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ON THE LOCAL DISTRIBUTION OF THE ITERATED DIVISOR FUNCTION

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Abstract: Let $d_2(n) = d(d(n))$, where $d(n)$ is the number of divisors of n. The asymptotic of $\#\{n \le x \mid d_2(n) = k\}$ is investigated.

1. Introduction

Notation. Let $d(n)$ be the number of divisors of n, $d_2(n) = d(d(n))$, and in general, $d_{r+1}(n) = d_r(d(n))$, that is $d_r(n)$ is the r'th iterate of $d(n)$. Let $\mathcal{P}$ be the set of primes. Let $\omega(n)$, $\Omega(n)$ be the number of prime divisors and the number of prime power divisors of n. The Liouville function $\lambda(n)$ is defined by $\lambda(n) = (-1)^{\Omega(n)}$. For the sake of simplicity we shall write $x_1 = \log x$, $x_2 = \log x_1$, $x_{r+1} = \log x_r$ $(r =$

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$= 2, 3, \dots)$. $\mathbb{N}$ = set of nonnegative integers. The letters ε, $\varepsilon_1, \varepsilon_2$, δ, δ_1 denote small positive constants.

Let

$$\rho_k(x) = \frac{1}{x_1} \frac{x_2^{k-1}}{(k-1)!}. \tag{1.1}$$

It is clear that

$$\sum_{k=1}^{\infty} \rho_k(x) = 1. \tag{1.2}$$

R. Bellman formulated the conjecture, namely that

$$\sum_{n \le x} d_r(n) = (1 + o(1)) x x_r \quad (x \to \infty) \tag{1.3}$$

holds for every $r = 1, 2, \dots$. This is proved for $r \le 4$. See [1], [3], [4], [6], [7], [11].

In [5] the limit distribution of $d_2(n)$ has been investigated. Our purpose in this short paper is to give the local distribution of $d_2(n)$, i.e. the asymptotic of

$$\#\{n \le x \mid d_2(n) = k\}. \tag{1.4}$$

K. Ramachandra [10] proved asymptotic for sums $\sum_{x \le n \le x+h} f(n)$, where f is a multiplicative function, the generating Dirichlet series $\sum \frac{f(n)}{n^s}$ of which can be expressed by the product of complex powers of L-functions, and $h = x^{7/12+\varepsilon}$, $\varepsilon > 0$. He applied the so called Hooley–Huxley contour in the proof. By using this technique, in [8] it was proved that

$$\sum_{\substack{\omega(n)=k \\ x \le n \le x+h}} 1 = (1 + o(1)) h \rho_k(x),$$

uniformly as $1 \le k \le x_2 + c_x \sqrt{x_2}$, $c_x \to \infty$ sufficiently slowly, and $x^{7/12+\varepsilon} \le h \le x$.

In [9] the following theorem is proved.

Theorem A. *Let $z \in \mathbb{C}$, $|z| \le c_1$, c_1 be a constant,*

$$U(z|x, h) := \sum_{x \le m \le x+h} z^{\omega(m)} |\mu(m)|, \tag{1.5}$$

(1.6) $$\prod\nolimits_l([x,x+h]) := \sum_{\substack{x\le n\le x+h\\ \omega(m)=l}} |\mu(n)|,$$

(1.7) $$\psi(x) = \frac{1}{\Gamma(z)}\prod_p\left(1-\frac{1}{p}\right)^z\left(1+\frac{z}{p}\right).$$

Assume that $x^{7/12+\varepsilon}\le h\le x^{0{,}66}$. *Then*

(1.8) $$\frac{U(z|x,h)}{h} = \psi(z)\cdot x_1^{z-1} + O\left(x_1^{z-2}\right),$$

consequently,

(1.9) $$\frac{1}{h}\prod\nolimits_l([x,x+h]) = \frac{6}{\pi^2}\left(1+O\left(\frac{1}{x_2}\right)\right)\rho_l(x),$$

uniformly as $1\le l\le cx_2$, c *is an arbitrary positive constant.*

2. The main auxiliary theorem

Every $n\in\mathbb{N}$ can be written uniquely as $n = Km$, where $(K,m) = 1$, K is square full, and m is square free. Let us write $d(K)$ as $2^\alpha k_1$, k_1 odd. Then $d(n) = d(Km) = k_1\cdot 2^{\alpha+\omega(m)}$, and so

(2.1) $$d_2(n) = d(k_1)(\alpha+1+\omega(m)).$$

For some fixed square full K, let

(2.2) $$\mathcal{E}_K = \{n \mid n = Km,\ (m,K)=1,\ |\mu(m)|=1\}.$$

For the integer $S = q_1^{u_1}\dots q_r^{u_r}$, $q_1,\dots,q_r\in\mathcal{P}$, $u_1,\dots,u_r\ge 1$, let $\mathcal{B}_S$ be the set of those integers m all the prime factors of which belong to the set $\{q_1,\dots,q_r\}$, i.e.

(2.3) $$\mathcal{B}_S = \{m \mid m = q_1^{v_1}\dots q_r^{v_r},\ v_1,\dots,v_r\in\mathbb{N}_0\}.$$

We define

(2.4) $$H_K(s,z) := \sum_{(m,K)=1}\frac{|\mu(m)|z^{\omega(m)}}{m^s} = \prod_{(p,K)=1}\left(1+\frac{z}{p^s}\right),$$

(2.5) $$T_K(s,z) := \prod_{p|K}\left(1+\frac{z}{p^s}\right)^{-1} = \sum_{v\in\mathcal{B}_K}\frac{\lambda(v)z^{\Omega(v)}}{v^s}.$$

From now on we assume that $|z|\le 2-\delta$, $\delta>0$ a constant. It is clear that

$$H_K(s,z) = T_K(s,z) \cdot H_1(s,z). \tag{2.6}$$

Let $H \geq 1$, and

$$V_K(z|x,H) := \sum_{\substack{x \leq m \leq x+H \\ (m,K)=1}} |\mu(m)| \cdot z^{\omega(m)}. \tag{2.7}$$

Then, by (2.6) we obtain immediately that

$$V_K(z|x,H) = \sum_{v \in \mathcal{B}_K} \lambda(v) z^{\Omega(v)} U\left(z \;\middle|\; \frac{x}{v}, \frac{H}{v}\right), \tag{2.8}$$

U is defined by (1.5).

Let

$$\prod\nolimits_l([x, x+H] \mid K) = \sum_{\substack{\omega(m)=l \\ (m,K)=1 \\ x \leq m \leq x+H}} |\mu(m)|. \tag{2.9}$$

By using (2.8)

$$\prod\nolimits_l([x, x+H] \mid K) = \sum_{v \in \mathcal{B}_K} \lambda(v) \prod\nolimits_{l-\Omega(v)} \left(\left[\frac{x}{v}, \frac{x+H}{v}\right]\right) \tag{2.10}$$

$(l = 1, 2, \dots)$.

Assume that $\delta > 0$, $\varepsilon > 0$ be fixed, $H = x^{\frac{7}{12}+2\varepsilon}$. Let $K \leq$ $\leq x_1^4$, $\frac{x_2}{2} \leq l \leq (\sqrt{2} - \delta) x_2$.

We define Y by $\log Y = x_2^{3/4}$, say.

Let $v \leq Y$. Then,

$$\begin{aligned} \rho_{l-\Omega(v)}\left(\frac{x}{v}\right) &= \rho_{l-\Omega(v)}(x)\left(1 + O\left(\frac{\log v}{x_1}\right)\right) = \\ &= \rho_l(x) \frac{(l-1)\dots(l-\Omega(v))}{x_2^{\Omega(v)}}\left(1 + O\left(\frac{\log v}{x_1}\right)\right) = \\ &= \rho_l(x) \cdot \left(\frac{l}{x_2}\right)^{\Omega(v)} \exp\left(-\frac{\Omega^2(v)}{2l}\right) \cdot \left(1 + O\left(\frac{\Omega^3(v)}{l^2} + \frac{\Omega(v)}{l}\right)\right). \end{aligned} \tag{2.11}$$

Let us apply (2.10) and (1.9). For $v \geq Y$ we shall use the obvious inequality

$$\prod\nolimits_{l-\Omega(v)}\left(\left[\frac{x}{v}, \frac{x+H}{v}\right]\right) \ll \frac{H}{v}. \tag{2.12}$$

We obtain that

$$\begin{aligned} &\prod\nolimits_{l}([x, x+H] \mid K) = \\ &= \frac{6}{\pi^2} H \rho_l(x) \sum_{\substack{v \in \mathcal{B}_K \\ v<Y}} \frac{\lambda(v)}{v} \cdot \left(\frac{l}{x_2}\right)^{\Omega(v)} \exp\left(-\frac{\Omega^2(v)}{2l}\right) + \\ &+ O(R_1) + O(R_2) + O(R_3). \end{aligned} \tag{2.13}$$

Here R_1 is the contribution of the error terms in (2.11), R_2 is that of (1.9) and R_3 is $H \sum_{\substack{v>Y \\ v \in \mathcal{B}_K}} 1/v \cdot \left(\frac{l}{x_2}\right)^{\Omega(v)}$.

We have

$$R_1 \ll H\rho_l(x) \sum_{\substack{v<Y \\ v \in \mathcal{B}_K}} \left(\frac{l}{x_2}\right)^{\Omega(v)} \left(\frac{\Omega^3(v)}{l^2} + \frac{\Omega(v)}{l}\right) \exp\left(-\frac{\Omega^2(v)}{2l}\right).$$

Since $\max_{y \geq 1} \left(\frac{y^3}{l^2} + \frac{y}{l}\right) \exp\left(-\frac{y^2}{2l}\right) \ll \frac{1}{\sqrt{l}}$, and $\Omega(v) \ll \log v$, we obtain that

$$\begin{aligned} R_1 &\ll \frac{H\rho_l(x)}{x_2} \sum_{\substack{v<e^{\sqrt{x_2}} \\ v \in \mathcal{B}_K}} \left(\frac{l}{x_2}\right)^{\Omega(v)} \cdot \frac{\Omega(v)}{v} + \\ &+ \frac{H}{\sqrt{x_2}} \rho_l(x) \sum_{\substack{e^{\sqrt{x_2}}<v<Y \\ v \in \mathcal{B}_K}} \left(\frac{l}{x_2}\right)^{\Omega(v)} \cdot \frac{1}{v} = R_1^{(1)} + R_2^{(2)}. \end{aligned} \tag{2.14}$$

Let $\frac{l}{x_2} = a$, shortly.

To estimate $R_1^{(1)}$, observe that

$$\sum_{\substack{v\in\mathcal{B}_K\\ v<e^{\sqrt{x_2}}}} \frac{a^{\Omega(v)}\cdot\Omega(v)}{v} \le \sum_{p|K}\sum_{\alpha=1}^{\infty}\frac{a^{\alpha}}{p^{\alpha}}\sum_{v_1\in\mathcal{B}_K}\frac{a^{\Omega(v_1)}}{v_1} \ll$$

$$\ll \prod_{p\in K}\left(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\dots\right)\sum_{p|K}1/p.$$

Thus, in the notation

$$\kappa(K) := \sum_{p|K} 1/p \tag{2.15}$$

we have

$$R_1^{(1)} \ll \frac{H\rho_l(x)}{x_2}\kappa(K)e^{a\kappa(K)}. \tag{2.16}$$

Furthermore

$$\sum_{\substack{e^{\sqrt{x_2}}<v<Y\\ v\in\mathcal{B}_K}} a^{\Omega(v)}\cdot\frac{1}{v} \ll e^{-\frac{1}{2}\sqrt{x_2}}\sum_{v\in\mathcal{B}_K}\frac{a^{\Omega(v)}}{\sqrt{v}} \ll$$

$$\ll e^{-\frac{1}{2}\sqrt{x_2}}\prod_{p|K}\left(1+\frac{a}{\sqrt{p}}+\left(\frac{a}{\sqrt{p}}\right)^2+\dots\right) \ll e^{-\frac{1}{2}\sqrt{x_2}}\frac{\sqrt{\log K}}{\log\log K} \ll e^{-\frac{1}{3}\sqrt{x_2}}.$$

Here we used that

$$\frac{a}{\sqrt{p}} \le 1-\frac{\delta}{\sqrt{2}}, \quad \sum_{p|K}\frac{1}{\sqrt{p}} \ll \frac{\sqrt{\log K}}{\log\log K}.$$

Thus

$$R_1^{(2)} \ll H\rho_l(x)e^{-1/3\sqrt{x_2}}\frac{\sqrt{x_2}}{x_3},$$

and so we have

$$R_1 \ll \frac{H\rho_l(x)}{x_2}\kappa(K)e^{a\kappa(K)}. \tag{2.17}$$

From (1.9),

$$R_2 \ll \frac{H}{x_2}\rho_l(x)\prod_{p|K}\left(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\dots\right), \text{ i.e. } R_2 \ll \frac{H}{x_2}\rho_l(x)e^{a\kappa(K)}.$$

Since

$$\sum_{p|K} 1/p \leq \log\log K \cdot 10 + c_1, \tag{2.18}$$

we obtain that

$$R_2 \ll \frac{H}{x_2}\rho_l(x)(\log\log 10K)^{\sqrt{2}}. \tag{2.19}$$

Since

$$\left|\sum_{\substack{v\geq Y\\ v\in\mathcal{B}_K}} \frac{a^{\Omega(v)}}{v}\right| \leq Y^{-1/2}\sum_{v\in\mathcal{B}_K}\frac{a^{\Omega(v)}}{v^{1/2}} \ll$$

$$\ll Y^{-1/2}\prod_{p|K}\left(1+\frac{a}{\sqrt{p}}+\left(\frac{a}{\sqrt{p}}\right)^2+\dots\right) \ll$$

$$\ll Y^{-1/2}\exp\left(\frac{\sqrt{\log K}}{\log\log K}\right) \ll Y^{-1/3},$$

and similarly

$$\sum_{\substack{v\geq Y\\ v\in\mathcal{B}_K}} \frac{1}{v} \ll Y^{-1/3},$$

we obtain thus

$$R_3 \ll HY^{-1/3}. \tag{2.20}$$

Consequently the following theorem holds.

Theorem 1. *Let* $\varepsilon, \delta > 0$ *be fixed,* $1 \leq K \leq x_1^4$, $H = x^{\frac{7}{12}+2\varepsilon}$, $\frac{x_2}{2} \leq l \leq$ $\leq (\sqrt{2}-\delta)x_2$, $\log Y = x_2^{3/4}$.

Let $a = \frac{l}{x_2}$. *Then*

$$\begin{aligned}\prod\nolimits_l([x,x+H]\mid K) = \frac{6H}{\pi^2}\rho_l(x)\sum_{\substack{v\in\mathcal{B}_K\\ v<Y}}\frac{\lambda(v)}{v}a^{\Omega(v)}\exp\left(-\frac{\Omega^2(v)}{2l}\right)+\\ +O\left(\frac{H}{x_2}\rho_l(x)(\log\log K)^{\sqrt{2}}\right)+O\left(HY^{-1/3}\right).\end{aligned} \tag{2.21}$$

Furthermore,

$$\left| \sum_{\substack{v\in\mathcal{B}_K\\ v<Y}} \frac{\lambda(v)}{v} a^{\Omega(v)} \exp\left(-\frac{\Omega^2(v)}{2l}\right) - \prod_{p|K} \frac{1}{1+\frac{a}{p}} \right| \ll \tag{2.22}$$
$$\ll \frac{1}{l}\left(1+\kappa^2(K)e^{a\kappa(K)}\right),$$

where $\kappa(K)$ *is defined by* (2.15).

It remains to prove only (2.22). The left-hand side is less than

$$\ll \frac{1}{2l} \sum_{\substack{v\in\mathcal{B}_K\\ v<Y}} \frac{1}{v} a^{\Omega(v)}\Omega^2(v) + \sum_{\substack{v\in\mathcal{B}_K\\ v>Y}} \frac{1}{v} a^{\Omega(v)}.$$

The second sum is $\ll H\cdot Y^{-1/3} \le \frac{H}{l}$, as we proved earlier.

Let

$$E_K(a) := \sum_{v\in\mathcal{B}_K} \frac{a^{\Omega(v)}}{v}\Omega^2(v).$$

Since

$$\Omega^2(v) = \sum_{p^\alpha\|v} \alpha^2 + \sum_{\substack{p\neq q\\ p^\alpha\|v\\ q^\beta\|v}} \alpha\cdot\beta,$$

therefore

$$E_K(a) \le \left\{\sum_{p|K}\left(\sum_{\alpha=1}^{\infty}\frac{\alpha^2}{p^\alpha}a^\alpha\right)\right\}\prod_{p|K}\left(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\dots\right)+$$
$$+\left\{\sum_{p|K}\sum_{q|K}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\frac{\alpha\cdot a^\alpha}{p^\alpha}\cdot\frac{\beta a^\beta}{q^\beta}\right\}\prod_{p|K}\left(1+\frac{a}{p}+\left(\frac{a}{p}\right)^2+\dots\right),$$

and so

$$E_K(a) \ll e^{a\cdot\kappa(K)}\kappa^2(K).$$

Thus (2.22) is true.

3. Formulation and the proof of the main theorem

Let $s\in\mathbb{N}$. We would like to estimate

$$Q(s\mid[x,x+H_0]) = \#\{n\in[x,x+H_0]\mid d_2(n)=s\}, \tag{3.1}$$

where $H_0 = x^{7/12+\varepsilon_1}$, $\varepsilon_1>0$.

For some $s \in \mathbb{N}$ let $\mathcal{T}_s = \{1 = l_0 < l_1 < \ldots < l_{r(s)}\}$ be the set of divisors of s. For every l_j let $\mathcal{B}_{l_j}$ be the set of those square-full integers K for which $d(k_1) = l_j$, and $K \le x_x^4$ holds. We shall use the notation (2.1).

It is clear that

$$Q(s \mid [x, x+H_0]) =$$

$$= \sum_{l_j \in \mathcal{T}_s} \sum_{K \in \mathcal{B}_{l_j}} \prod\nolimits_{\frac{s}{l_j} - \alpha - 1} \left(\left[\frac{x}{K}, \frac{x+H_0}{K} \right] \,\Big|\, K \right) + O\left(\frac{H_0}{x_1^2} \right), \tag{3.2}$$

where the error term comes from the contribution of square full numbers $K > x_1^4$.

It is known that

$$\sum_{x \le n \le x+y} r^{\omega(n)} \ll y \cdot x_1^{r-1} \tag{3.3}$$

if $y \ge x^{7/12+\varepsilon_2}, \varepsilon > 0$.

The inequality (3.3) can be proved by sieve methods (see [2]). (3.3) follows from our Th. A, as well.

From (3.3) we obtain that

$$\# \{n \in [x, x+H] \mid \omega(n) \ge \alpha x_2\} \ll$$

$$\ll \frac{1}{r^{\alpha x_2}} \sum_{n \le n \le x+H} r^{\omega(n)} \ll \frac{H}{x_1} \exp((r - \alpha \log r) x_2).$$

By choosing $r = 5$, $\alpha = 5$, we obtain that

$$\# \{n \in [x, x+H] \mid \omega(n) \ge 5x_2\} \ll \frac{H}{x_1^2}. \tag{3.4}$$

Thus (3.2) remains valid if we drop all those l_j for which

$$\frac{s}{l_j} - \alpha - 1 \ge 5x_2.$$

Let $t = \dfrac{s}{l_j} - \alpha - 1$, $K \in \mathcal{B}_{l_j}$. From Th. A we obtain that

$$\prod\nolimits_t \left(\left[\frac{x}{K}, \frac{x+H_0}{K} \right] \,\Big|\, K \right) \le \prod\nolimits_t \left(\left[\frac{x}{K}, \frac{x+H_0}{K} \right] \right) \ll r \frac{H}{K} \rho_t(x),$$

which is valid for $t < cx_2$.

Since

$$\rho_t(x) \ll \frac{1}{\sqrt{x_2}} \exp\left(-\frac{\sqrt{x_2}}{2}\right) \quad \text{if} \quad |t - x_2| \geq x_2^{3/4},$$

we obtain that

$$Q(s \mid [x, x+H_0]) =$$

$$= \sum_{l_j \in \mathcal{T}_s} \sum_{K \in \mathcal{B}_{l_j}}^{*} \prod\nolimits_{\frac{s}{l_j}-\alpha-1} \left(\left[\frac{x}{K}, \frac{x+H_0}{K}\right] \,\middle|\, K\right) + \tag{3.5}$$

$$+ O\left(H \exp\left(-\frac{x_2^{1/2}}{2}\right)\right),$$

where $*$ means that we sum over those K only for which

$$\left|\frac{s}{l_j} - \alpha - 1 - x_2\right| \leq x_2^{3/4} \tag{3.6}$$

holds.

If no such K exists, then on the right-hand side of (3.5) we have an empty sum.

From Th. 1 we can deduce the following

Lemma 1. *Assume that the conditions of* Th. 1 *are satisfied, and let* $l = x_2 + \Delta$, *where* $|\Delta| \ll x_2^{3/4}$. *Then*

$$\left|\prod\nolimits_l ([x, x+H] \mid K) - \frac{6}{\pi^2} H \rho_l(x) \eta(K)\right| \ll \tag{3.7}$$

$$\ll H\rho_l(x) \left\{\frac{|\Delta|}{x_2}(1+\kappa(K)) + \frac{1}{x_2}\left(1+\kappa^2(K)\exp\left(\left(1+\frac{|\Delta|}{x_2}\right)\kappa(K)\right)\right)\right\} +$$

$$+ \frac{H\rho_l(x)}{x_2}(\log\log K)^{\sqrt{2}} + O\left(H \cdot Y^{-1/3}\right).$$

If $K \leq x_1^c$, *then the right-hand side of* (3.7) *is less than*

$$H\rho_l(x) \left\{\frac{|\Delta|}{x_2}(1+x_4) + \frac{1}{x_2}\left(1+x_4^2 x_3 + x_5^{\sqrt{2}}\right)\right\} + O\left(H \cdot Y^{-1/3}\right). \tag{3.8}$$

Proof. (3.7) is a straightforward consequence of Th. 1. We observe that

$$\prod_{p|K} \frac{1}{1+\frac{a}{p}} = \eta(K) \prod\nolimits_1, \quad \prod\nolimits_1 = \prod_{p|K} \frac{1}{1 + \frac{\Delta}{x_2} \cdot \frac{1}{(p+1)}},$$

and so

$$\log \prod\nolimits_1 = \frac{\Delta}{x_2}\left(\kappa(K) + O(1)\right) + O\left(\left(\frac{\Delta}{x_2}\right)^2\right),$$

$$\prod\nolimits_1 - 1 = \frac{\Delta}{x_2}\left(\kappa(K) + O(1)\right) + O\left(\left(\frac{\Delta}{x_2}\right)^2\right).$$

Thus the left-hand side of (3.7) is less than

$$H\rho_l(x)\eta(K)\left(\frac{|\Delta|}{x_2}\left(\kappa(K) + O(1)\right)\right) +$$
$$+ \frac{H\rho_l(x)}{l}\left(1 + \kappa^2(K)\exp\left(\left(1 + \frac{|\Delta|}{x_2}\right)\kappa(K)\right)\right) +$$
$$+ O\left(H \cdot Y^{-1/3}\right) + O\left(\frac{H}{x_2}\rho_l(x)(\log\log K)^{\sqrt{2}}\right).$$

(3.8) directly follows from (3.7).

Thus Lemma 1 is true.

Lemma 2. *Let f be additive, for prime power p^α is defined by*

$$f(p^\alpha) = \begin{cases} 0 & \textit{if } \alpha \geq 2, \\ 0 & \textit{if } p^\alpha = 2, \\ 1 & \textit{if } \alpha = 1, \ p > 2. \end{cases}$$

Then

$$\max_{x^{7/12+\varepsilon} \leq y \leq x} \frac{1}{y} \#\{n \in [x, x+y] \mid |f(n) - x_2| \geq x_2^{3/4}\} \leq$$
$$\leq c_1 \exp\left(-c_2\sqrt{x_2}\right),$$

where ε is an arbitrary positive constant, c_1 and c_2 are suitable constants.

Proof. By using the method of Erdős [2] we obtain that

$$\#\left\{n \in [x, x+y] \mid f(n) > x_2 + x_2^{3/4}\right\} \leq$$
$$\leq z^{-x_2 - x_2^{3/4}} \sum_{n \in [x, x+y]} z^{f(n)} \ll z^{-x_2 - x_2^{3/4}} \cdot x_1^{z-1}.$$

Choosing $z = 1 + \dfrac{c^*}{x_2^{1/4}}$ with a small positive constant c^*, we obtain that

$$\#\left\{n \in [x, x+y] \mid f(n) => x_2 + x_2^{3/4}\right\} \ll y \exp\left(-\frac{c^*}{2}\sqrt{x_2}\right).$$

Similarly, by sieve one can prove that

$$\#\left\{n\in[x,x+y]\mid f(n)=k\right\}\le c_1 y\frac{(x_2+c_2)^{k-1}}{x_1\cdot(k-1)!},$$

whence, by summing the right-hand side for $k\le x_2-x_2^{3/4}$, Lemma 2 immediately follows.

Let δ_1 be a fixed positive number, $s\le x_2^{1+\delta_1}$, $\delta_1<1/4$.

Let $n=Km$, $d(K)=2^\alpha\cdot k_1$,

$$d(k_1)(\alpha+1+m)=s.$$

Assume that $|m-x_2|\le x_2^{3/4}$, $\alpha<x_2^{3/4}$. Then

$$d(k_1)=\frac{s}{x_2+O\left(x_2^{3/4}\right)}=\frac{s}{x_2}+O\left(\frac{s}{x_2^{5/4}}\right).$$

Since $\frac{s}{x_2^{5/4}}=o_x(1)$ $(x\to\infty)$, therefore for some $s\le x_2^{1+\delta_1}$ if $|m-x_2|\le x_2^{3/4}$, $\alpha<x_2^{3/4}$, then

$$d(k_1)=T,$$

where

$$|s-Tx_2|\le\frac{1}{2}x_2.$$

The contribution of those n for which $\alpha\ge\left[x_2^{3/4}\right]=\beta$ is less than

$$H_0\sum_{d(K)>2^\beta}\frac{1}{K}\ll H_0\exp\left(-cx_2^{3/4}\right).$$

The contribution of those n for which $|m-x_2|\ge x_2^{3/4}$ is less than $H_0\exp\left(-\frac{1}{2}\sqrt{x_2}\right)$.

If $x_2^{3/4}\le|s-Tx_2|\le\frac{1}{2}x_2$, then

$$Q(s\mid[x,x+H_0])\ll H_0\exp\left(-\frac{1}{2}\sqrt{x_2}\right). \tag{3.9}$$

We estimate now

$$\sum:=\sum_{s\ge x_2^{1+\delta_1}}Q\left(s\mid[x,x+H_0]\right). \tag{3.10}$$

If n is counted in (3.10), then either $\omega(m)\ge 2x_2$, or $\alpha\ge x_2$, or $d(k_1)\ge x_2^{\delta_1}$. We have

$$\sum_{\substack{\omega(m)\geq 2x_2\\ m\in[x,x+H_0]}} \ll 2^{-2x_2} \sum_{x\leq m\leq x+H_0} 2^{\omega(m)} \ll H_0 \exp((1-2\log 2)x_2) \ll \frac{H_0}{x_1^c}$$

with a suitable constant $c>0$. If $\alpha \geq x_2$, then $2^\alpha | d(K)$, $K \gg x_1^{1/\varepsilon}$, and so

$$\sum_{\substack{x\leq n\leq x+H_0\\ K>x_1^{1/\varepsilon}}} 1 \ll \frac{H_0}{x_1^{10}}, \quad \text{say.}$$

If $d(k_1) \geq x_2^{\delta_1}$, then $K > x_2^{1/\varepsilon}$, and so $\sum_{K>x_2^{1/\varepsilon}} \frac{H_0}{K} \ll x_2^{-1/2\varepsilon}$, consequently

$$\sum \ll \frac{H_0}{x_2^B}, \tag{3.11}$$

where B is an arbitrary large constant. Let

$$|s-Tx_2| \leq x_2^{3/4}, \quad T \text{ integer}, \quad s\leq x_2^{1+\delta_1}. \tag{3.12}$$

By using Lemma 1, we obtain that

$$Q(s \mid [x,x+H_0]) = \tag{3.13}$$

$$= \frac{6}{\pi^2} H_0 \sum_{K\in\mathcal{B}_T}^{**} \frac{\rho_{T-\alpha-1}(x)\eta(K)}{K} + O\left(\frac{x_4}{x_2^{1/4}} H_0 \sum_{K\in\mathcal{B}_T}^{**} \frac{\rho_{T-\alpha-1}(x)\eta(K)}{K}\right) +$$
$$+O\left(H_0 Y^{-1/3}\right) + O\left(H_0 \exp\left(-\frac{1}{2}\sqrt{x_2}\right) + O\left(\frac{H_0}{x_2^B}\right)\right),$$

where $**$ means that we sum only over those K for which $\alpha \leq x_2^{3/4}$. The last three order term can be simplified by $O\left(H_0 \cdot x_2^{-B}\right)$.

We proved

Theorem 2. *The following relations hold.*

a) *Let $\delta_1 > 0$ be a small positive constant. Then*

$$\sum_{s>x_2^{1+\delta_1}} Q(s \mid [x,x+H_0]) \ll \frac{H_0}{x_2^B},$$

where B is an arbitrary positive constant.

b) *Assume that $s \leq x_2^{1+\delta_1}$, $0<\delta_1<1/4$, and T is an integer for which $x_2^{3/4} \leq |s-Tx_2| < \frac{1}{2}x_2$. Then (3.9) holds.*

c) *Assume that* $s \leq x_2^{1+\delta_1}$, $0 < \delta_1 < 1/4$, T *is the integer for which* $|s - Tx_2| < x_2^{3/4}$. *Then*

$$Q(s \mid [x, x+H_0]) = \left(1 + O\left(\frac{x_4}{x_2^{1/4}}\right)\right) H_0 \sum_{K \in \mathcal{B}_T}^{**} \frac{\rho_{T-\alpha-1}(x)\eta(K)}{K} + \\ + O\left(H_0 x_2^{-B}\right).$$

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