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THE FACIAL STRUCTURE OF THE FINITE DIMENSIONAL LATTICIAL CONE

A. B. **Németh**

Faculty of Mathematics and Computer Science, Babeş–Bolyai University, 400084 Cluj-Napoca, Romania

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Abstract: The latticial cone $K \subset \mathbb{R}^n$ is called stiff, if it has a proper face K' of maximal dimension such that each its element is comparable (with respect to the order relation induced by K) with each element of the set $K \setminus K'$. The face K'' of the latticial cone K is a stiff face if it is a stiff cone in the subspace $K'' - K''$; it is a maximal stiff face if it is not contained in any other stiff face of K. It is proved that each latticial cone K in $\mathbb{R}^n$ is the direct sum of its maximal stiff faces; if K is not stiff, then it is the direct sum of its proper maximal stiff faces. If K possesses r different maximal stiff faces, then its conjugate K^* is a cone generated by r linearly independent vectors.

1. Introduction

The theorem of Yudin [9] which asserts that a closed cone $K \subset \subset \mathbb{R}^n$ induces a latticial ordering in $\mathbb{R}^n$ if and only if it is the positive orthant of a coordinate system in $\mathbb{R}^n$, is one of the first important results in the vector lattice theory. (A such cone is called a Yudin cone.) A different proof of Yudin's theorem was given by Szőkefalvi Nagy in [7]. Another interesting result concerning the latticially ordered Euclidean space is the assertion that each totally ordered Euclidean space is lexicographically ordered. Due to Schaefer [8], this theorem

E-mail address: nemab@math.ubbcluj.ro

was rediscovered by Martinez-Legaz and Singer [2]. A different proof of its is given by us in [3] and is included also into the present paper.

We call latticial cone the positive cone of the latticially ordered Euclidean space. The Yudin cone and the lexicographic cone are extremal cases for latticial cones. Both above cited characterization theorems are related to the facial structures of the underlying cones. But what about the latticial structure of the general latticial cone in $\mathbb{R}^n$? Our aim is to answer this question.

To do this we have to mobilize a great amount of known results from the convex geometry, linear algebra and vector lattice theory (Sections 3, 4, 7 and 10), and to augment them with some of their consequences. We have to introduce and handle new notions (Sections 2 and 5), and to prove their properties (Sections 5, 8 and 9), results which can have independent usages. The resulting material becomes this way a contribution in searching the relation between the vectorial ordering and the geometry of the Euclidean space.

2. Terminology and main results

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. For the interior, the closure and the boundary of the set $A \subset \mathbb{R}^n$ we shall use the notations A^o, A^- and A^b respectively.

The nonempty set $W \subset \mathbb{R}^n$ is called a wedge if it possesses the properties:

(i) $W + W \subset W$,

(ii) $tW \subset W \; \forall\, t \in \mathbb{R}_+ = [0, +\infty)$.

The wedge $K \subset \mathbb{R}^n$ is called a cone, if it satisfies the condition

(iii) $K \cap (-K) = \{0\}$.

The space $\mathbb{R}^n$ endowed with a reflexive, transitive, antisymmetrical relation "$\leq$", which is translation invariant (i.e., from $u \leq v$ it follows $u + z \leq v + z$, $\forall z$) and invariant with respect to multiplication with non-negative scalars (i.e., $u \leq v$ implies $tu \leq tv$, $\forall\; t \in \mathbb{R}_+$,) is called ordered Euclidean space and is denoted with $(\mathbb{R}^n, \leq)$. We say that the elements x, $y \in \mathbb{R}^n$ are comparable if either $x \leq y$ or $y \leq x$.

If $(\mathbb{R}^n, \leq)$ is an ordered Euclidean space, then the set $K = \{x \in \in \mathbb{R}^n : x \geq 0\}$ is a cone, which is called its positive cone.

With the aid of the cone $K \subset \mathbb{R}^n$ it can be defined a binary relation $\leq_K$ by putting $u \leq_K v$ whenever $v - u \in K$. Endowed with this relation $\mathbb{R}^n$ becomes an ordered Euclidean space, whose positive cone is K. Thus the order relation of the ordered Euclidean space is completely determined by its positive cone K. This circumstance motivates the parallel usage for the ordered Euclidean space $(\mathbb{R}^n, \leq)$ also of the notation $(\mathbb{R}^n, K)$, where K is the positive cone of the space.

The cone K in $\mathbb{R}^n$ is called generating if $\mathrm{sp}K = K - K$, the vector space spanned by K is $\mathbb{R}^n$.

If $(\mathbb{R}^n, \leq)$ is ordered Euclidean space, then upper and lower bounds of sets are defined as usual. For u, $v \in \mathbb{R}^n$ the supremum $u \vee v = = \sup\{u, v\}$ is defined as being the least upper bound of the set $\{u, v\}$ (if it exists). We define similarly the infimum $u \wedge v = \inf\{u, v\}$ as being the greatest lower bound of the set $\{u, v\}$ (if it exists).

The ordered Euclidean space $(\mathbb{R}^n, \leq)$ is called latticially ordered, if $u \vee v$ (and hence also $u \wedge v$) exists for every u and v. The positive cone of a latticially ordered Euclidean space is called latticial cone. It is easy to see that a latticial cone is generating.

The subset K' of the cone K is called face of K if it is a cone and if from $0 \leq y \leq x$, $x \in K'$, it follows that $y \in K'$, where $\leq$ is the order relation induced by K. The one dimensional faces are called edges of K. The cone K is its own face. The face $K' \subset K$ is called proper face, if $K' \neq K$. The set $\{0\}$ is always a face of K, which is said to be its trivial face. The face K' of the latticial cone K is itself a latticial cone in $\mathrm{sp}K' = K' - K'$.

Definition 1. A latticial cone $K \subset \mathbb{R}^n$ is called *stiff*, if it possesses a proper face K_0 of maximal dimension, such that each element of K_0 is comparable with every element of $K \setminus K_0$. The positive cone $K = \mathbb{R}_+$ of $\mathbb{R}$ is stiff. The face of this K for which the condition in the definition holds is $K_0 = \{0\}$.

The positive cone of the totally ordered Euclidean vector space is a stiff latticial cone (see Cor. 8).

Let $H \subset \mathbb{R}^n$ be a hyperplane through 0. Let us denote by H^+ one of the open semispaces determined by it. Let $K_0 \subset H$ be a latticial cone in the $n-1$-dimensional Euclidean space H. Then $K = K_0 \cup \cup H^+$ is a stiff cone in $\mathbb{R}^n$. Every stiff cone can be represented in this form. (The assertions follow from Lemma 24 and Lemma 25.) Hence the single closed stiff cone is the one dimensional one.

Since each face K' of the latticial cone K is a latticial cone in the space $\mathrm{sp}K'$ it spans, the term "stiff face" make a sense. A stiff face is maximal, if it is not contained properly in another stiff face. The edges of the latticial cone are its stiff faces.

We say that a cone K is the direct sum of cones $K_1, \ldots, K_m$, if each element of K can be uniquely represented as sum of elements of these cones. (In this case the cones K_j are faces of K.)

Our main result is the following one:

Theorem 1. *Each latticial cone in the Euclidean space is the direct sum of its maximal stiff faces.*

If the latticial cone is stiff, then it is the single maximal stiff face it contains. Hence the "direct sum" in the theorem reduces to a single term.

Other important result in this regard is the following:

Theorem 2. *For the latticial cone $K \subset \mathbb{R}^n$ the following assertions are equivalent:*

1. *K is not a stiff cone;*
2. *K is the sum of its proper faces;*
3. *K is the direct sum of its proper maximal stiff faces.*

It will be shown that if the latticial cone is closed, then every its stiff face is an edge. Hence from this theorem it follows the

Corollary 1. *Each closed latticial cone in the Euclidean space is the direct sum of its edges.*

This assertion is the theorem of Yudin [9].

Denote by $\langle x, y\rangle$ the scalar product of the elements x and y. If K is a cone, then the set

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0, \ \forall\, y \in K\}$$

is called the dual of K.

Theorem 3. *Suppose that the latticial cone K has the representation*

$$K = K_1 \dot{+} \ldots \dot{+} K_r$$

with K_i maximal stiff face of K, $\forall\, i$ and $\dot{+}$ denoting direct sum. Then the dual K^* of K is a cone in $\mathbb{R}^n$ engendered by r linearly independent vectors.

3. Preliminaries from the convex geometry

If H is a hyperplane in $\mathbb{R}^n$, then there exists a nonzero vector x^* in $\mathbb{R}^n$ and a real number α such that

$$H = \{x \in X : \langle x^*, x\rangle = \alpha\}.$$

x^* is called the normal of H. The set

$$H_+ = \{x \in X : \langle x^*, x\rangle \geq \alpha\},$$

and respectively

$$H^+ = \{x \in X : \langle x^*, x\rangle > \alpha\}$$

is called the closed, respectively the open semispace determined by H (in the direction of x^*). The set H_+ is closed, the set H^+ is open in $\mathbb{R}^n$, $H^+ = (H_+)^o = H_+ \setminus H$, $H_+ = H \cup H^+$.

The hyperplane H is a supporting hyperplane of the set A if $A \subset \subset H_+$ and $H \cap A^b \neq \emptyset$.

The relative interior ri$C$ of a convex set $C \subset \mathbb{R}^n$ is the interior of C with respect to its affine hull.

Some standard results from the convex geometry of the Euclidean space as well as their immediate consequences constitute basic tools in our proofs. They can be get in monographs as [5], or [6], and are the following ones:

1. The closure of a cone is a wedge.
2. The cone K in $\mathbb{R}^n$ is generating if and only if $K^o \neq \emptyset$. Hence the relative interior of every cone in $\mathbb{R}^n$ is not empty.
3. If W is a wedge with $W^o \neq \emptyset$, then $W^o + W^- \subset W^o$.
4. Every boundary point of the wedge (cone) is contained in some of its supporting hyperplane. Every supporting hyperplane of the wedge (cone) contains the point 0.
5. If W is a wedge, $x \notin W^-$, then there exists its supporting hyperplane

$$H = \{y \in \mathbb{R}^n : \langle x^*, y\rangle = 0\}$$

such that $\langle x^*, x\rangle < 0$, that is, $x \in -H^+$.

We shall use these results next in the proofs with no special references.

4. The geometry of faces

In this section we insert, for further usage, a lot of results on faces of a general cone in the Euclidean space. Most of them are standard results of the theory of cones and can be easily verified (hence we list them without proofs).

Lemma 1. *If $K \subset \mathbb{R}^n$ is a cone and $\{K_i : i \in I\}$ is a family of faces of K, then $\cap_{i\in I} K_i$ is a face of K.*

The subset K_0 of the cone K is called subcone of K, if it is a cone. The subcone $K_0 \subset K$ is a section subcone of K, if $K_0 = (\mathrm{sp}K_0) \cap K$.

Lemma 2. *Each face of K is a section subcone of K.*
Lemma 3. *Let K_0 be a section subcone of K, u, $v \in \mathrm{sp}K_0$. Then $u \leq_K v$ holds if and only if $u \leq_{K_0} v$.*
Lemma 4. *The subcone K_1 of the face K_0 of the cone K is face of K_0 if and only if it is face of K.*
Lemma 5. *Let K_0 be a face of the generating cone $K \subset \mathbb{R}^n$, $K_0 \neq K$. Then $K_0 \cap K^o = \emptyset$. Hence $\dim K_0 < n$ (where $\dim K_0$ is the dimension of $\mathrm{sp}K_0$).*
Corollary 2. *If K_1 and K_2 are faces of the cone $K \subset \mathbb{R}^n$ and $K_2 \subset \subset K_1$, $K_2 \neq K_1$, then $\dim K_2 < \dim K_1$.*
Lemma 6. *If H is a supporting hyperplane of the cone K, then $K' = = H \cap K$ is a face of K.*

A face of K obtained as an intersection of K with a supporting hyperplane of its, is called a supporting face of K.

Lemma 7. *Let H' be a supporting hyperplane of the generating cone $K \subset \mathbb{R}^n$, $x \in K \cap H'$ and $-x \notin K^-$. Then there exists a supporting hyperplane H'' of K, such that*

$$\dim K \cap H'' < \dim K \cap H'.$$

Proof. Let H be a supporting hyperplane of K for which $-x \in \in -H^+$. Let x^* be the normal of H, and x' be the normal of H'. Put $x'' = x' + x^*$, and

$$H'' = \{y \in \mathbb{R}^n : \langle x'', x\rangle = 0\}.$$

Since $K \subset H'_+ \cap H_+ \subset H''_+$, H'' is a supporting hyperplane of K. From the relation $(H'_+ \cap H_+) \cap H'' = H' \cap H$ we have

$$K \cap H'' \subset (H'_+ \cap H_+) \cap H'' = H' \cap H,$$

wherefore

$$K \cap H'' \subset (K \cap H') \cap H \subset K \cap H'.$$

At the same time $x \in K \cap H'$ and $x \notin H''$, since $\langle x'', x\rangle > 0$. This shows that the supporting face $K \cap H''$ is a proper subset of the face $K \cap H'$. Putting this together with Lemma 2, the conclusion of the lemma follows. ◊

As consequences of this lemma we have the following results:

Corollary 3. *Let H be a supporting hyperplane of the generating cone K. The face $K \cap H$ is with respect to the inclusion a minimal supporting face of K if and only if there is no $x \in K \cap H$ for which $-x \notin K^-$.*
Proof. Indeed, if it would exist an $x \in K \cap H$ such that $-x \notin K^-$, it would follow from the lemma that $K \cap H$ is not a minimal supporting face.

On the other hand, each supporting hyperplane of the cone K is the supporting hyperplane of the wedge K^-. If for each element x of $K_0 = K \cap H$ $-x \in K^-$, it follows that sp$K_0 = K_0 - K_0 \subset K^-$ and hence it will be contained in every supporting hyperplane of the wedge K^-. Accordingly $K_0 - K_0$ is contained in every supporting hyperplane of K, hence no proper face of K_0 can be a supporting face. $\Diamond$

Corollary 4. *If $K \subset \mathbb{R}^n$ is a closed cone, then it has a supporting hyperplane H with the property $K \cap H = \{0\}$.*

Let K_0 be a subcone of the cone K. Then the set

$$\{y \in K : \exists x \in K_0 \text{ for which } y \leq x\}$$

is a face of K called the face engendered by K_0. If K_0 is generated by a single element $y \in K$, then the above defined face will be called the face engendered by y.

Lemma 8. *The face of K engendered by the subcone $K_0 \subset K$ does not meet the interior K^o of K, if and only if K_0 has this property.*

Proof. If y would be the common element of the engendered face and of K^o, then for $x \geq y$ there would be $x \in K^o$. $\Diamond$

Lemma 9. *If the boundary K^b of the cone K possesses a nonzero element of K, then K possesses a supporting face different from $\{0\}$.*

Proof. Let H be the supporting hyperplane of K containing the element $y \in (K^b \cap K) \setminus \{0\}$ Then $K \cap H$ is the requested supporting face. $\Diamond$

5. The anti-Archimedean subcone of a cone

Definition 2. Let K be the positive cone of the ordered Euclidean space $(\mathbb{R}^n, \leq)$. The element $a \in K$ is called *anti-Archimedean*, if there exists $b \in \mathbb{R}^n$ with the property $na \leq b$, $\forall n \in \mathbb{N}$. The element $a \in$ $\in K$ is anti-Archimedean if and only if one of the following conditions hold: There exists $b \in K$ such that: (i) $ta \leq b$, $\forall\, t \in \mathbb{R}^+ = (0, +\infty)$; (ii) $a \leq tb\, \forall\, t \in \mathbb{R}^+$; (iii) $ra \leq sb$, $\forall\, r,\, s \in \mathbb{R}^+$.

Denote by K_a the subset in K of the anti-Archimedean elements. Then K_a is a cone, called the *anti-Archimedean subcone of* K. Obviously, $\{0\} \subset K_a$. If $K_a = \{0\}$ we say that the anti-Archimedean cone is trivial.

Lemma 10. *Let K_a be the anti-Archimedean subcone of the generating cone K. If $x \in K_a$, $y \in K^o$, then $k \leq y$.*

Proof. There exists by definition $b \in K$ such that $tx \leq b$, $\forall t \in \mathbb{R}^+$.

Since $y \in K^o$, $\exists s \in \mathbb{R}^+$, for which $sb \leq y$, whence $stx \leq sb \leq y$, $\forall t \in \in \mathbb{R}^+$. For a suitable t the assertion follows. ◊

Lemma 11. *Let K_a be the anti-Archimedean subcone of the cone $K \subset \subset \mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is in K_a if and only if x is in K and $-x$ is in K^-.*

Proof. Suppose first that K is a generating cone, $x \in K_a$, $y \in K^o$. Then from Lemma 10, $nx \leq y$, $\forall n \in \mathbb{N}$. Hence

$$\frac{1}{n}y - x \in K,\ \forall n \in \mathbb{N},\ \text{whence } \frac{1}{n}y - x \to -x,\ n \to \infty,\ \Rightarrow -x \in K^-.$$

Suppose now that $x \in K$, $-x \in K^-$. Then for $y \in K^o$, $\frac{1}{n}y \in K^o$ and then $\frac{1}{n}y - x \in K$, $\forall n \in \mathbb{N}$, hence $nx \leq y, : \forall n \in \mathbb{N}$, thus $x \in K_a$.

If K is not generating, then we restrict the above reasoning to the subspace sp$K = K - K$, where K is generating. ◊

Corollary 5. *If the cone K is closed, then $K_a = \{0\}$.*

Lemma 12. *The anti-Archimedean cone K_a is the minimal supporting face of the cone K. Hence it is contained in every supporting hyperplane of K.*

Proof. The proof follows from Lemma 7, the Cor. 3 and Lemma 11. ◊

Lemma 13. *Suppose that the dimension of the anti-Archimedean subcone K_a of the cone $K \subset \mathbb{R}^n$ is $n-1$. Then*

1. *the cone K possesses a single supporting hyperplane H,*
2. *$K_a = K \cap H$,*
3. *$K^o = H^+$,*
4. *$K = K^o \cup K_a$.*

Proof. The subcone K_a is a proper face of K, hence from Cor. 2 one has that K is n-dimensional, hence $K^o \neq \emptyset$.

From Lemma 12, K possesses a supporting hyperplane H for which

$$K_a = K \cap H.$$

From Lemma 11, $-K_a \subset K^-$ and since K^- is a wedge,

$$H = K_a - K_a \subset K^- \subset H_+.$$

Let us show that

$$K^o = H^+,$$

whence it will follow that H is the single supporting hyperplane of K.

Let x^* be the normal of the supporting hyperplane H, and $k \in \in K^o$ satisfying $\langle x^*, k\rangle = 1$. Assume that there exists $y \in H^+ \setminus K$. Then $\langle x^*, y\rangle > 0$ and since by a multiplication of y with a positive scalar it remains in $H^+ \setminus K$, we can suppose that $\langle x^*, y\rangle = 1$. Then $y - k \in H \subset K^-$, and then $y \in k + K^- \subset K$, which is impossible.

Thus $H^+ \subset K$ and hence $H^+ \subset K^o$.

On the other hand $K \subset H_+$, and hence $K^o \subset (H_+)^o = H^+$ and our assertion follows.

Since $K \cap H = K_a$, $K \subset H_+$ and $K^o = H^+$, we have finally

$$K = K^o \cup K_a. \qquad \Diamond$$

Definition 3. Let $(\mathbb{R}^n, \leq)$ be an ordered Euclidean space. Denote by L_a the subset in $\mathbb{R}^n$ of the elements x for which exists $y \in \mathbb{R}^n$ such that $-ty \leq x \leq ty$, $\forall t \in \mathbb{R}^+$. L_a is a vector space, called the *anti-Archimedean subspace* of the ordered space.

Lemma 14. *Let K the positive cone of the ordered Euclidean space $(\mathbb{R}^n, \leq)$, and K_a be its anti-Archimedean subcone, L_a the anti-Archimedean subspace of the ordered space. Then $K_a = K \cap L_a$ and hence L_a does not meet the interior of the cone K.*

Proof. Let be $x \in K \cap L_a$. If $y \in \mathbb{R}^n$ is the element in $\mathbb{R}^n$ for which $-ty \leq x \leq ty$, $\forall t \in \mathbb{R}^+$, then $nx \leq y$, $\forall n \in \mathbb{N}$, and hence $x \in K_a$. That is, $K \cap L_a \subset K_a$.

The inclusion $K_a \subset L_a$ is obvious. $\Diamond$

Corollary 6. sp$K_a \subset L_a$.

6. The totally ordered Euclidean space

The ordered Euclidean space $(\mathbb{R}^n, \leq)$ is called totally ordered, if for any two elements u and v of its either $u \leq v$, or $v \leq u$.

If K is the positive cone of the ordered Euclidean space $(\mathbb{R}^n, \leq)$, then the space is totally ordered if and only if $\mathbb{R}^n$ is the reunion of K and $-K$.

The order relation $\leq$ defined in $\mathbb{R}^n$ is called lexicographic if there exists a base $e_1, \dots, e_n$ of the space such that if $u = u^1e_1 + \cdots + u^ne_n$ and $v = v^1e_1 + \cdots + v^ne_n$ are elements of the space, then $u \leq v$ if and only if either $u = v$, or $u^k < v^k$ for the first superscript k for which $u^k \neq v^k$. We shall say in this case that $e_1, \dots, e_n$ realizes the lexicographic ordering. From its definition it follows that the lexicographic ordering is a full ordering in $\mathbb{R}^n$.

Our aim is to prove that the total ordering in $\mathbb{R}^n$ can always realized as a lexicographic ordering.

The positive cone K of the lexicographically ordered Euclidean space is the following one:

$K = \{x = (x^1, ..., x^n) : x^1 = ... = x^{k-1} = 0,\ x^k > 0,\ k = 1, ..., n\} \cup \{0\}$.

The totally ordered Euclidean space is obviously a vector lattice.

Lemma 15. *Let $(\mathbb{R}^n, \leq)$ be totally ordered. If K_a is the anti-Archimedean subcone of the positive cone K of this ordered vector space, then* $\dim K_a = n - 1$.

Proof. Since K is a latticial cone, $K^o \neq \emptyset$. Let be $b \in K^o$, and let us consider the set

$$K_0 = \{x \in K :\ x \leq tb,\ \forall\, t \in \mathbb{R}^+\}.$$

We shall show that $\dim K_0 = n - 1$, and then that $K_0 = K_a$.

Consider the elements $k_1, \dots, k_{n-1}$ in K which together with b form a linearly independent system.

Consider the open line segments $(-k_i, b)$, $i = 1, \dots, n-1$ which necessarily meet the boundary K^b of the cone K. That is, there exist the scalars $t_i \in (0, 1)$ such that $y_i = t_i k_i + (1 - t_i) b \in K^b$, $i = 1, \dots, n-1$. The system of vectors $y_1, \dots, y_{n-1}$ is linearly independent. Since $K \cup (-K) = \mathbb{R}^n$, for each index i either y_i or $-y_i$ is element of K. Denote by z_i those of them which is in K. Then $z_1, \dots, z_{n-1}$ are linearly independent elements of K.

Assume that among the above determined elements there exists a z_i, for which there exists the scalar $t > 0$ such that $tb \leq z_i$. Then $z_i \in K^o$, which furnishes a contradiction with the hypothesis that $y_i \in K^b$.

From the fact that the space is totally ordered, we have then that $z_i \leq tb$, $\forall\, t \in \mathbb{R}^+$, $i = 1, \dots, n-1$. Accordingly $z_1, \dots, z_{n-1} \in K_0$.

From the definition of K_0 follows that $K_0 \subset K_a$ and since $\dim K_0 \geq n - 1$, one has $\dim K_a \geq n - 1$.

On the other hand, since $K_a \cap K^o = \emptyset$, one has $\dim K_a \leq n - 1$. Thus $\dim K_a = n - 1$. Each element $y \in K_a$ can be represented as $y = t_1 z_1 + \cdots + t_{n-1} z_{n-1}$ and hence $y \leq tb$, $\forall\, t \in \mathbb{R}^+$. Thus $y \in K_0$. ◊

Corollary 7. *If L_a is the anti-Archimedean subspace in the totally ordered Euclidean space $(\mathbb{R}^n, K)$, then $L_a = \mathrm{sp}K_a$ and hence $K_a = \mathrm{sp}K_a \cap K$.*

Proof. We know that $K^o \neq \emptyset$ and $L_a \cap K^o = \emptyset$ (Lemma 14), whereby $\dim L_a \leq n - 1$. On the other hand $\mathrm{sp}K_a \subset L_a$ (Cor. 6), and $\dim K_a = n - 1$. ◊

Lemma 16. *If $(\mathbb{R}^n, \leq)$ is totally ordered Euclidean space, and K is its positive cone, then for each section cone K_0 of K $(\mathrm{sp}K_0, \leq_{K_0})$ is a totally ordered vector space.*

Proof. Let be u, $v \in \mathrm{sp}K_0$. We can suppose that $u \leq v$. Then by Lemma 3, $u \leq_{K_0} v$. ◊

Theorem 4. *If* $(\mathbb{R}^n, \leq)$ *is totally ordered Euclidean space, then the ordering is lexicographic.*

Proof. We proceed by induction with respect to the dimension of the space. If $n = 1$, each total ordering in $\mathbb{R}^1$ is lexicographic.

Suppose that each total ordering in $\mathbb{R}^{n-1}$ is lexicographic.

Let $(\mathbb{R}^n, \leq)$ be totally ordered. Then the anti-Archimedean subcone K_a of the positive cone K of the space is of dimension $n-1$ by Lemma 15. From Lemma 16, $(\mathrm{sp}K_a, \leq_{K_a})$ is a totally ordered vector space. Let be $e_1 \in K \setminus K_a$. Then each element $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = t^1 e_1 + y,\ t^1 \in \mathbb{R},\ y \in \mathrm{sp}K_a.$$

(This follows from the fact that $e_1 \notin \mathrm{sp}K_a$. Since from $e_1 \in \mathrm{sp}K_a$, Lemma 12 and Lemma 2 via the relation $K_a = \mathrm{sp}K_a \cap K$ proved in Lemma 7 it would follow $e_1 \in K_a$.)

Let us see that if in the above representation of x one has $t^1 > 0$, then $x \in K$. Assume the contrary. Then $t^1 e_1 + y \in -K$, from where $0 \leq t^1 e_1 \leq -y$. Since $-y \in \mathrm{sp}K_a = L_a$, with L_a the anti-Archimedean subspace of the ordered space (see the Cor. 7), there exists the element $z \in \mathbb{R}^n$ such that $-tz \leq -y \leq tz$, $\forall t \in \mathbb{R}^+$. But then $-tz \leq t^1 e_1 \leq$ $\leq tz$, $\forall t \in \mathbb{R}^+$, whence $t^1 e_1 \in L_a = \mathrm{sp}K_a$. Since $K_a = \mathrm{sp}K_a \cap K$, it would follow $t^1 e_1 \in K_a$, which is impossible.

If $x \in K$, then in the above representation of x, $t^1 \geq 0$.

Indeed, $x \in K$ and $t^1 < 0$ would mean that $-t^1 e_1 + x = y \in$ $\in K$, $y \in \mathrm{sp}K_a$, hence $y \in K_a$ and $0 \leq -t^1 e_1 \leq -t^1 e_1 + x = y \in$ $\in K_a$. Since by Lemma 12, K_a is a face, it would be $e_1 \in K_a$, which is impossible.

Suppose that $x \in K$ and $t^1 = 0$. Then $x = y \in \mathrm{sp}K_a$ and $x \in$ $\in \mathrm{sp}K_a \cap K = K_a$ (Cor. 7).

Accordingly x is then and only then an element of K, if either in its above representation $t^1 > 0$, or if $x \in K_a$.

We know that $(\mathrm{sp}K_a, \leq_{K_a})$ is a totally ordered vector space of dimension $n-1$ in $\mathbb{R}^n$, which can be identified with the totally ordered Euclidean space $\mathbb{R}^{n-1}$. From the induction hypothesis, this space possesses a base $e_2, \dots, e_n$ which realizes the ordering $\leq_{K_a}$ as a lexicographic one. But then, from the above reasonings x is in K if and only if in its representation as $x = t^1 e_1 + t^2 e_2 + \cdots + t_n e^n$ either every t^i

is zero, or if the first t^i which is not zero, is positive. Accordingly the base $e_1, e_2, \ldots, e_n$ realizes $\leq$ as a lexicographic ordering. $\diamondsuit$

Corollary 8. *The positive cone K of the totally ordered Euclidean vector space $(\mathbb{R}^n, \leq)$ is a stiff latticial cone.*

Proof. According to the just proved theorem, there exists a base $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that

$$K = \{x = (x^1, ..., x^n) : x^1 = ... = x^{k-1} = 0,\ x^k > 0,\ k = 1, ..., n\} \cup \{0\},$$

where x^i are the coordinates of x with respect to this base. Then

$$K_0 = \{x \in K \text{ for which } x^1 = 0\}$$

is the face of K for which the condition in Def. 1 holds. $\diamondsuit$

7. Faces of the latticial cone

Beside the results from the convex geometry wee need in our proofs some standard results from the vector lattice theory. They can be get in monographs on vector lattice theory as [8] or [1] and are the following ones:

Lemma 17. *The cone $K \subset \mathbb{R}^n$ is latticial if and only if for any a, $b \in \mathbb{R}^n$ there exists $c \in \mathbb{R}^n$ such that*

$$(K + a) \cap (K + b) = K + c.$$

Lemma 18. *If $(\mathbb{R}^n, K)$ is vector lattice, then K fulfils the Riesz sum condition:*

$$(S)\qquad \begin{cases} \textit{whenever}\, u_1,\ u_2,\ v_1,\ v_2 \in K,\ u_1 + u_2 = v_1 + v_2 \\ \exists w_{11},\ w_{12},\ w_{21},\ w_{22} \in K \\ \textit{such that}\, u_1 = w_{11} + w_{12},\ u_2 = w_{21} + w_{22}, \\ v_1 = w_{11} + w_{21},\ v_2 = w_{12} + w_{22}. \end{cases}$$

Lemma 19. *In the vector lattice $(\mathbb{R}^n, K)$ holds the Riesz condition*

$$(R)\qquad \begin{cases} \textit{whenever}\, x_1,\ x_2,\ y \in K,\ y \leq x_1 + x_2 \\ \exists y_1,\ y_2 \in K \\ \textit{such that}\, y = y_1 + y_2 \\ \textit{and}\, y_1 \leq x_1,\ y_2 \leq x_2. \end{cases}$$

The faces of the latticial cone possesses some important properties which we shall prove for the sake of completeness.

Lemma 20. *Let K_1 and K_2 be faces of the latticial cone $K \subset \mathbb{R}^n$. Then $K_1 + K_2$ is a face of K.*

Proof. Obviously, K_1+K_2 is a cone. Let be $x \in K_1+K_2$ and $0 \leq y \leq \leq x$, where $\leq$ is the order relation induced by K. Then $x = x_1 + x_2$, where $x_i \in K_i$, $i = 1,2$. From Lemma 19 there exist the elements y_1, y_2 such that $y = y_1 + y_2$ and $0 \leq y_i \leq x_i$, $i = 1,2$. Since K_i is face, $y_i \in K_i$, $i = 1,2$, hence $y \in K_1 + K_2$. ◊

Lemma 21. *If K_1 and K_2 are faces of the latticial cone K with the property $K_1 \cap K_2 = \{0\}$, then*

$$K_1 + K_2 = K_1 \dot{+} K_2,$$

where the symbol $\dot{+}$ in the right term means that each element in $K_1 + + K_2$ is uniquely representable as the sum of elements of K_1 and K_2. We have further

$$\text{sp}(K_1 + K_2) = \text{sp}K_1 \dot{+} \text{sp}K_2, \tag{1}$$

accordingly

$$\dim(K_1 + K_2) = \dim K_1 + \dim K_2.$$

Proof. It is sufficient to prove the relation (1). To do this is enough to show that

$$(\text{sp}K_1) \cap (\text{sp}K_2) = \{0\}.$$

Suppose that z is an element of the above intersection,

$$z = u - v = x - y, \quad u,\ v \in K_1;\ x,\ y \in K_2.$$

Then $u+y = x+v$ and from Lemma 18 there exist w_{11}, w_{12}, w_{21}, $w_{22} \in \in K$ with the properties

$$u = w_{11} + w_{12}$$
$$y = w_{21} + w_{22}$$
$$x = w_{11} + w_{21}$$
$$v = w_{12} + w_{22}.$$

From the first relation $0 \leq w_{11} \leq u$, hence $w_{11} \in K_1$, from the third one $0 \leq w_{11} \leq x$, hence $w_{11} \in K_2$. But then $w_{11} = 0$. From the second and the fourth relation we deduce in a similar way that $w_{22} = 0$. By substitution we get $u = v$ and $x = y$ whereby we conclude that $z = 0$. ◊

8. The stiff latticial cone

Lemma 22. *Suppose that the latticial cone $K \subset \mathbb{R}^n$ is not closed. Then the anti-Archimedean subcone K_a of K is not trivial.*

Proof. Suppose that $x \in K^- \setminus K$ and let be $y \in K^o$. Then

$$tx + (1-t)y \in K,\ \forall\, t \in (0,1) \Rightarrow x \geq \frac{t-1}{t}y,\ \forall\, t \in (0,1) \Rightarrow$$
$$\Rightarrow -sy \leq x,\ \forall\, s \in \mathbb{R}^+.$$

Let be $z = x \wedge 0$. Then $z \neq 0$.
Since $-ty \leq 0$, and $-ty \leq x\ \forall\, t \in \mathbb{R}^+$,

$$-ty \leq z \leq 0 \leq ty,\ \forall\, t \in \mathbb{R}^+.$$

Hence

$$0 \leq -z \leq \frac{1}{n}y,\ \forall n \in \mathbb{N},$$

and thus $0 \neq -z \in K_a$. ◊

Lemma 23. *If K_0 is such a face of the generating cone $K \subset \mathbb{R}^n$ that $K = K^o \cup K_0$ and $\dim K_0 \leq n-2$, then there exists $u \in \mathbb{R}^n$ with the property that*

$$K \cap (u + K)$$

is not a translate of K.

Proof. Since K_0 is a proper face, every its point is a boundary point of K. Let be $x \in \mathrm{ri}K_0$ and let H be a supporting hyperplane of K containing the element x. Then $K_0 \subset H$. From the condition $\dim K_0 \leq$ $\leq n-2$, K_0 does not generate the hyperplane H, and hence there exists $u \in H$ with the property that

$$K_0 \cap (u + K_0) = \emptyset.$$

Since $u + K_0 \subset H$ and $H \cap K^o = \emptyset$, $K^o \cap (u + K_0) = \emptyset$. Similarly $K_0 \cap (u + K^o) = \emptyset$, because $(u + K^o) \cap H = \emptyset$. Accordingly

$$K \cap (u + K) = (K^o \cup K_0) \cap ((u + K^o) \cup (u + K_0)) =$$
$$= (K^o \cap (u + K^o)) \cup (K_0 \cap (u + K_0)) = K^o \cap (u + K^o).$$

Hence

$$K \cap (u + K) = K^o \cap (u + K^o).$$

The set at the right-hand side of this relation is open as the intersection of two open sets. Hence it cannot be the translate of K, which is not open ($0 \in K$ is not an interior point of K). ◊

Corollary 9. *If $K \subset \mathbb{R}^n$ is a latticial cone with the property that $K = K^o \cup K_0$, where K_0 is a proper face of K, then* $\dim K_0 = n - 1$.

Proof. From the preceding lemma and Lemma 17, we have $\dim K_0 \geq$ $\geq n-1$. From Cor. 2, $\dim K_0 < n$. ◊

The stiff latticial cone was defined in Def. 1. We shall call it simply stiff cone. Regarding this notion we have the following important result:
Lemma 24. *Let $K \subset \mathbb{R}^n$ be a latticial cone. The following statements are equivalent:*

1. *K is a stiff cone.*
2. *K possess a proper face K_0 with the property that $K = K^o \cup K_0$.*
3. *K possesses a single proper face K_0 which contains any other proper face of K.*
4. $\dim K_a = n-1$, *where K_a is the anti-Archimedean subcone of K.*

Proof. If $n = 1$, $\dim K = 1$ and $K_0 = \{0\}$ is the face satisfying all the assertions of the lemma.

Suppose that $n > 1$, K is a stiff cone and K_0 is the face from Def. 1.

Assume that $K \setminus K_0 \neq K^o$. Then K has boundary points without K_0. Let x be such a point and let H be the supporting hyperplane of K through x. Then $K \cap H$ is a proper face of K. Since each point of $(K \cap H) \setminus K_0$ is comparable with any point of K_0, for $k \in K_0$ we must have $k \leq x$ and then $K_0 \subset K \cap H$, since $K \cap H$ is a face and $x \in K \cap H$. But K_0 is a proper face of maximal dimension and $K \cap H$ is a proper face, thus we must have $K_0 = K \cap H$. From this it would follow $x \in K_0$, contrary with the definition of x.

The obtained contradiction shows that $K \setminus K_0 = K^o$, whereby $K = K^o \cup K_0$ and this proves the implication $1 \Rightarrow 2$.

The equivalence $2 \Leftrightarrow 3$ is obvious.

$2 \Rightarrow 4$. By a comparison of the assertion 2 with Cor. 9 it follows that $\dim K_0 = n - 1$. Let H be the supporting hyperplane of K engendered by K_0. Then $K \subset H_+$ and $K^o \subset {H_+}^o = H^+$.

Assume that there exists $x \in H^+ \setminus K^o$, and let y be the element with

$$(x + K) \cap K = y + K \tag{1}$$

(Lemma 17).

We have $x + H_+ \subset H^+$, $K_0 \cap H^+ = \emptyset$, hence $x + K \subset H^+$ and

$$(x + K) \cap K = (x + K) \cap K^o.$$

Then $y \in K^o$. As element of $x + K$, y is on a closed halfline of this set issuing from x. A such closed halfline intersects K^o in an open interval since $x \notin K^o$. Thus y is element of an open interval of the intersection $(x + K) \cap K$. But then y cannot be the vertex of the translated cone $y + K$ in the formula (1).

The obtained contradiction shows that $K^o = H^+$ and hence $H \subset \subset K^-$. This means that $-u \in K^-$ for each $u \in K_0$, and by Lemma 11, $u \in K_a$. Hence $K_a = K_0$ and $\dim K_a = n - 1$.

$4 \Rightarrow 1$. The implication follows from Lemma 13. ◊

This lemma has some important consequences.

Corollary 10. *The stiff latticial cone cannot be represented as sum of its proper faces.*

Corollary 11. *If K is a latticial cone which is not stiff, K' is its proper face of dimension $n-1$, then K possesses points in $K \setminus K'$ which are also boundary points.*

Proof. Indeed, if it would not so, then it would be $K = K^o \cup K'$ which according the point 3 of Lemma 24 would imply that K is stiff. ◊

Corollary 12. *If K is a stiff cone and $K_a = \{0\}$, then K is of dimension one. If K is a stiff cone of dimension greater then one then $K_a \neq \{0\}$ and K cannot be closed.*

Lemma 25. *Let H be a hyperplane in $\mathbb{R}^n$ through 0, $K_0 \subset H$ a latticial cone in the subspace H. Then*

$$K = H^+ \cup K_0$$

is a stiff latticial cone in $\mathbb{R}^n$.

Proof. A formal verification shows that K is a cone and that $K^o = = H^+$.

Consider a coordinate system in $\mathbb{R}^n$ with the first axis in the direction of the normal of H, the other axes being placed in H. For the sake of simplicity we shall represent the points in $\mathbb{R}^n$ in the form (x^1, ξ), where x^1 is the first coordinate of the point, ξ is the collection of the other coordinates; ξ is considered to be a point in H. Let us see that for $a = (a^1, \alpha)$, $b = (b^1, \beta) \in \mathbb{R}^n$ there exists a point $c = (c^1, \gamma)$ with the property

$$(K + a) \cap (K + b) = K + c. \tag{2}$$

If $a^1 < b^1$, then $K + b \subset K + a$ and we can take $c = b$.

If $a_1 = b_1$, let γ be the point in H, for which $(K_0 + \alpha) \cap (K_0 + + \beta) = K_0 + \gamma$ (see Lemma 17). Then taking $c^1 = a^1 = b^1$ it can be shown that $c = (c^1, \gamma)$ satisfies the relation (2).

By Lemma 17, K is a latticial cone. K_0 is the single its face of dimension $n-1$. From Lemma 24, we have then that K is a stiff cone. ◊

9. The relationship of stiff faces

The faces K_1 and K_2 of the cone K are said comparable, if either $K_1 \subset K_2$, or $K_2 \subset K_1$. We know from Lemma 4 that $K_1 \subset K_2$ is equivalent with the fact that K_1 is the face of K_2.

Lemma 26. *If K_1 and K_2 are stiff faces of the latticial cone $K \subset \mathbb{R}^n$ and $K_{1a} \cap K_{2a} \neq \{0\}$ (with K_{ia} the anti-Archimedean subcone of K_i), then K_1 and K_2 are comparable.*

Proof. Since from Lemma 20 K_1+K_2 is a face of K, it is a latticial cone in the space it spans. For the sake of simplicity we can then suppose that $K = K_1 + K_2$.

(a) Assume that K_1 and K_2 are not comparable. Then K_1 and K_2 are proper faces of K, and since K is the sum of its proper faces, it isn't stiff (Lemma 10). Hence $\dim K_a \leq n-2$ (Lemma 24).

(b) From the definition of the anti-Archimedean cone it follows that

$$K_{1a} + K_{2a} \subset K_a.$$

(c) According to Lemma 12, the cone K possesses a supporting hyperplane H such that

$$K \cap H = K_a.$$

(d) For any $k \in K_{1a} \cap K_{2a}$ and any $x \in K \setminus K_a$ one has

$$k \leq x.$$

Indeed, $x = x_1 + x_2$, $x_i \in K_i, i = 1, 2$, and by (b) at least for an index i, $x_i \in K_i \setminus K_{ia}$, whereby $k \leq x_i \leq x_1 + x_2 = x$.

(e) Let H be the supporting hyperplane defined at (c). By (a) there exists $u \in H$ such that

$$u \in H \setminus (K_a - K_a) \text{ and hence } (K_a + u) \cap K_a = \emptyset.$$

Let be $v = 0 \vee u$, with $\vee$ the lattice operation induced by K. Let us show that $v - u \in K \setminus K_a$.

Indeed, if it would be $v - u \in K_a$, then $v \in K_a + u \subset H$, and since $v \in K$, it should be $v \in K \cap H = K_a$. Hence from the definition of u we would come to the contradiction $v \in (K_a + u) \cap K_a = \emptyset$.

(f) We have also $v \in K \setminus K_a$, since if it would be $v \in K_a$, then from $v - u \in K$ and $v - u \in H$ it would be $v - u \in K_a$ and then $u \in v - K_a \subset K_a - K_a$, which would contradict the definition in (e) of u.

(g) Since by (e) and (f) v and $v - u$ are both in $K \setminus K_a$, for the element $k \in (K_{1a} \cap K_{2a}) \setminus \{0\}$ it holds according (d) the relations $k \leq v$

and $k \leq v - u$. Hence $0 \leq v - k$ and $u \leq v - k$ and from the last two relations it follows the contradiction

$$v = 0 \vee u \leq v - k. \ \lozenge$$

Lemma 27. *If K_1 and K_2 are stiff faces of the latticial cone K, then they are either comparable, or $K_1 \cap K_2 = \{0\}$.*

Proof. Suppose that $x \in (K_1 \cap K_2) \setminus \{0\}$.

If K_1 (or K_2) is of dimension one, then K_1 (or K_2) is generated by $x \in K_2$ ($x \in K_1$) and hence $K_1 \subset K_2$ (or $K_2 \subset K_1$).

We can suppose next that $K_{1a} \neq \{0\}$ and $K_{2a} \neq \{0\}$.

If $x \in K_{1a} \cap K_{2a}$, by Lemma 26 K_1 and K_2 are comparable.

Let be $x \in K_{1a}$ and $x \in K_2 \setminus K_{2a}$. Then for each $k \in K_{2a} \setminus \{0\}$ it holds $k \leq x$. But x is also in the face K_{1a}, hence $k \in K_{1a}$, and by Lemma 26 K_1 and K_2 are comparable.

Suppose finally that $x \in (K_1 \setminus K_{1a}) \cap (K_2 \setminus K_{2a})$. Then for $k \in \in K_{1a} \setminus \{0\}$ $nk \leq x$, $\forall\, n \in \mathbb{N}$, and in the same time $nk \in K_2$, $\forall\, n \in \mathbb{N}$, consequently $k \in K_{2a}$. Thus Lemma 26 can be applied also in this case. $\lozenge$

10. Proofs of the main results

Lemma 28. *Let $K \subset \mathbb{R}^n$ be a latticial cone. Then K possesses faces K^i of dimension i, $i = 1, \ldots, n-1$ such that*

$$K^1 \subset K^2 \subset \ldots \subset K^{n-1}.$$

Proof. If $n = 2$ and K is closed, then it possesses a supporting hyperplane (line) H through the nonzero boundary point x of its. Then $K \cap H$ will be a one-dimensional face of K. If K is not closed, then by Lemma 22, $K_a \neq \{0\}$ will be a one-dimensional face of K.

Suppose that the assertion of the lemma holds for latticial cones of dimension less than n. It is sufficient to prove that the latticial cone $K \subset \mathbb{R}^n$ possesses a face of dimension $n-1$.

If K is a stiff cone, then by Lemma 24, $\dim K_a = n - 1$ and the assertion of the lemma follows from the induction hypothesis.

Assume that K' is a face of maximal dimension of K and $\dim K' \leq \leq n-2$. Then K cannot be a stiff cone and in the set $K \setminus K'$, K possesses points which are its boundary points by Cor. 11. Let x be such a point and let K'' be the face of K engendered by it. Then K'' is a proper face of K (Lemma 8), which isn't contained in K', hence $K' + K''$ is a face of K of dimension greater then the dimension of K'. Hence, by the definition of K' it must hold $K' + K'' = K$. The face K'' is a latticial cone in the subspace it spans, subspace of dimension less then n. In this subspace the induction hypothesis works and hence K'' possesses the proper faces $K^1 \subset K^2 \subset \ldots \subset K^i$ such that $\dim K^j = j$, $j = 1, \ldots, i$, $i = \dim K'' - 1$. The cones $K' + K^j$ are by Lemma 20 faces of K. If $K^1 \not\subset K'$, then $K' + K^1$ would be a face of dimension $n-1$ of K, if $K^1 \subset K'$, and $K^2 \not\subset K'$, then $K' + K^2$ would have this property, if $K^2 \in K'$ we can proceed similarly with K^3 and so on, by this way we can get a face K^j for which $\dim K' < \dim(K' + K^j) < n$. The obtained contradiction shows that the dimension of K' must be $n - 1$. $\Diamond$

The proof of Theorem 1. We carry the proof by induction.

The assertion of the theorem is trivial for $n = 1$.

Suppose that it holds for each latticial cone of dimension less than n.

If K is a stiff cone, then it is its own maximal stiff face and the assertion of the theorem holds.

If K is not a stiff cone, then by Lemma 28, it possesses a face K' of dimension $n - 1$ and according Lemma 11, K has points which are its boundary points in the set $K \setminus K'$. Let x be such a point and let K'' be a proper face of K containing the point x. Then $K' + K''$ is a face (Lemma 20), and K' is a proper subset of its. Hence it must be equal with K (Cor. 2). The induction hypothesis works for the faces K' and K'', hence K' possesses maximal stiff faces K'_i and K'' possesses maximal stiff faces K''_j such that

$$K' = K'_1 \dotplus \ldots \dotplus K'_p,$$

respectively

$$K'' = K''_1 \dotplus \ldots \dotplus K''_q.$$

Then

$$K = K' + K'' = (K'_1 \dotplus \ldots \dotplus K'_p) + (K''_1 \dotplus \ldots \dotplus K''_q).$$

Consider the sum

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1.$$

If for every index i one has $K'_i \cap K''_1 = \{0\}$, then according to Lemma 21 it follows that

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1 = K'_1 \dotplus \ldots \dotplus K'_p \dotplus K''_1.$$

If there exists a single i, with $K'_i \cap K''_1 \neq \{0\}$, then from Lemma 27, K'_i and K''_1 are comparable. If $K''_1 \subset K'_i$, then

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1 = K'_1 \dotplus \ldots \dotplus K'_p.$$

If $K'_i \subset K''_1$, then

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1 = K'_1 \dotplus \ldots \dotplus K'_{i-1} \dotplus K'_{i+1} \dotplus \ldots \dotplus K'_p \dotplus K''_1.$$

If for the indexes $i_1, \ldots, i_s$ $(s > 1)$ $K'_{i_r} \cap K''_1 \neq \{0\}$, then

$$K'_{i_1} \dotplus \ldots \dotplus K'_{i_s} \subset K''_1.$$

In this case

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1 = \dotplus_{i \notin \{i_1,\ldots,i_s\}} K'_i \dotplus K''_1.$$

Hence for any possible case the sum

$$(K'_1 \dotplus \ldots \dotplus K'_p) + K''_1$$

can be represented as a direct sum of stiff faces of K.

Following this procedure, we can see that

$$(K'_1 \dotplus \ldots \dotplus K'_p \dotplus K''_1) + K''_2$$

can be represented as direct sum of stiff faces and so on, finally also K can be represented as direct sum of stiff faces. From Lemma 27 the stiff faces in this direct sum are maximal stiff faces of K. ◊

Corollary 13. *If the latticial cone K is closed, then it is the direct sum of its edges. Hence the closed latticial cone $K \subset \mathbb{R}^n$ possesses n edges.*

Proof. Since K is closed, by Cor. 5, $K_a = \{0\}$, and by Cor. 12, the cone K cannot have stiff faces of dimension greater as one. ◊

Corollary 14 [Yudin's theorem]. *The closed cone $K \in \mathbb{R}^n$ is latticial if and only if it is engendered by n linearly independent vectors.*

Proof. It is easy to see that for the linearly independent system of vectors $\{e_1, \ldots, e_n\}$ the engendered cone

$$K = \{t^1 e_1 + \cdots + t^n e_n : \ t^i \in \mathbb{R}_+, \ i = 1, \ldots, n\}$$

is latticial. (It is the positive orthant of the reference system with base vectors $e_1, \ldots, e_n$ and thus the very classical example of a latticial cone.)

If K is a closed latticial cone in $\mathbb{R}^n$, then by Cor. 13 it is the direct sum of its n edges and hence is the convex hull of these edges. Accordingly K is engendered by any n linearly independent vectors placed on these edges. ◊

The proof of Theorem 2. According Lemma 28, the cone K possesses a face K' of dimension $n-1$. From Cor. 11 since K is not stiff, it has boundary points in the set $K \setminus K'$. Let x be such a point and K'' a proper face of K containing x. Then by Lemma 20 and Cor. 2 we must have $K = K' + K''$, and the statement 2. is proved. By the proof of Th. 1 it follows that K is the direct sum of their maximal proper stiff faces. Thus the implications $1 \Rightarrow 2 \Rightarrow 3$ were proved.

If K is the direct sum of its maximal proper faces, then by Cor. 10 it cannot be stiff, whereby we have the implication $3 \Rightarrow 1$. $\Diamond$

The ordered Euclidean spaces $(\mathbb{R}^n, K')$ and $(\mathbb{R}^n, K'')$ are called isomorphic if there exists a linear isomorphism (a linear bijection) $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(K') = K''$.

We shall use next without special references the following standard statements from the linear algebra:

1. Let be $\mathbb{R}^n = L_m \dotplus L_{n-m} = M_m \dotplus M_{n-m}$ two representations of the Euclidean space as direct sum of subspaces, where L_m and M_m are m-dimensional, L_{n-m} and M_{n-m} are $n - m$-dimensional subspaces, $1 < m < n$. If $A_1 : L_m \to M_m$, and $A_2 : L_{n-m} \to M_{n-m}$ are linear isomorphisms, then the mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ defined for $x = x_1 + x_2$, $x_1 \in L_m$, $x_2 \in L_{n-m}$ as $Ax = A_1x_1 + A_2x_2$, is a linear isomorphism.
2. If $K \subset \mathbb{R}^n$ is a cone and $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, then $A(K)$ is a cone. The operator A transforms the faces of K onto faces of $A(K)$. If A is an isomorphism, then it preserves the whole face structure of K, that is, A realizes a one to one (linear) correspondence between the faces of K and the faces of $A(K)$.
3. The conjugate A^* of the liner isomorphism A of $\mathbb{R}^n$ (i.e. the linear operator defined by the relation $\langle Au, v\rangle = \langle u, A^*v\rangle$, $\forall\, u, v \in \mathbb{R}^n$) is a linear isomorphism.
4. If A is the isomorphism of the ordered Euclidean spaces $(\mathbb{R}^n, K')$ and $(\mathbb{R}^n, K'')$, then the conjugate A^* of A transforms linearly and bijectively the conjugate K''^* of K'' onto K'^*, the conjugate of K' (and hence $A^*(K''^*) = K'^*$.)
5. If two ordered Euclidean spaces are isomorphic, then they are identical from the point of view of their ordered vector space theoretic structure.

Let $K \subset \mathbb{R}^n$ be a latticial cone and consider the representation of the positive cone K in the form

$$K = K_1 \dotplus \ldots \dotplus K_r \tag{1}$$

with K_i its maximal stiff faces (Th. 1).

Let $L_i = \mathrm{sp}K_i = K_i - K_i$ be the linear space engendered by the face K_i. Then by Lemma 21 we have

$$\mathbb{R}^n = L_1 \dot{+} \ldots \dot{+} L_r.$$

Consider the subspaces $M_i,\ i = 1, \ldots, r$ of $\mathbb{R}^n$ such that $\dim M_i = \dim L_i$ and

$$\mathbb{R}^n = M_1 \oplus \ldots \oplus M_r, \tag{2}$$

where $\oplus$ denotes orthogonal sum.

Denote by A_i a linear isomorphism of L_i onto M_i. Let be $K_i' = A_i(K_i)$. Then the operator defined for $x = x_1 + \cdots + x_r,\ x_i \in L_i$ by the formula

$$Ax = A_1x_1 + \cdots + A_rx_r$$

is a linear isomorphism and

$$A(K) = K_1' \oplus \ldots \oplus K_r'$$

is a cone having the same facial structure as K (hence $K_i',\ i = 1, \ldots, r$ are maximal stiff faces of $A(K)$).

The proof of Theorem 3. According to the remarks above we can suppose that K is represented as

$$K = K_1 \oplus \ldots \oplus K_r$$

with K_i its maximal stiff faces, $i = 1, \ldots, r$.

Let K_a be the anti-Archimedean subcone of K and K_{ia} be the anti-Archimedean subcone of K_i, $i = 1, \ldots, r$. Then

$$K_a = K_{1a} \oplus \ldots \oplus K_{ra}.$$

Let be $x^* \in K^* \setminus \{0\}$. Then x^* is the normal of a supporting hyperplane H of K and in the same time a supporting hyperplane of the wedge K^-. Then $K_a - K_a \subset H$ by Lemma 12, and hence $\langle x^*, a\rangle = 0,\ \forall\, a \in K_a$.

Since K_i is a stiff face, $\dim K_{ia} = \dim K_i - 1$. Let $k_i \in K_i \setminus K_{ia}$ be the element for which $\langle k_i, a'\rangle = 0,\ \forall\, a' \in K_{ia}$ and whose Euclidean norm $\|k_i\| = 1$. We have then $\langle k_i, k_j\rangle = 0$ if $i \neq j$ and $\langle k_i, k_i\rangle = 1$.

Each element x_i of the face K_i is representable in the form $x_i = a_i + t^i k_i$, where $a_i \in K_{ia}$, $t^i \in \mathbb{R}_+$. Since $a_i \in K_a$, we have $\langle x^*, x_i\rangle = t^i\langle x^*, k_i\rangle$.

Let be

$$x_i^* = \langle x^*, k_i\rangle k_i.$$

We claim that

$$x^* = \sum_{i=1}^{r} x_i^* = \sum_{i=1}^{r} \langle x^*, k_i \rangle k_i.$$

Indeed, an arbitrary element x of K can be represented in the form $x = x_1 + \cdots + x_r$, where $x_i \in K_i$, $x_i = a_i + t^i k_i$, $a_i \in K_{ia} \subset \subset K_a$, $t^i \in \mathbb{R}_+$, $\langle k_i, a_i \rangle = 0$. Hence

$$\langle x^*, x \rangle = \sum_{i=1}^{r} \langle x^*, x_i \rangle = \sum_{i=1}^{r} t^i \langle x^*, k_i \rangle.$$

On the other hand, $\langle k_i, x_i \rangle = t^i$, and using the relations $\langle k_i, x_j \rangle = = 0$, $i \neq j$ it follows that

$$\left\langle \sum_{i=1}^{r} \langle x^*, k_i \rangle k_i, \sum_{i=1}^{r} x_i \right\rangle = \sum_{i=1}^{r} \langle x^*, k_i \rangle \langle k_i, x_i \rangle = \sum_{i=1}^{r} t^i \langle x^*, k_i \rangle.$$

Since the scalar product of the vectors x^* and $\sum_{i=1}^{r} \langle x^*, k_i \rangle k_i$ with each x in K coincide, and since K is a generating cone, our claim is verified.

Hence every element $x^* \in K^*$ can be written in the form

$$x^* = \sum_{i=1}^{r} \langle x^*, k_i \rangle k_i.$$

The converse of this assertion is obvious: each sum of form

$$\sum_{i=1}^{r} s^i k_i, \;\; s_i \in \mathbb{R}_+, \;\; i = 1, \ldots, r$$

is an element of K^*. $\Diamond$

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References

[1] ALIPRANTIS, C. D. and BURKINSAW, O.: Positive Operators, Academic Press, Orlando, 1985.

[2] MARTINEZ-LEGAZ, I. E. and SINGER, I.: The structure of hemispaces in $\mathbb{R}^n$, *Linear Algebra Appl.* **110** (1988), 117–179.

[3] NÉMETH, A. B.: The axiom of Archimedes, hemispaces, hypercones and ε-efficiency, Babes-Bolyai Univ. Faculty of Mathematics and Informatics Research Seminars, Seminar on Mathematical Analysis, Preprint Nr. 7, 1994, 79–94.

[4] PERESSINI, A.: Ordered Topological Vector Spaces, Harper and Row, New York, 1967.

[5] ROCKAFELLAR, R. T.: Convex Analysis, Princeton University Press, Princeton, 1970.
[6] STOER, J., WITZGALL, C.: Convexity and Optimization in Finite Dimension I, Springer-Verlag, Berlin, 1970.
[7] SZ. NAGY, B.: Sur les lattis linéaires de dimension finie, *Comment. Math. Helv.* **17** (1944), 209–213.
[8] SCHAEFER, H. H.: Banach Lattices and Positive Operators, Springer-Verlag, New York–Heidelberg, 1974.
[9] YOUDINE, A.: Solution des deux problèmes de la théorie des espaces semi-ordonnés, *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS* **23** (1939), 418–422.