(a_l \otimes b)$.

2. *$\forall a \in A, b, b' \in B, r \in R$, the following are satisfied in $A \otimes_R B$:*

i) $(a)_l \otimes (b + b') = a \otimes_r b + a \otimes_r b'$,

ii) $ar \otimes_r b = a \otimes_r rb$,

iii) $a \otimes_{rB} b = 0$,

iv) $a \otimes_r (-b) = -(a \otimes_r b)$.

Remarks. **1.** In general $(a+a')r \neq ar+a'r$ in A, as A is an R-module. But we have

$$(a + a')r_l \otimes b = (a + a')_l \otimes rb = a_l \otimes rb + a'_l \otimes rb =$$
$$= ar_l \otimes b + a'r_l \otimes b = (ar + a'r)_l \otimes b$$

This shows that in $A_R \otimes B$, with $b \neq 0$.

2. It is possible that $a \otimes_r b$ is zero in $A \otimes_R B$, with $b \neq 0$.

Later on we will show these by examples (Ex. 4 and Ex. 1 respectively).

We generalize the definition of a middle linear map.

Definition. Let R, A and B be as before and let C be any group with a map $f : A \times B \longrightarrow C$. Then we call f:

1. a *left R-middle linear map* if:

$(a+a', b)f = (a, b)f + (a', b)f,\ (ar, b)f = (a, rb)f,\ \forall a, a' \in A,\ b \in B,\ r \in R$;

2. a *right R-middle linear map* if:

$(a, b+b')f = (a, b)f + (a, b')f,\ (ar, b)f = (a, rb)f,\ \forall a \in A,\ b, b' \in B,\ r \in R$.

From now on we will write LRMLM and RRMLM for a left and a right R-middle linear map respectively.

It is easy to see that

$$\theta_l = j\pi_L : A \times B \longrightarrow A_R \otimes B = F/L,\ (a, b) \longmapsto a_l \otimes b$$

and

$$\theta_l = j\pi_K : A \times B \longrightarrow A \otimes_R B = F/K,\ (a, b) \longmapsto a \otimes\ rb$$

are LRMLM and RRMLM respectively. Here $j : A \times B \longrightarrow F$ is the inclusion map, and

$$\pi_L : F \longrightarrow F/L \text{ and } \pi_K : F \longrightarrow F/K$$

are the natural homomorphisms. We call $\theta_l(\theta_r)$ the canonical LRMLM (RRMLM).

Now we prove the universal property of the tensor products.

Theorem 2.2. *Let R be a left near-ring, A be an R-module and B an R-comodule. Let C be a group with a function $f : A \times B \longrightarrow C$, and θ_l (θ_r) be as above.*

1. *If f is a LRMLM, then there exists a unique group homomorphism $g : A_R \otimes B \longrightarrow C$, such that $\theta_l g = f$.*

2. *If f is an RRMLM, then there exists a unique group homomorphism $g : A \otimes {}_R B \longrightarrow C$, such that $\theta_r g = f$.*

Proof. We prove only (1) as the proof of (2) is similar. Consider the following diagram

$$\begin{array}{ccccc} A\times B & \overset{j}{\longrightarrow} & F & \overset{\pi_L}{\longrightarrow} & F/L = A_R\otimes B \\ & f\searrow & h\downarrow & \swarrow g & \\ & & C & & \end{array}$$

where h is the unique homomorphism extending f, as F is the free group on $A \times B$. Since f is an LRMLM, $L \subseteq \operatorname{Ker} h$. This gives us a unique group homomorphism $g : F/L \longrightarrow C$, such that $\pi_L g = h$. It follows then $\theta_l g = j\pi_L g = jh = f$. Now for the uniqueness let g' be another homomorphism from F/L to C with $\theta_l g' = f$. Then:

$$(a\ {}_l\otimes b)g' = (a,b)\theta_l g' = (a,b)f = (a,b)jh = (a,b)j\pi_L g =$$
$$= (a,b)\theta_l g = (a\ {}_l\otimes b)g.$$

Therefore g and g' agree on the generators of $A\ {}_R\otimes B$ and hence are equal. $\Diamond$

Corollary. *Let A, A' be R-modules and B, B' be R-comodules, $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ be R-homomorphisms of R-modules and R-comodules respectively. Then there are unique group homomorphisms*

$$\phi : A\ {}_R\otimes B \longrightarrow A'\ {}_R\otimes B' \text{ and } \psi : A \otimes {}_R B \longrightarrow A' \otimes {}_R B'$$

such that $(a\ {}_l\otimes b)\phi = af\ {}_l\otimes bg$ and $(a \otimes {}_r b)\psi = af \otimes {}_r bg$.

The homomorphism ϕ and ψ in the above corollary are denoted by $f\ {}_l\otimes g$ and $f \otimes {}_r g$ respectively.

If A, A', A'' are R-modules, B, B', B'' are R-comodules with R-homomorphisms $f : A \longrightarrow A'$, $f' : A \longrightarrow A''$, $g : B \longrightarrow B'$ and $g' : B \longrightarrow B''$, then $(f\ {}_l\otimes g)(f'\ {}_l\otimes g') = ff'\ {}_l\otimes gg'$ and $(f \otimes {}_r g)(f' \otimes$ $\otimes {}_r g') = ff' \otimes {}_r gg'$.

Moreover, if f and g are isomorphisms then $f\ {}_l\!\otimes g$ and $f \otimes_r g$ are isomorphisms.

Next we consider some special cases.

Theorem 2.3. *Let R and S be left near-rings, A an $(R-S)$-bimodule and B an S-comodule. Then $A\ {}_S\!\otimes B$ and $A \otimes{}_S B$ are R-comodules.*

Proof. For each $r \in R$, define $\alpha_r : A \times B \longrightarrow A\ {}_S\!\otimes B$ by

$$(a,b)\alpha_r = ra\ {}_l\!\otimes b, \forall (a,b) \in A \times B.$$

We claim that α_r is a LSMLM. $\forall a, a' \in A,\ b \in B$ and $s \in S$ we have:

$$(a+a',b)\alpha_r = r(a+a')\ {}_l\!\otimes b = (ra+ra')\ {}_l\!\otimes b = ra\ {}_l\!\otimes b + ra'_l \otimes b =$$
$$= (a,b)\alpha_r + (a',b)\alpha_r$$
$$(as,b)\alpha_r = r(as)\ {}_l\!\otimes b = (ra)s\ {}_l\!\otimes b = ra\ {}_l\!\otimes sb = (a,sb)\alpha_r$$

By Th. 2.2 there is a unique endomorphism β_r of $A_S \otimes B$ such that $\theta_l\beta_r = \alpha_r$, where θ_l is the canonical LSMLM: $A \times B \longrightarrow A\ {}_S\!\otimes B$. The action of R on $A_S \otimes B$ is now defined by $ru = u\beta_r$, for $r \in R$ and $u \in A_S \otimes B$. We claim that this action defines $A_S \otimes B$ as an R-comodule. For all $u, u' \in A_S \otimes B$ and $r, r' \in R$ we have:

$$r(u+u') = (u+u')\beta_r = u\beta_r + u'\beta_r = ru + ru'.$$

In order to prove that $(rr')u = r(r'u)$, it is enough to prove that $\beta_{rr'} = = \beta_{r'}\beta_r, \forall r, r' \in R$. We look at their action on the generators of $A\ {}_S\!\otimes B$.

$$(a\ {}_l\!\otimes b)\beta_{rr'} = (a,b)\theta_l\beta_{rr'} = (a,b)\alpha_{rr'} = (rr')a\ {}_l\!\otimes b = r(r'a)\ {}_l\!\otimes b =$$
$$= (r'a,b)\alpha_r = (r'a\ {}_l\!\otimes b)\beta_r = (a,b)\alpha_{r'}\beta_r = (a\ {}_l\!\otimes b)\beta_{r'}\beta r.$$

This completes the proof of the fact that $A\ {}_S\!\otimes B$ is an R-comodule. Similarly it can be proved that $A \otimes{}_S B$ is an R-comodule. ◊

Corollary. *$R_R \otimes B$ and $R \otimes_R B$ are R-comodules for any R-comodule B.*

Remark. Let R be a left near-ring with 1, and B be any unital R-comodule. Then for $r \in R$ and $b \in B$, we have $r\ {}_l\!\otimes b = 1\ {}_l\!\otimes rb$ and $r \otimes_r b = 1 \otimes_r rb$.

Therefore we have:

i) $R\ {}_R\!\otimes B$ is generated by $\{1\ {}_l\!\otimes b | b \in B\}$.

ii) $R \otimes{}_R B$ is generated by $\{0 \otimes_r b, 1 \otimes_r b | b \in B \backslash \{0\}\}$.

iii) $\theta_b : R \longrightarrow R\ {}_R\!\otimes B$ defined by $r\theta_b = r\ {}_l\!\otimes b$ is a group homomorphism.

iv) $\theta_b : R \longrightarrow R \otimes{}_R B$ defined by $r\theta_b = r \otimes_r b$ is not a group homomorphism in general.

Theorem 2.4. *Let R and S be left near-rings, A an R-module and B an $(R-S)$-bimodule. Then $A_R \otimes B$ is an S-comodule with S acting on* *the right. If in addition B is a right strong S-module then $A \otimes {}_R B$ is also an S-comodule with S acting on the right.*

Proof. For $s \in S$ define $\alpha_s : A \times B \longrightarrow A_R \otimes B$ by $(a,b)\alpha_s = a \,{}_l\otimes \otimes\, bs$. It is easy to see that α_s is a LRMLM. This gives us a unique endomorphism β_s of $A_R \otimes B$ such that $\theta_l \beta_s = \alpha_s$, where $\theta_l : A \times B \longrightarrow \longrightarrow A_R \otimes B$ is the canonical LRMLM. For all $(a \,{}_l\otimes b) \in A_R \otimes B$, we have:

$$(a \,{}_l\otimes b)\beta_s = (a,b)\theta_l\beta_s = (a,b)\alpha_s = a \,{}_l\otimes (bs).$$

As in Th. 2.3, we define an action of S on $A_R \otimes B$ by $us = u\beta_s$, $\forall u \in A_R \otimes B$. Clearly $(u+u')s = us + u's$, $\forall u,\ u' \in A_R \otimes B$.

For the other condition we need to show that $\beta_{ss'} = \beta_s\beta_{s'}, \forall s, s' \in \in S$. It is enough to look at their behavior on the generators.

$$(a \,{}_l\otimes b)\beta_{ss'} = a \,{}_l\otimes b(ss') = a \,{}_l\otimes (bs)s' = (a \,{}_l\otimes (bs))\beta_{s'} = (a \,{}_l\otimes b)\beta_s\beta_{s'}.$$

The second part is proved similarly. ◊

Corollary. *$A_R \otimes R$ is an R-comodule with R acting on the right. If R is distributive then $A \otimes {}_R R$ is an R-comodule with R acting on the right.*

Remarks. 1. Let R be a left near-ring with 1 and A be a unital R-module. Then we have: $A_R \otimes R$ is generated by $\{a \,{}_l\otimes 1,\ a \,{}_l\otimes 0 | a \in \in A \backslash \{0\}\}$ and $A \otimes {}_R R$ is generated by $\{a \otimes {}_r 1 | a \in A\}$.

2. If A is any group and B is an abelian group then $A_Z \otimes B$ and $A \otimes {}_Z B$ are right Z-comodules and hence are abelian groups.

3. If R is a ring and A is an R-module in the near-ring sense, i.e $(A,+)$ is not necessarily abelian, then $A_R \otimes B$ and $A \otimes {}_R B$ can be constructed, which may be different. An example of this will be given in the next section.

3. Examples

Example 1. By Remark 1 following Th. 2.4, $(S_3)_Z \otimes Z_2$ and $S_3 \otimes \otimes\, {}_Z(Z_2)$ are abelian groups. We look into their structure further. S_3 has presentation:

$$\langle a, b \mid a3,\ b2,\ (a+b)2 \rangle.$$

Using Th. 2.1 and the definition of $(S_3)_Z \otimes Z_2$ we see that it is generated by $\{a \,{}_l\otimes 0,\ b \,{}_l\otimes 0,\ a \,{}_l\otimes 1,\ b \,{}_l\otimes 1\}$. These generators satisfy the following:

$$(a\ {}_l\!\otimes 0)2 = a2\ {}_l\!\otimes 0 = a\ {}_l\!\otimes 2(0) = a\ {}_l\!\otimes 0, (a\ {}_l\!\otimes 1)2 = a2\ {}_l\!\otimes 1 = \\ = a\ {}_l\!\otimes 2 = a\ {}_l\!\otimes 0, (a\ {}_l\!\otimes 1)3 = a3\ {}_l\!\otimes 1 = 0\ {}_l\!\otimes 1 = 0.$$

This shows that $a\ {}_l\!\otimes 0 = 0$ and $a\ {}_l\!\otimes 1 = 0$. Hence we have the following presentation:

$$(S_3)_Z \otimes Z_2 = \langle b\ {}_l\!\otimes 0,\ b\ {}_l\!\otimes 1 \mid (b\ {}_l\!\otimes 0)2,\ (b\ {}_l\!\otimes 1)2\rangle.$$

Therefore $(S_3)_Z \otimes Z_2 \cong Z_2 \oplus Z_2$.

On the other hand a set of generators of $S_3 \otimes_Z (Z_2)$ is $\{x \otimes_r 1 \mid x \in \in S_3\}$. Moreover $(x \otimes_r 1)2 = x \otimes_r 2 = x \otimes_r 0 = 0$. Therefore $S_3 \otimes_Z (Z_2)$ is an abelian group generated by six elements each of which is of order 2. Hence $S_3 \otimes_Z (Z_2) \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$.

We note that $a2 \neq 0$ in A but $a2 \otimes_r 1 = a \otimes_r 2 = 0$ in $S_3 \otimes_Z (Z_2)$.

Example 2. By the corollary to Th. 2.4, $(S_3)_Z \otimes Z$ and $S_3 \otimes_Z Z$ are Z-comodules with Z acting on the right and hence are abelian groups. A presentation of $(S_3)_Z \otimes Z$ is:

$$\langle a\ {}_l\!\otimes 0,\ b\ {}_l\!\otimes 0,\ a\ {}_l\!\otimes 1,\ b\ {}_l\!\otimes 1 \mid (a\ {}_l\!\otimes 0)3,\ (a\ {}_l\!\otimes 1)3,\ (b\ {}_l\!\otimes 0)2,\ (b\ {}_l\!\otimes)2\rangle.$$

Therefore it is isomorphic to $Z_6 \oplus Z_6$, a presentation of which can be:

$$\langle a\ {}_l\!\otimes 0 + b\ {}_l\!\otimes 1,\ a\ {}_l\!\otimes 1 + b\ {}_l\!\otimes 0 \mid (a\ {}_l\!\otimes 0 + b\ {}_l\!\otimes 1)6,\ (a\ {}_l\!\otimes 1 + b\ {}_l\!\otimes 0)6\rangle.$$

On the other hand the order of $0 \otimes_r 1$ in $S_3 \otimes_Z Z$ is infinite. Therefore $S_3 \otimes_Z Z$ is an infinite abelin group with a presentation $\langle x \otimes \otimes_r 1 \mid x \in S_3\rangle$.

Example 3. Consider the dihedral group D_8 of order 8 with a presentation $\langle a, b \mid a4,\ b2,\ (a+b)2\rangle$ $R = \{0, I, f, g\}$ is a semigroup of endomorphisms of D_8 defined by the following table:

$\alpha(x)$	0	a	a2	a3	b	a+b	a2+b	a3+b
0	0	0	0	0	0	0	0	0
I	0	a	a2	a3	b	a+b	a2+b	a3+b
f	0	a2	0	a2	a2	0	a2	0
g	0	a3	a2	a	b	a3+b	a2+b	a+b

It is easy to verify that $(R, \oplus, \circ) \cong Z_4$, is a ring, where $\oplus$ and $\circ$ are given by the tables:

⊕	0	I	f	g
0	0	I	f	g
I	I	f	g	0
f	f	g	0	I
g	g	0	I	f

∘	0	I	f	g
0	0	0	0	0
I	0	I	f	g
f	0	f	0	f
g	0	g	f	I

For details about this ring we refer to Ex. 3.3.10 of [5]. Here we just mention that $\oplus$ is not the ordinary pointwise addition of mappings, because:

$$f(b) + g(b) = a2 + b \neq b = I(b) = (f \oplus g)(b)$$

D_8 is a unitary R-comodule, because every element of R is an endomorphism of D_8. Since $(f \oplus g)(b) \neq f(b) + g(b)$, D_8 is not a left strong R-module. As R is an $(R - R)$-bimodule, by Th. 2.3, $R_R \otimes D_8$ and $R \otimes {}_R D_8$ are R-comodules.

$R_R \otimes D_8$ is generated by $\{I\ _l \otimes x \mid x \in D_8\}$, with each generator of order 4. But $I_l \otimes a + I_l \otimes b$ is of infinite order. Hence $R_R \otimes D_8$ is an infinite non abelian group.

$R \otimes {}_R D_8$ is generated by $I \otimes {}_r a$, $I \otimes {}_r b$, $0 \otimes {}_r a$, $0 \otimes {}_r b$, of orders 4, 2, 4 and 2 respectively. In this case also $I \otimes {}_r a + 0 \otimes {}_r a$ is of infinite order. Hence $R_R \otimes D_8$ is an infinite nonabelian group.

Example 4 (Ex. 14 of [2]). Let $V = \{0,\ a,\ b,\ a + B\}$ be the Klein 4-group and let $R = \{\beta_0,\ \beta_1,\ \beta_2,\ \beta_3\}$ be the set of four endomorphisms of V given by the following table:

$\beta(x)$	0	a	b	a+b
β_0	0	0	a+b	a+b
β_1	0	a	b	a+b
β_2	0	b	a	a+b
β_3	0	a+b	0	a+b

$\oplus$ and $\circ$ are defined on R by the following tables:

$\oplus$	β_0	β_1	β_2	β_3
β_0	β_0	β_1	β_2	β_3
β_1	β_1	β_0	β_3	β_2
β_2	β_2	β_3	β_0	β_1
β_3	β_3	β_2	β_1	β_0

$\circ$	β_0	β_1	β_2	β_3
β_0	β_0	β_0	β_3	β_3
β_1	β_0	β_1	β_2	β_3
β_2	β_0	β_2	β_1	β_3
β_3	β_0	β_3	β_0	β_3

β_0 is the zero for $\oplus$ and β_1 is the identity for $\circ$. We note that β_0 is not the zero endomorphism. It is easy to see that $(R, \oplus, \circ)$ is a left near-ring with identity. It is not a ring, since: $(\beta_3 \oplus \beta_1) \circ \beta_2 = \beta_2 \circ \beta_2 = \beta_1$, whereas, $\beta_3 \circ \beta_2 \oplus \beta_1 \circ \beta_2 = \beta_0 \oplus \beta_2 = \beta_2$

Moreover, V is a unitary R-comodule with R acting on the left. Therefore we can construct $R_R \otimes V$ and $R \otimes_R V$ which are R-comodules by the corollary of Th. 2.3. By the remark after that corollary we have the following:

$R_R \otimes V$ is generated by $\{\beta_1\ {}_l\otimes x \mid x \in V\}$ and $R \otimes_R V$ is generated by $\{\beta_1 \otimes_r x,\ 0 \otimes_r x \mid x \in V\}$.

Moreover, each generator of $R_R \otimes V$ and $R \otimes_R V$ is of order 2. We consider the two cases seperately.

Easy calculations show that the generators of $R_R \otimes V$ commute. Also

$$\beta_1\ {}_l \otimes (a+b) = \beta_l\ {}_l \otimes \beta_0(b) = \beta_1 \circ \beta_0\ {}_l \otimes b = \beta_0\ {}_l \otimes b = 0.$$

Therefore $R_R \otimes V$ is an abelian group generated by the following elements:

$$\beta_1\ {}_l \otimes 0,\ \beta_1\ {}_l \otimes a,\ \beta_1\ {}_l \otimes b,$$

each of which is of order 2. Hence $R_R \otimes V \cong Z_2 \oplus Z_2 \oplus Z_2$.

We note here that even though $\beta_1 \neq 0_R$, $\beta_1\ {}_l \otimes (a+b) = 0$ in $R_R \otimes V$.

$R \otimes_R V$ is an infinite non abelian group because the elements $\beta_1 \otimes \otimes_r a$, $\beta_0 \otimes_r a$ of $R \otimes_R V$ do not commute and $\beta_1 \otimes_r a + \beta_0 \otimes_r a$ is of infinite order.

The structure of tensor products of near-ring modules can be explored further. These examples suffice here to show the importance of this concept.

References

[1] CLAY, J. R.: Near-rings-Geneses and Applications, Oxford University Press, Oxford, 1992.

[2] GRAINGER, G.: Left modules for left near-rings, Doctoral Thesis, Univ. of Arizona, 1988.

[3] HUNGERFORD, T. W.: Algebra, Springer-Verlag, New York, 1974.

[4] MAHMOOD, S. J. and MANSOURI, M. F.: Tensor Product of Near-ring Modules, Kluwer Academic Publishers, Netherland, 1997.

[5] Mathna, N. M. Q.: Near-rings and their Modules, Master's Thesis, King Saud Univ., KSA, 1990.