Mathematica Pannonica
15/1 (2004), 141–148

SOLUTIONS WITH BIG GRAPH OF ITERATIVE FUNCTIONAL EQUATIONS OF THE SECOND ORDER

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Received: October 2003

MSC 2000: 39 B 12, 39 B 22, 26 A 30

Keywords: Functional equation of the second order, big graph, orbit.

Abstract: Given commuting bijections f, g of an arbitrary set X we construct very irregular solutions of the general functional equation of the second order

$$F(x, \varphi(x), \varphi(f(x)), \varphi(g(x))) = 0.$$

The graph of such a solution is connected on the plane and big in the sense of measure and topology.

1. Introduction

The idea of constructing solutions with big graph go back to F. B. Jones [6] and concerns the Cauchy equation (see also [8, Ch. 12, §4], [1] and [5]). P. Kahlig and J. Smítal [7] were the first who obtained solutions with big graph of an equation in a single variable. Since the latter paper several types of functional equations were considered in

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This research was supported by the Silesian University Mathematics Department (Functional Equations in a Single Variable program).

this direction, see [3] and references therein. It is the aim of the present paper to elaborate this theory considering the general equation of the second order

$$F(x, \varphi(x), \varphi(f(x)), \varphi(g(x))) = 0. \tag{1}$$

Accept the following definition. Given sets X, Y and a family $\mathcal{R}$ of subsets of $X \times Y$, we say that $\varphi\colon X \to Y$ has a big graph with respect to $\mathcal{R}$ if its graph $\mathrm{Gr}\,\varphi$ meets every set of $\mathcal{R}$.

We are interested in finding conditions under which equation (1) has a solution with big graph with respect to a sufficiently large family.

The paper is organized as follows. In the second section we classify orbits. Next we discuss our assumptions and present examples. The fourth part contains a proof of the main result. We complete the paper by topological and measure-theoretical properties of functions with big graph chosen from [2]. From that properties the existence of solutions of (1) with a connected graph almost covering the plane follows.

2. Description of orbits

Let X be a nonempty set and suppose that $f : X \to X$ and $g : X \to X$ are commuting bijections (one-to-one and onto). For every $x \in X$ denote by $C(x)$ the orbit of the point x generated by f and g; i.e. the equivalence class, containing x, of the relation $\sim$ on X defined by

$$x \sim y \iff y = f^p(g^l(x)) \text{ for some } p, l \in \mathbb{Z}.$$

Clearly,

$$C(x) = \{f^p(g^l(x)) : p, l \in \mathbb{Z}\}. \tag{2}$$

Our purpose is to construct a solution $\varphi : X \to Y$ of (1) which has a big graph with respect to a given family $\mathcal{R}$. We will do it separately on each orbit: we fix a point $x \in X$ and put $\varphi(x) = y$ (such that $(x, y) \in R$ for an $R \in \mathcal{R}$) and next we define φ on the whole orbit $C(x)$; this is possible in many cases but it depends on the structure of the orbit. Although the structure of orbits was discussed in [4], for the purpose of the present paper we have to describe them more precisely.

Definition. Let $x \in X$, let m, n be positive integer and let k be an integer.

(i) The orbit $C(x)$ is of the type $(0, 0)$ if

$$f^p(x) \neq g^l(x) \text{ for any } p, l \in \mathbb{Z} \text{ such that } |p| + |l| \neq 0.$$

(ii) The orbit $C(x)$ is of the type $(0, n, k)$ if

$$f^l(x) \neq x \text{ for any } l \in \mathbb{N},$$

$$g^n(x) = f^k(x),$$

and

$$g^l(x) \neq f^p(x) \text{ for any } 0 < l < n \text{ and } p \in \mathbb{Z}.$$

(iii) The orbit $C(x)$ is of the type $(m, 0, k)$ if

$$g^l(x) \neq x \text{ for any } l \in \mathbb{N},$$

$$f^m(x) = g^k(x),$$

and

$$f^l(x) \neq g^p(x) \text{ for any } 0 < l < m \text{ and } p \in \mathbb{Z}.$$

(iv) The orbit $C(x)$ is of the type (m, n) if

$$f^m(x) = x \text{ and } f^l(x) \neq x \text{ for any } 0 < l < m,$$

$$g^n(x) = f^p(x) \text{ for some } p \in \mathbb{Z},$$

and

$$g^l(x) \neq f^q(x) \text{ for any } 0 < l < n \text{ and } q \in \mathbb{Z}.$$

3. General assumptions and the key lemma

Our general hypothesis on the given function F reads as follows:

(H_1) The set X is uncountable, T is a set with a distinguished element 0, Y is a set and $F : X \times Y^3 \to T$ is a function such that for every $x \in X$, $j \in \{1,2,3\}$, $i \in \{1,2,3\} \setminus \{j\}$ and $y_i \in Y$ there exists a $y_j \in Y$ with

(3) $$F(x, y_1, y_2, y_3) = 0.$$

The following is the key lemma for our construction.

Lemma 1. *Assume* (H_1), *let* $f, g : X \to X$ *be commuting bijections and suppose that an* $x \in X$ *satisfies one of the following conditions:*

(i) $C(x)$ *is of the type* $(0, 0)$;

(ii) $C(x)$ *is of the type* $(0, n, k)$ *for some* $n \in \mathbb{N}$ *and* $k \notin \{0, n\}$;

(iii) $C(x)$ *is of the type* $(m, 0, k)$ *for some* $m \in \mathbb{N}$ *and* $k \notin \{0, m\}$.

Then for every $y \in Y$ *there exists a solution* $\varphi : C(x) \to Y$ *of* (1) *such that* $\varphi(x) = y$.

Proof. Fix a $y \in Y$ and put $\varphi(x) = y$.

(i) We define φ arbitrarily on the set $\{f^p(x) : p \in \mathbb{Z} \setminus \{0\}\} \cup \cup \{g^{-l}(x) : l \in \mathbb{N}\}$. Next using solvability of (3), we choose suitable $\varphi(f^p(g^l(x)))$ inductively: first – using solvability of (3) with respect to y_1 – for negative integers p and l in such a manner that

$$F(x, \varphi(f^p(g^l(x))), \varphi(f^{p+1}(g^l(x))), \varphi(f^p(g^{l+1}(x))) = 0,$$

then for positive p and negative l applying solvability of (3) with respect to y_2, and, finally, for any integer p and positive l using solvability of (3) with respect to y_3.

(ii) In this case we consider three subcases:

(a) $0 < k < n$;

(b) $0 < n < k$;

(c) $k < 0 < n$.

In the case (a) we define φ arbitrarily on the set $\{f(x), \ldots, f^{n-1}(x)\}$. Next using (H_1) we define φ on the rest of the orbit in the following way: First using solvability of (3) with respect to y_3 we define φ on the "triangle" $\{f^p(g^l(x)) : p, p+l \in \{0, \ldots, n-1\}\}$, then using solvability of (3) with respect to y_2 we define φ on the set $\{f^p(g^l(x)) : : p \geq 0 \text{ and } n \leq p+l\}$, and using solvability of (3) with respect to y_1 we define φ on the set $\{f^p(g^l(x)) : p < 0\}$.

In the case (b) we define φ arbitrarily on the set $\{f(x), \ldots, f^{k-1}(x)\}$ and we proceed similarly to the previous case, using the solvability condition with respect to y_3 and y_1 only.

In the last case (c) we put φ arbitrarily on the set $\{f(x), \ldots, f^{n-k-1}(x)\}$ and we define φ on the rest of the orbit $C(x)$ using the solvability condition with respect to y_3 and y_2 only.

(iii) We argue analogously to the case (ii). $\Diamond$

In the assumption of the above lemma we omitted some orbits; e.g. orbits of type (m, n). In the case of such an orbit finding a solution of (1) is equivalent to finding $m \cdot n$ values of the unknown function and leads to the problem of solving a system of $m \cdot n$ algebraic equations. As we will see in Ex. 1 this system may have no solution; but even though this system has a solution, the equation itself may have no solution with big graph with respect to a reasonable big family $\mathcal{R}$. The case of orbits of types $(0, n, 0)$, $(0, n, n)$, $(m, 0, 0)$ and $(m, 0, m)$ leads to an infinite system of algebraic equations.

Example 1. Let $X = Y = \mathbb{C}$ and consider the linear equation of the second order

$$\varphi(x) = p\varphi(\alpha x) + q\varphi(\beta x) + r \tag{4}$$

with $\alpha = -1$ and $\beta = i$.

For every $x \neq 0$ the orbit $C(x) = \{x, -x, ix, -ix\}$ is of the type $(2,2)$. The problem of finding a solution $\varphi : C(x) \to \mathbb{C}$ of (4) reduces to the problem of finding a solution of the following matrix equation

$$\begin{bmatrix} 1 & -p & 0 & -q \\ -p & 1 & -q & 0 \\ -q & 0 & 1 & -p \\ 0 & -q & -p & 1 \end{bmatrix} \begin{bmatrix} \varphi(x) \\ \varphi(-x) \\ \varphi(-ix) \\ \varphi(ix) \end{bmatrix} = \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix}.$$

It is easy to see that if $p = q = \frac{1}{2}$ and $r \in \mathbb{C} \setminus \{0\}$, then equation (4) has no solution $\varphi : \mathbb{C} \to \mathbb{C}$, and if $p = q \neq \frac{1}{2}$ then for every $r \in \mathbb{C}$ it has exactly one solution $\varphi : \mathbb{C} \to \mathbb{C}$ and this solution takes at most four values. Hence in both cases there is no solution with big graph.

Example 2. Let $X = Y = \mathbb{R}$ and consider equation (4) with $\alpha = \sqrt{2}$ and $\beta = -1$.

For every $x \neq 0$ the orbit $C(x) = \{(\sqrt{2})^l x, -(\sqrt{2})^l x : l \in \mathbb{Z}\}$ is of the type $(0,2,0)$.

A simple calculations show that if $p = 2$, $q = 1$ and $r = 0$, then the zero function is the only solution of (4); in particular there is no solution of (4) with big graph.

Now formulate our main assumptions on the given functions f and g.

(H_2) The functions f and g are commuting bijections of X such that for any positive integer m and n there is no orbit of the types (m,n), $(m,0,0)$, $(m,0,m)$, $(0,n,0)$ and $(0,n,n)$.

Below we give some examples for which hypothesis (H_2) is fulfilled.

Example 3. Assume $\operatorname{card} X = \mathfrak{c}$, let h be a bijection of X onto $\mathbb{R}$, suppose that a, b are nonzero reals and put

$$f(x) = h^{-1}(h(x) + a), \qquad g(x) = h^{-1}(h(x) + b).$$

If a and b are noncommensurable (i.e. $\frac{a}{b} \notin \mathbb{Q}$), then every orbit is of the type $(0,0)$, and so (H_2) holds.

Example 4. Let $X = (0, +\infty)$ and given $n \in \mathbb{N}$, $k \in \mathbb{Z}$ consider $\alpha, \beta \in (0, +\infty)$ such that $\alpha \neq 1$ and $\beta^n = \alpha^k$. Putting

$$f(x) = \alpha x, \qquad g(x) = \beta x,$$

we see that for every $x \in (0, +\infty)$ the orbit $C(x)$ is of the type $(0, n, k)$. Thus (H_2) holds iff $k \neq 0, n$.

4. Main result

We are now in a position to formulate and prove our main result. Let $\pi : X \times Y \to X$ be the projection.

Theorem. *Assume* (H_1), (H_2) *and let* $\mathcal{R}$ *be a family of subset of* $X \times Y$ *such that*

$$\operatorname{card} \mathcal{R} \leq \operatorname{card} X \tag{5}$$

and

$$\operatorname{card} \pi(R) = \operatorname{card} X \quad \textit{for every } R \in \mathcal{R}. \tag{6}$$

Then there exists a solution $\varphi : X \to Y$ *of* (1) *which has a big graph with respect to the family* $\mathcal{R}$.

Proof. The family $\mathcal{C}$ of all the orbits is a partition of X and a function $\varphi : X \to Y$ is a solution of (1) iff for every $C \in \mathcal{C}$ the function $\varphi|_C$ does. This allows us to define a solution φ of (1) by defining it on each orbit.

Let γ be the smallest ordinal such that its cardinal $|\gamma|$ equals that of $\mathcal{R}$ and let $(R_\alpha : \alpha < \gamma)$ be a one-to-one transfinite sequence of all the elements of $\mathcal{R}$. Using the transfinite induction we will define a sequence $((x_\alpha, y_\alpha) : \alpha < \gamma)$ of elements of $X \times Y$ such that for every $\alpha < \gamma$ we have

$$(x_\alpha, y_\alpha) \in R_\alpha \tag{7}$$

and

$$x_\alpha \in \pi(R_\alpha) \setminus \bigcup \{C \in \mathcal{C} : x_\beta \in C \text{ for some } \beta < \alpha\}. \tag{8}$$

Fix $\alpha < \gamma$ and suppose that we have already defined suitable (x_β, y_β) for $\beta < \alpha$. It follows from (H_2), (H_1) and (5) that

$$\operatorname{card} \bigcup \{C \in \mathcal{C} \colon x_\beta \in C \text{ for some } \beta < \alpha\} \leq$$
$$\leq \aleph_0 \cdot |\alpha| = \max\{\aleph_0, |\alpha|\} < \operatorname{card} X$$

which jointly with (6) ensures that the set occurring in (8) is nonempty and we can choose a point x_α from it. In particular, $x_\alpha \in \pi(R_\alpha)$ and so there exists a y_α such that (7) holds.

Fix now an orbit $C \in \mathcal{C}$. If the set

$$C \cap \{x_\alpha \colon \alpha < \gamma\} \tag{9}$$

is nonempty, then, according to (8), it consists of exactly one point x_α and we put

$$(x, y) = (x_\alpha, y_\alpha).$$

If the set (9) is empty, then we choose $(x, y) \in C \times Y$ arbitrarily. In both these cases $C = C(x)$. Assumption (H_2) allows us to apply Lemma 1 and we get a solution $\varphi_C : C \to Y$ of (1) such that

$$\varphi_C(x) = y.$$

Putting

$$\varphi = \bigcup_{C \in \mathcal{C}} \varphi_C$$

we obtain a solution of (1) satisfying $\varphi(x_\alpha) = y_\alpha$for every $\alpha < \gamma$, which jointly with (7) shows that φ has a big graph with respect to the family $\mathcal{R}$. ◊

The following corollary is an immediate consequence of the Theorem.

Corollary. *Let X be an uncountable set. Assume* (H_2) *and $\mathcal{R}$ be a family of subset of $X \times \mathbb{R}$ such that* (5) *and* (6) *hold. Then for any functions $p, q : X \to \mathbb{R} \setminus \{0\}$ and $r : X \to \mathbb{R}$ there exists a solution $\varphi : X \to \mathbb{R}$ of the equation*

$$\varphi(x) = p(x)\varphi(f(x)) + q(x)\varphi(g(x)) + r(x)$$

which has a big graph with respect to the family $\mathcal{R}$.

5. Properties of functions with big graph

Given two topological spaces X and Y, consider the family

$$(10) \qquad \{R \in \mathcal{B}(X \times Y): \ \pi(R) \text{ is uncountable}\},$$

where $\mathcal{B}(X \times Y)$ denotes the σ−algebra of all Borel subsets of $X \times Y$.

Observe that if X and Y are Polish spaces and X is uncountable, then family (10) satisfies conditions (5) and (6) of the Theorem (all the cardinals occurring there are equal to $\mathfrak{c}$).

Following [8, p. 289] it is easy to prove that if a function $\varphi: X \to Y$ has a big graph with respect to this family (10), then its graph is big from the topological point of view:

Proposition 1. *Assume X is a T_1-space and has no isolated point. If $\varphi: X \to Y$ has a big graph with respect to the family* (10), *then the set $(X \times Y) \setminus \mathrm{Gr}\,\varphi$ contains no subset of $X \times Y$ of second category having the property of Baire.*

Such a graph is also big from the point of view of measure theory. Namely we have the following.

Proposition 2. *Assume X is a T_1-space and λ is a measure on $\mathcal{B}(X \times Y)$ vanishing on all the vertical lines $\{x\} \times Y$, $x \in X$. If $\varphi\colon X \to Y$ has a big graph with respect to the family* (10), *then the set $(X \times Y) \setminus \operatorname{Gr}\varphi$ contains no Borel subset of $X \times Y$ of positive measure λ.*

In other words complement of the graph is of the zero inner measure, and, consequently, the graph is of the full outer measure, i.e., $\lambda^*(B \cap \operatorname{Gr}\varphi) = \lambda(B)$ for every $B \in \mathcal{B}(X \times Y)$.

Applying Lemmas 1 and 2 from the paper of W. Kulpa [9] we obtain what follows.

Proposition 3. *Assume that X and Y are connected spaces and every nonempty open subset of X is uncountable. If $\varphi\colon X \to Y$ has a big graph with respect to the family* (10), *then $\operatorname{Gr}\varphi$ is dense and connected in $X \times Y$.*

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