

DENDRITES AND MONOTONE MAPPINGS

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Abstract: Structure of dendrites Y is studied such that for each dendrite X and for each mapping $f : X \rightarrow Y = f(X)$ that satisfies some special conditions (mainly of monotoneity type) the dendrite X contains a homeomorphic copy of Y .

1. Introduction

Krzysztof Omiljanowski has proved [2, Th. 6.12, p. 183] the following theorem.

Theorem 1.1. *Let a dendrite Y be such that*

(1.1.1) *all ramification points of Y are of order 3;*

(1.1.2) *the set $R(Y)$ of all ramification points of Y is discrete.*

If a dendrite X can be mapped onto Y under a monotone mapping, then

(1.1.3) *X contains a homeomorphic copy of Y .*

The result resembles the well known theorem of Gordon Thomas Whyburn for light open mappings (see [13, (2.4), p. 188] and compare [5, Cor. 4, p. 1839]) that runs as follows.

Theorem 1.2. *Let D be a dendrite. For every compact space X and for every light open surjective mapping $f : X \rightarrow Y$ with $D \subset Y$ there*

exists a homeomorphic copy D' of D in X such that the restriction $f|_{D'} : D' \rightarrow f(D') = D$ is a homeomorphism.

Since each nonconstant open mapping defined on a dendrite is light, see [7, Cor. 6, p. 216], if X is a dendrite, then the assumption of lightness in Th. 1.2 can be omitted. For more general results see [5, Cor. 10, p. 1842] and [6, Th. 16].

The two results, namely Ths. 1.1 and 1.2, play the key role in proving some results in dynamical systems about preserving PR-property and Ω EP-property under monotone as well as under open mappings of dendrites, see [4, Ths. 3.6, 3.9 and 3.11]. Since the inverse implication to that of Th. 1.2 is known to be true (and thus characterizations of dendrites are obtained, see [5, Cor. 10, p. 1842] and [6, Th. 16]), it is natural to ask if Th. 1.1 can be reversed; in other words, we are interested in solving the following problem.

Problem 1.3. Characterize all dendrites Y having the property that if a dendrite X can be mapped onto Y under a monotone mapping, then X contains a homeomorphic copy of Y .

Observe that, contrary to open mappings, not all dendrites Y have the above formulated property. If X is of the shape of the letter H then shrinking the horizontal bar of X to a point we get a monotone mapping of X onto a simple 4-od Y , that is, onto a dendrite of the shape of the letter X, and we see that X does not contain any copy of Y .

Note also that, since the image of a dendrite under a monotone mapping is again a dendrite, the obtained characterization is valid for all continua, not only for dendrites.

In this paper we present a partial answer to the above formulated Problem 1.3.

2. Preliminaries

All considered spaces are assumed to be metric, and a mapping means a continuous function. We denote by \mathbb{N} the set of all positive integers and by \mathbb{C} the set of all complex numbers. For $A \subset X$ we denote $\text{cl}_X(A)$ and $\text{bd}_X(A)$ the closure and the boundary of A in X , correspondingly. The symbol $\text{card}(A)$ stands for the cardinality of A , and $\text{diam}(A)$ means the diameter of A .

A concept of an order of a point p in a continuum X (in the sense

of Menger-Urysohn), written $\text{ord}(p, X)$, is defined as follows.

Let \mathfrak{n} stand for a cardinal number. We write:

- (•) $\text{ord}(p, X) \leq \mathfrak{n}$ provided that for every $\varepsilon > 0$ there is an open neighborhood U of p such that $\text{diam}(U) \leq \varepsilon$ and $\text{card}(\text{bd}(U)) \leq \mathfrak{n}$;
- (•) $\text{ord}(p, X) = \mathfrak{n}$ provided that $\text{ord}(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\text{ord}(p, X) \leq \mathfrak{m}$ does not hold;
- (•) $\text{ord}(p, X) = \omega$ provided that the point p has arbitrarily small open neighborhoods U with finite boundaries $\text{bd}(U)$ and $\text{card}(\text{bd}(U))$ is not bounded by any $n \in \mathbb{N}$.

Thus, for any continuum X we have

$$\text{ord}(p, X) \in \{1, 2, \dots, n, \dots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [8, §51, I, p. 274].

A point $p \in X$ is called an end point of X provided that $\text{ord}(p, X) = 1$, and it is called a ramification point of X provided that $\text{ord}(p, X) \geq 3$. For a space X we denote the sets of end points of X and of ramification points of X by $E(X)$ and $R(X)$, respectively.

A continuum X is said to be a dendrite if it is locally connected and contains no simple closed curve.

A surjective mapping $f : X \rightarrow Y$ between topological spaces is said to be:

— light provided that for each point $y \in Y$ the set $f^{-1}(y)$ has one-point components (note that if the point-inverses are compact, this condition is equivalent to the property that they are zero-dimensional);

— open (\mathcal{O}) provided that the images of open sets under f are open;

— monotone (\mathcal{M}) provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected;

— an OM-mapping (or an MO-mapping) (\mathcal{OM} or \mathcal{MO} , respectively) provided that there exist mappings f_1 and f_2 such that $f = f_2 \circ f_1$, where f_1 is monotone and f_2 is open (or f_1 is open and f_2 is monotone, respectively);

— locally MO-mapping ($\text{Loc}(\mathcal{MO})$) provided that for each point $y \in Y$ there is a closed neighborhood V of y in Y such that the partial mapping $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is an MO-surjection;

— almost monotone (\mathcal{AM}) provided that for each subcontinuum Q of Y with nonempty interior the preimage $f^{-1}(Q)$ is connected;

— feebly monotone (\mathcal{FM}) provided that X and Y are continua, and if A and B are proper subcontinua of Y such that $Y = A \cup B$, then the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected;

— quasi-monotone (\mathcal{QM}) provided that for each subcontinuum Q of Y with a nonempty interior the preimage $f^{-1}(Q)$ has a finite number of components and each of them is mapped onto Q under f ;

— weakly monotone (\mathcal{WM}) provided that for each subcontinuum Q of Y with a nonempty interior each component of the preimage $f^{-1}(Q)$ is mapped onto Q under f ;

— confluent (\mathcal{C}) provided that for each subcontinuum Q of Y each component of the preimage $f^{-1}(Q)$ is mapped onto Q under f ;

— locally confluent ($\text{Loc}(\mathcal{C})$) provided that for each point $y \in Y$ there is a closed neighborhood V of y in Y such that the partial mapping $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$ is a confluent surjection (equivalently, if for each point $x \in X$ there is a closed neighborhood U of x in X such that the partial mapping $f|_U : U \rightarrow f(U)$ is confluent, see [11, Th. 4.24, p. 19]);

— semi-confluent (\mathcal{SC}) provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of the preimage $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

The following implications between these classes of mappings are known, and none of them can be reversed in general, see e.g. [11, Table II, p. 28, p. 28], [9] and [3, Prop. 2.1, p. 16, and Remarks 2.3, p. 17].

$$\begin{array}{ccccccc}
 & \mathcal{O} & & & & \mathcal{SC} & \\
 & \downarrow & & & & \uparrow & \\
 & \mathcal{MO} \Rightarrow \text{Loc}(\mathcal{MO}) \Rightarrow \mathcal{OM} \Rightarrow & & & \mathcal{C} \Rightarrow \text{Loc}(\mathcal{C}) & & \\
 (2.1) & \uparrow & & & \downarrow & & \\
 & \mathcal{M} \Rightarrow \mathcal{AM} \Rightarrow \mathcal{QM} \Rightarrow \mathcal{WM} & & & & & \\
 & & & & \downarrow & & \\
 & & & & \mathcal{FM} & &
 \end{array}$$

Further, if the range space $Y = f(X)$ is locally connected, then all locally confluent mappings and all weakly monotone mappings f are \mathcal{OM} -mappings and quasi-monotone ones (see [11, (6.2), p. 51]), whence the statement below follows.

Statement 2.1. *If $f : X \rightarrow Y$ is a surjective mapping onto a locally connected space, then the conditions $f \in \mathcal{A}$ are equivalent for the following classes \mathcal{A} :*

(2.1.1) $OM, C, \text{Loc}(C), \mathcal{QM}, WM.$

3. Extensions to other classes of mappings and of continua

A natural question related to Th. 1.1 is what classes of mappings (in particular, the ones larger than the class of monotone mappings) the result can be extended to. Further, one can ask if it is necessary to assume that both continua, X and Y , are dendrites. The results below give answers to these questions.

Theorem 3.1. *Let a continuum Y be such that*

(1.1.1) *all ramification points of Y are of order 3;*

(1.1.2) *the set $R(Y)$ of all ramification points of Y is discrete.*

If a dendrite X can be mapped onto Y under a mapping that belongs to one of the following classes of mappings:

(3.1.1) *open, monotone, MO-mappings, or OM-mappings,*
then

(1.1.3) *X contains a homeomorphic copy of Y .*

Proof. Recall that open as well as monotone mappings preserve the property of being a dendrite (see [11, Table IV, p. 69 and 70]), whence it follows that MO- and OM-mappings do. Thus if X is a dendrite and $f : X \rightarrow Y$ is as in (3.1.1), then Y is a dendrite as well.

Consequently, the conclusion (1.1.3) for monotone mappings follows from Th. 1.1. Further, since each nonconstant open mapping defined on a dendrite is light, see [7, Cor. 6, p. 216], it follows that "open" and "light open" are equivalent, so the conclusion for open mappings follows from Th. 1.2. Finally, for MO-mappings and OM-mappings it is a consequence of the previous assertions using compositions of mappings. \diamond

Corollary 3.2. *Let a continuum Y be such that*

(1.1.1) *all ramification points of Y are of order 3;*

(1.1.2) *the set $R(Y)$ of all ramification points of Y is discrete.*

If a dendrite X can be mapped onto Y under a mapping that belongs to one of the classes of mappings listed in (2.1) (except the class SC), then

(1.1.3) *X contains a homeomorphic copy of Y .*

Proof. By Cor. 3.2 the conclusion (1.1.3) holds for mappings in (3.1.1). Since any locally MO-mapping is an OM-mapping, we have the conclusion for locally MO-mapping.

Again it follows that Y is a dendrite, since any one of the mappings of (2.1) preserves the class of dendrites, see [11, Table IV, p. 69 and 70] and [3, Prop. 2.5, p. 18]. Applying Statement 2.1 we get the conclusion for mappings listed in (2.1.1). Further, since each almost monotone mapping is quasi-monotone just by the definitions, the conclusion for almost monotone mappings follows from the previous assertion for quasi-monotone mappings. Finally, since each feebly monotone mapping from a continuum onto a locally connected one is confluent (thus weakly monotone), see [3, Prop. 2.5, p. 18], the conclusion holds for feebly monotone mapping as well. \diamond

Recall that the class of semi-confluent mappings is essentially larger than that of confluent mappings, even for mappings between locally connected continua, [11, Ex. 3.12, p. 14], and thus semi-confluent mappings cannot be attached to ones in (2.1.1) of Statement 2.1. However, the image of a dendrite under a semi-confluent mapping is again a dendrite, [10, Cor. 5.4, p. 262]. In connection with this the following question is interesting.

Question 3.3. Let a continuum Y satisfy conditions (1.1.1) and (1.1.2), and let a dendrite X can be mapped onto Y under a semi-confluent mapping. Must then X contain a homeomorphic copy of Y ?

Remark 3.4. Let us note that that Th. 1.2 and, consequently, Th. 3.1 and Cor. 3.2 cannot be extended to (locally connected) continua X (or linear graphs even) that contain simple closed curves. This can be seen by the (known) example below.

Example 3.5. *There are a planar linear cyclic graph X all ramification points of which are of order 3, and a local homeomorphism (thus an open mapping) $f : X \rightarrow Y$ which is 2-to-1, onto a non-planar linear cyclic graph Y such that both conditions (1.1.1) and (1.1.2) are satisfied, while (1.1.3) is not.*

Proof. The example is constructed in [13, Ch. X, §3, Ex., p. 189]. Conditions (1.1.1) and (1.1.2) follow from the construction. Since X is planar and Y is not (in fact, Y is homeomorphic to one of the Kuratowski non-planar graphs, see [8, §51, VII, Fig. 11 (left), p. 305]), (1.1.3) does not hold. \diamond

Remark 3.6. Note that the possibility of the construction as in Ex. 3.5 follows from the following result, which is the inverse implication to that of Th. 1.2 (see [5, Th. 9, p. 1842]).

(3.6.1) *Let a continuum D be such that for every compact space X , for every light open mapping $f : X \rightarrow Y = f(X)$ with $D \subset Y$*

there exists a homeomorphic copy D' of D in X such that the restriction $f|_{D'} : D' \rightarrow f(D') = D$ is a homeomorphism. Then D is a dendrite.

In this way, combining Th. 1.2 with (3.6.1) one gets a characterization of dendrites in terms of light open mappings, see [5, Cor. 10, p. 1842].

It is interesting to know if a similar characterization of dendrites can be obtained in terms of monotone mappings. More precisely, one can ask the following question.

Question 3.7. Let a continuum Y satisfy conditions (1.1.1) and (1.1.2), and let a continuum X be such that

(3.7.1) if X can be mapped onto Y under a monotone mapping, then X contains a homeomorphic copy of Y .

Must then X be a dendrite? If not, under what additional assumptions does the implication hold?

4. Inverse implications

Another important and interesting question related to Th. 1.1 is if the implication in the result can be reversed. Below we give a partial affirmative answer.

Theorem 4.1. *Let a dendrite Y have the following property:*

(4.1.1) *for each dendrite X if X can be mapped onto Y under a monotone mapping, then X contains a homeomorphic copy of Y .*

Then either Y is an arc or

(1.1.1) *all ramification points of Y are of order 3.*

Proof. Suppose on the contrary that neither Y is an arc nor (1.1.1) is true, i.e., that Y contains a ramification point of order greater than 3. We will construct (using the inverse limit procedure) a dendrite X and a monotone mapping $f : X \rightarrow Y$ such that all ramification points of X are of order 3, and thus no homeomorphic copy of Y is contained in X .

Recall the following known facts.

(4.1.2) Each ramification point of a dendrite is of order at most ω , see [8, §51, VI, Th. 4, p. 301 and IV, Th. 9, p. 287].

(4.1.3) The set of all ramification points of a dendrite is at most countable, see [8, §51, VI, Th. 7, p. 302].

(4.1.4) A continuum Y is a dendrite if and only if $\text{ord}(y, Y)$ is equal to the number of components of $Y \setminus \{y\}$ whenever either of these

is finite, see [8, §51, VI, Th. 6, p. 302] and [13, Ch. V, (1.1), (iv), p. 88].

Thus there exists a sequence

$$(4.1.5) \quad y_1, y_2, \dots, y_k, y_{k+1}, \dots$$

of points of Y such that, for each $k \in \mathbb{N}$, the following conditions are satisfied.

(4.1.6) If $\text{ord}(y, Y) \geq 4$, then the point y appears in the sequence (4.1.5), i.e., $y = y_k$ for some $k \in \mathbb{N}$.

(4.1.7) Each point y_k is a ramification point of Y of order at least 4.

(4.1.8) Each point y_k appears in the sequence (4.1.5) exactly once, so that the members of the sequence (4.1.5) are mutually distinct.

Now we define the above mentioned inverse sequence $\{X_k, f_j^k\}$ of dendrites X_k and monotone bonding mappings $f_j^k : X_j \rightarrow X_k$, where $j, k \in \mathbb{N}$ with $j \leq k$ and with $f_k^k : X_k \rightarrow X_k$ being the identity, as follows.

Put $X_1 = Y$. Let C_m^1 be the one-point compactification of a component of the set $X_1 \setminus \{y_1\}$, where $m \in \{1, 2, \dots, \text{ord}(y_1, X_1)\}$ if $\text{ord}(y_1, X_1) \geq 4$ is finite, and $m \in \mathbb{N}$ otherwise. Denote by c_m^1 the point being the remainder in the compactification. It follows that, in the latter case, $\lim_{m \rightarrow \infty} \text{diam}(C_m^1) = 0$ since X_1 is locally connected. Define a dendrite X_2 as follows. Let A_1 be an arc with end points a_1, b_1 such that $\text{diam} A_1 < 1$. Consider three cases.

CASE 1.1. If $\text{ord}(y_1, X_1) = 4$, then we identify c_1^1 and c_2^1 with the point a_1 , identify c_3^1 and c_4^1 with the point b_1 , and define

$$X_2 = A_1 \cup C_1^1 \cup C_2^1 \cup C_3^1 \cup C_4^1.$$

Note that X_2 is a dendrite, and that $\text{ord}(c_i^1, X_2) = 3$ for $i \in \{1, 2, 3, 4\}$.

CASE 1.2. If $\text{ord}(y_1, X_1) = m$ for certain integer $m > 4$, then we choose in the set $A_1 \setminus \{a_1, b_1\}$ some $m - 4$ pairwise distinct points p_5^1, \dots, p_m^1 and identify c_1^1 and c_2^1 with a_1 , identify c_3^1 and c_4^1 with b_1 (as in Case 1.1), and identify c_i^1 with p_i^1 for each $i \in \{5, \dots, m\}$. Then define

$$X_2 = A_1 \cup \bigcup \{C_i^1 : i \in \{1, 2, \dots, m\}\}.$$

Again X_2 is a dendrite, and $\text{ord}(c_i^1, X_2) = 3$ for $i \in \{1, 2, \dots, m\}$.

CASE 1.3. If $\text{ord}(y_1, X_1) = \omega$, then we choose in the set $A_1 \setminus \{a_1, b_1\}$ a sequence of distinct points p_5^1, p_6^1, \dots such that $a_1 < p_5^1 < p_6^1 < \dots < b_1$ (where $<$ means the ordering on A_1 from a_1 to b_1)

converging to b_1 . Identify, as previously, c_1^1 and c_2^1 with a_1 , c_3^1 and c_4^1 with b_1 , and identify c_i^1 with p_i^1 for each $i \in \{5, 6, \dots\}$. Then define

$$X_2 = A_1 \cup \bigcup \{C_i^1 : i \in \mathbb{N}\}.$$

Thus, as previously, X_2 is a dendrite, and $\text{ord}(c_i^1, X_2) = 3$ for $i \in \mathbb{N}$.

In this way the dendrite X_2 is defined. Let a mapping $f_1^2 : X_2 \rightarrow X_1$ shrink the arc A_1 back to the point y_1 and be one-to-one on $X_2 \setminus A_1$. Thus $(f_1^2)^{-1}(y) = A_1$ if $y = y_1$ and $(f_1^2)^{-1}(y)$ is a singleton otherwise. Therefore f_1^2 is monotone. Finally, since $y_1 \neq y_2$, the inverse image $f_1^{-1}(y_2)$ is a singleton in $X_2 \setminus A_1$. Denote by y_2^2 the copy of y_2 in X_2 , that is, $y_2^2 \in X_2 \setminus A_1$ is defined by the condition $f_1^2(y_2^2) = y_2$.

Assume now that the dendrites X_k, X_{k+1} , the arc $A_k \subset X_{k+1}$ and a monotone bonding mapping $f_k^{k+1} : X_{k+1} \rightarrow X_k$ that shrinks the arc A_k to a point $y_k^k \in X_k$ are defined for some $k \in \mathbb{N}$, and let the point $y_{k+1}^{k+1} \in X_{k+1}$ be determined by $f_1^{k+1}(y_{k+1}^{k+1}) = y_k \in X_1 = Y$. We will define X_{k+2} . Note that, since the elements of the sequence (4.1.5) are distinct, we have $y_{k+1}^{k+1} \in X_{k+1} \setminus A_k$. Let C_m^{k+1} be the one-point compactification of a component of the set $X_{k+1} \setminus \{y_{k+1}^{k+1}\}$, where $m \in \{1, 2, \dots, \text{ord}(y_{k+1}^{k+1}, X_{k+1})\}$ if $\text{ord}(y_{k+1}^{k+1}, X_{k+1}) \geq 4$ is finite, and $m \in \mathbb{N}$ otherwise. Denote by c_m^{k+1} the point being the remainder in the compactification. It follows that, in the latter case, $\lim_{m \rightarrow \infty} \text{diam}(C_m^{k+1}) = 0$ since X_{k+1} is locally connected. Define a dendrite X_{k+2} as follows. Let A_{k+1} be an arc with end points a_{k+1}, b_{k+1} such that $\text{diam} A_{k+1} < \frac{1}{k+1}$. Consider three cases.

CASE $k + 1.1$. If $\text{ord}(y_{k+1}^{k+1}, X_{k+1}) = 4$, then we identify c_1^{k+1} and c_2^{k+1} with the point a_{k+1} , identify c_3^{k+1} and c_4^{k+1} with the point b_{k+1} , and define

$$X_{k+2} = A_{k+1} \cup C_1^{k+1} \cup C_2^{k+1} \cup C_3^{k+1} \cup C_4^{k+1}.$$

Thus X_{k+2} is a dendrite, and $\text{ord}(c_i^{k+1}, X_{k+2}) = 3$ for $i \in \{1, 2, 3, 4\}$.

CASE $k + 1.2$. If $\text{ord}(y_{k+1}^{k+1}, X_{k+1}) = m$ for certain integer $m > 4$, then we choose in the set $A_1 \setminus \{a_1, b_1\}$ some $m - 4$ pairwise distinct points $p_5^{k+1}, \dots, p_m^{k+1}$ and identify c_1^{k+1} and c_2^{k+1} with a_{k+1} , identify c_3^{k+1} and c_4^{k+1} with b_{k+1} (as in Case $k + 1.1$), and identify c_i^{k+1} with p_i^{k+1} for each $i \in \{5, \dots, m\}$. Then define

$$X_{k+2} = A_{k+1} \cup \bigcup \{C_i^{k+1} : i \in \{1, 2, \dots, m\}\}.$$

Again X_{k+2} is a dendrite, and $\text{ord}(c_i^{k+1}, X_{k+2}) = 3$ for $i \in \{1, 2, \dots, m\}$.

CASE $k + 1.3$. If $\text{ord}(y_{k+1}^{k+1}, X_{k+1}) = \omega$, then we choose in the set $A_{k+1} \setminus \{a_{k+1}, b_{k+1}\}$ a sequence of distinct points $p_5^{k+1}, p_6^{k+1}, \dots$ such that $a_{k+1} < p^{k+1}1_5 < p_6^{k+1} < \dots < b_{k+1}$ (where $<$ means the ordering on A_{k+1} from a_{k+1} to b_{k+1}) converging to b_{k+1} . Identify, as previously, c_1^{k+1} and c_2^{k+1} with a_{k+1} , c_3^{k+1} and c_4^{k+1} with b_{k+1} , and identify c_i^{k+1} with $p^{k+1}1_i$ for each $i \in \{5, 6, \dots\}$. Then define

$$X_{k+2} = A_{k+1} \cup \bigcup \{C_i^{k+1} : i \in \mathbb{N}\}.$$

Thus, as previously, X_{k+2} is a dendrite, and $\text{ord}(c_i^{k+1}, X_{k+2}) = 3$ for $i \in \mathbb{N}$.

Let a mapping $f_{k+1}^{k+2} : X_{k+2} \rightarrow X_{k+1}$ shrink the arc A_{k+1} back to the point y_{k+1} and be one-to-one on $X_{k+2} \setminus A_{k+1}$. Thus $(f_{k+1}^{k+2})^{-1}(y) = A_{k+1}$ if $y = y_{k+1}$ and $(f_{k+1}^{k+2})^{-1}(y)$ is a singleton otherwise. Therefore f_{k+1}^{k+2} is monotone. Consequently, the mapping $f_1^{k+2} : X_{k+2} \rightarrow X_1$ is monotone as the composition of monotone mappings, and it follows from the construction that $(f_1^{k+2})^{-1}(y_{k+2})$ is a singleton in $X_{k+2} \setminus A_{k+1}$. Denote this point by y_{k+2}^{k+2} .

Since all the cases are considered, the inductive procedure is finished, and thus the inverse sequence $\{X_k, f_j^k\}$ of dendrites with monotone bonding mappings has been defined. Let $X = \varprojlim \{X_k, f_j^k\}$ be the inverse limit space. Then X is a dendrite, see [12, Th. 10.36, p. 180]. Since all ramification points of order greater than 3 were consecutively eliminated in the process of constructing the dendrites X_k , the dendrite X contains ramification points of order 3 only. This statement can be more precisely shown using the standard argument of the Anderson–Choquet Embedding Theorem, because the construction can be provided so that all dendrites X_k are lying in the plane \mathbb{R}^2 and that X is homeomorphic to $\bigcap \{\text{cl}_{\mathbb{R}^2}(\bigcup \{X_k : k \geq j\}) : j \in \mathbb{N}\}$ (see [12, Th. 2.10, p. 23]; the details are left to the reader).

To complete the proof note that the projection $f : X \rightarrow X_1 = Y$ is a monotone mapping, see [1, Lemma 4.2, p. 241]. \diamond

We close the paper with the following question.

Question 4.2. Let a dendrite Y have property (4.1.1). Must then Y either be an arc or have the set $R(Y)$ of its ramification points discrete?

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