

SOME INEQUALITIES FOR RANDOM VARIABLES WHOSE PROBABILITY DENSITY FUNCTIONS ARE BOUNDED

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Abstract: In this paper, we improve some inequalities obtained by N. S. Barnett and S. S. Dragomir in [1] for random variables whose probability density functions are bounded. Our approach is based on the use of a variant of the Grüss inequality which is recently obtained by X. L. Cheng and J. Sun (see [3]).

1. Introduction

In a recent paper (see [1]), N. S. Barnett and S. S. Dragomir, using the pre-Grüss inequality obtained by Matić, Pečarić and Ujević in [4], have established some inequalities for random variables whose probability density functions are bounded. More precisely, we can find in [1] the following result:

Theorem 1. *Let X be a random variable having the probability density function $f : [a, b] \rightarrow \mathbb{R}$. Let us denote $F(x) := \int_a^x f(t) dt$ the distribution function and $E(X)$ the expectation of X . Assume that there exist two*

constants γ and ϕ such that $0 \leq \gamma \leq f(t) \leq \varphi$ a.e. on $[a, b]$. Then we have the inequality

$$(1.1) \quad \left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{4\sqrt{3}}(\varphi - \gamma)(b-a)^2,$$

for all $x \in [a, b]$. In particular, if in (1.1) we choose either $x = a$ or $x = b$, we get

$$(1.2) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{4\sqrt{3}}(\varphi - \gamma)(b-a)^2,$$

The following result also is proved in [1].

Theorem 2. Let X , f , γ , ϕ and F as above. Then we have the inequalities

$$(1.3) \quad \left| E(X) + \frac{b-a}{2}F(x) - \frac{b+x}{2} \right| \leq \frac{1}{2\sqrt{3}}(\varphi - \gamma) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \leq \\ \leq \frac{1}{4\sqrt{3}}(\varphi - \gamma)(b-a)^2,$$

for all $x \in [a, b]$.

The aim of this paper is to give some improvements to the inequalities established in the above theorems. Our approach will be based on a recent result obtained by X. L. Cheng and J. Sun (see [3]).

2. The results

Before we proceed to the main results of this paper we need to recall the following variant of the Grüss inequality which is recently obtained by X. L. Cheng and J. Sun in their paper [3].

Theorem 3. Let $h, g : [a, b] \rightarrow \mathbb{R}$ be two integrable functions such that $\gamma \leq g(x) \leq \varphi$ for some real constants and for all $x \in [a, b]$. Then

$$(2.1) \quad \left| \int_a^b h(x)g(x) dx - \frac{1}{b-a} \int_a^b h(x) dx \int_a^b g(x) dx \right| \leq \\ \leq \frac{1}{2} \left(\int_a^b \left| h(x) - \frac{1}{b-a} \int_a^b h(y) dy \right| dx \right) (\varphi - \gamma).$$

Our first result is the following theorem:

Theorem 4. Let X be a random variable having the probability density function $f : [a, b] \rightarrow \mathbb{R}$. Let us denote $F(x) := \int_a^x f(t) dt$ the distribution function and $E(X)$ the expectation of X . Assume that there exist two constants γ and φ such that $0 \leq \gamma \leq f(t) \leq \varphi$ a.e. on $[a, b]$. Then we have the inequality

$$(2.2) \quad \left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{8}(\varphi - \gamma)(b-a)^2,$$

for all $x \in [a, b]$. In particular, if in (2.2) we choose either $x = a$ or $x = b$, we get

$$(2.3) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{8}(\varphi - \gamma)(b-a)^2.$$

Furthermore, the constant $\frac{1}{8}$ in (2.2) (resp. (2.3)) is the best possible.

Proof. Let X , f , γ , φ and F as above. Let $p(x, t)$ be the kernel defined on $[a, b] \times [a, b]$ by setting

$$p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x], \\ t - b & \text{if } t \in]x, b]. \end{cases}$$

Making integration by parts, it is easy to derive the following identity:

$$(2.4) \quad (b-a)F(x) + E(X) - b = \int_a^b p(x, t)f(t) dt.$$

Applying the inequality (2.1) to f and $g(t) := p(x, t)$, we get

$$(2.5) \quad \left| \int_a^b f(t)p(x, t) dt - \frac{1}{b-a} \int_a^b p(x, t) dt \cdot \int_a^b f(t) dt \right| \leq \frac{1}{2} \left(\int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt \right) (\varphi - \gamma).$$

We observe that

$$\int_a^b f(t) dt = 1, \quad \frac{1}{b-a} \int_a^b p(x, t) dt = x - \frac{a+b}{2}.$$

It remains to compute the integral $\int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt$. To this end, we shall discuss two cases:

(i) The case where $x \in [a, \frac{a+b}{2}]$. We set $t_1(x) := x - \frac{b-a}{2}$ and $t_2(x) := x + \frac{b-a}{2}$. We observe that $t_1(x) \leq a$ and $x < t_2(x) \leq b$. We observe also that

$$\int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt = \int_a^x [t - t_1(x)] dt + \int_x^b |t - t_2(x)| dt$$

$$:= I_1 + I_2.$$

However,

$$I_1 = \int_a^x [t - t_1(x)] dt = \frac{(b-a)^2}{8} - \frac{1}{2} \left[x - \frac{a+b}{2} \right]^2$$

and

$$I_2 = \int_x^{t_2(x)} [t_2(x) - t] dt + \int_{t_2(x)}^b [t - t_2(x)] dt = \frac{(b-a)^2}{8} + \frac{1}{2} \left[x - \frac{a+b}{2} \right]^2.$$

We deduce that, in this case, we have

$$(2.6) \quad \int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt = \frac{(b-a)^2}{4}.$$

(ii) The case where $x \in]\frac{a+b}{2}, b]$. We set $t_1(x) := x - \frac{b-a}{2}$ and $t_2(x) := x + \frac{b-a}{2}$. We observe that $t_2(x) \geq b$ and $a \leq t_1(x) < x$. We observe also that

$$\int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt = \int_a^x |t - t_1(x)| dt + \int_x^b [t_2(x) - t] dt$$

$$:= J_1 + J_2.$$

However,

$$J_2 = \int_x^b [t_2(x) - t] dt = \frac{(b-a)^2}{8} - \frac{1}{2} \left[x - \frac{a+b}{2} \right]^2$$

and

$$J_1 = \int_a^{t_1(x)} [t_1(x) - t] dt + \int_{t_1(x)}^b [t - t_1(x)] dt =$$

$$= \frac{1}{2} \left[x - \frac{a+b}{2} \right]^2 + \frac{(b-a)^2}{8}.$$

We deduce that, in this case, we have

$$(2.7) \quad \int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt = \frac{(b-a)^2}{4}.$$

Therefore, for all $x \in [a, b]$, we have the following identity

$$(2.8) \quad \int_a^b \left| p(x, t) - \frac{1}{b-a} \int_a^b p(x, s) ds \right| dt = \frac{(b-a)^2}{4}.$$

From (2.4), (2.5) and (2.8) we derive the inequality (2.2).

To prove the sharpness of the constant $\frac{1}{8}$ in (2.2), assume that (2.2) holds with a constant $C > 0$, that is,

$$(2.9) \quad \left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq C(\varphi - \gamma)(b-a)^2,$$

for every finite interval $[a, b]$, for all $x \in [a, b]$ and all random variable X taking values in $[a, b]$ and having a probability density function f such that $0 \leq \gamma \leq f(t) \leq \varphi$ a.e. on $[a, b]$. Set $[a, b] = [0, 1]$ and consider the random variable X_0 taking values in $[0, 1]$ having the probability density function f_0 defined by:

$$f_0(x) := \begin{cases} \frac{1}{2} & \text{if } t \in [0, \frac{1}{2}], \\ \frac{3}{2} & \text{if } t \in]\frac{1}{2}, 1]. \end{cases}$$

Then $\phi = \frac{3}{2}$ and $\gamma = \frac{1}{2}$. We have $E(X_0) = \frac{5}{8}$. Consequently, by (2.9), we get

$$\left| \frac{1}{8} + F(x) - x \right| \leq C,$$

for every $x \in [0, 1]$. For $x = 0$, we get $\frac{1}{8} \leq C$. Therefore the constant $\frac{1}{8}$ is sharp in (2.2). The same example will show that the constant $\frac{1}{8}$ is the best possible in (2.3). So our result is completely proved. \diamond

Corollary 5. *Let X , f , γ , ϕ and F as above. Then we have the inequality*

$$(2.10) \quad \left| E(X) + (b-a)Pr\left(X \leq \frac{a+b}{2}\right) - b \right| \leq \frac{1}{8}(\varphi - \gamma)(b-a)^2,$$

for all $x \in [a, b]$. Furthermore, the constant $\frac{1}{8}$ in (2.10) is the best possible.

(2.10) is obtained from (2.2) by choosing $x = \frac{a+b}{2}$. The example used in the proof of Th. 4 can be used to show the sharpness of the constant $\frac{1}{8}$ in (2.10).

Our second result is the following theorem.

Theorem 6. *Let X , f , γ , φ and F as above. Then we have the inequalities*

$$(2.11) \quad \left| E(X) + \frac{b-a}{2} F(x) - \frac{b+x}{2} \right| \leq \\ \leq \frac{1}{4}(\varphi - \gamma) \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \leq \frac{1}{8}(\varphi - \gamma)(b-a)^2,$$

for all $x \in [a, b]$. Furthermore, the constant $\frac{1}{4}$ multiplying $\varphi - \gamma$ (resp. appearing in the bracket) in the second member of (2.11) is the best possible.

Proof. Let $x \in (a, b)$. We know by the identity (2.4) that

$$(b-a)F(x) + E(X) - b = \int_a^b (t-a)f(t) dt + \int_a^b (t-b)f(t) dt.$$

We apply the inequality (2.1) on the interval $[a, x]$ with f and $g(t) := t - a$. Then we get

$$(2.12) \quad \left| \int_a^x f(t)[t-a] dt - \frac{1}{x-a} \int_a^x [t-a] dt \cdot \int_a^x f(t) dt \right| \leq \\ \leq \frac{1}{2} \left(\int_a^x \left| t-a - \frac{1}{x-a} \int_a^x (s-a) ds \right| dt \right) (\varphi - \gamma).$$

We have

$$\int_a^x f(t) dt = F(x), \quad \frac{1}{x-a} \int_a^x (t-a) dt = \frac{x-a}{2}.$$

By setting set $t_1(x) := \frac{x+a}{2}$, we observe that

$$\int_a^x \left| t-a - \frac{1}{x-a} \int_a^x (s-a) ds \right| dt = \int_a^x |t-t_1(x)| dt = \frac{(x-a)^2}{4}.$$

Thus we get

$$(2.13) \quad \left| \int_a^x f(t)[t-a] dt - \frac{x-a}{2} F(x) \right| \leq \frac{1}{8}(\varphi - \gamma)(x-a)^2.$$

We apply the inequality (2.1) on the interval $[x, b]$ with f and $g(t) := t - b$. Then we get

$$(2.14) \quad \left| \int_x^b f(t)[t-b] dt - \frac{1}{b-x} \int_x^b [t-b] dt \cdot \int_x^b f(t) dt \right| \leq \\ \leq \frac{1}{2} \left(\int_x^b \left| t-b - \frac{1}{b-x} \int_x^b (s-b) ds \right| dt \right) (\varphi - \gamma).$$

We have

$$\int_x^b f(t) dt = 1 - F(x), \quad \frac{1}{b-x} \int_x^b (t-b) dt = \frac{x-b}{2}.$$

By setting set $t_2(x) := \frac{x+b}{2}$, we observe that

$$\int_x^b \left| t - a - \frac{1}{b-x} \int_x^b (s-b) ds \right| dt = \int_a^x |t - t_2(x)| dt = \frac{(b-x)^2}{4}.$$

Thus we get

$$(2.15) \quad \left| \int_x^b f(t)[t-b] dt + \frac{b-x}{2}[1-F(x)] \right| \leq \frac{1}{8}(\varphi - \gamma)(b-x)^2,$$

for all $x \in [a, b]$.

Summing (2.13) and (2.15) and using the triangle inequality, we deduce that

$$(2.16) \quad \left| \int_a^x (t-a)f(t) dt + \int_x^b (t-b)f(t) dt - \frac{b-a}{2}F(x) + \frac{b-x}{2} \right| \leq \frac{1}{8}(\varphi - \gamma) [(x-a)^2 + (b-x)^2],$$

however,

$$\frac{1}{2} [(x-a)^2 + (b-x)^2] = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2.$$

From (2.4), (2.16) and the above identity, we deduce the first inequality of (2.11). Now, by using the example introduced in the proof of Th. 4, one can prove easily that the constant $\frac{1}{4}$ multiplying $\varphi - \gamma$ (resp. appearing in the bracket) in the second member of (2.11) is the best possible. The second inequality of (2.11) is clear. Thus, our result is completely proved. \diamond

Remark 7. If in (2.11) we choose either $x = a$ or $x = b$ then we get

$$(2.17) \quad \left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{8}(\varphi - \gamma)(b-a)^2,$$

and we recapture (2.3).

Remark 8. If in (2.11) we choose $x = \frac{a+b}{2}$ then we get

$$(2.18) \quad \left| E(X) + \frac{b-a}{2} Pr \left(X \leq \frac{a+b}{2} \right) - \frac{a+3b}{4} \right| \leq \frac{1}{16}(\varphi - \gamma)(b-a)^2,$$

which is the best inequality that can be obtained from (2.11).

3. A related result

Let X be a random variable as in section two. We set $\mu_0 := \frac{a+b}{2}$, and define $A_{\mu_0} := \int_a^b |t - \mu_0| f(t) dt$. In [1], the following inequality was established:

$$(3.1) \quad \left| A_{\mu_0} - \frac{b-a}{4} \right| \leq \frac{1}{8\sqrt{3}}(\varphi - \gamma)(b-a)^2.$$

In this section, using Cheng–Sun inequality, we shall give an improvement to this inequality. More precisely, we have

Theorem 9. *Let X, f, γ, φ and F as above. Then we have the inequality*

$$(3.2) \quad \left| A_{\mu_0} - \frac{b-a}{4} \right| \leq \frac{1}{16}(\varphi - \gamma)(b-a)^2.$$

Proof. By easy computation, we have

$$\frac{1}{b-a} \int_a^b |t - \mu_0| dt = \frac{b-a}{4}.$$

By using Cheng–Sun inequality, we get

$$\left| A_{\mu_0} - \frac{b-a}{4} \right| \leq \frac{1}{2}(\varphi - \gamma) \int_a^b \left| |t - \mu_0| - \frac{b-a}{4} \right| dt.$$

However,

$$\begin{aligned} \int_a^b \left| |t - \mu_0| - \frac{b-a}{4} \right| dt &= \int_a^{\frac{a+b}{2}} \left| \frac{3a+b}{4} - t \right| dt + \int_{\frac{a+b}{2}}^b \left| t - \frac{a+3b}{4} \right| dt = \\ &= \int_a^{\frac{3a+b}{4}} \left(\frac{3a+b}{4} - t \right) dt + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left(t - \frac{3a+b}{4} \right) dt + \\ &+ \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left(\frac{a+3b}{4} - t \right) dt + \int_{\frac{a+3b}{4}}^b \left(t - \frac{a+3b}{4} \right) dt = K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Easy computations will show that $K_1 = K_2 = K_3 = K_4 = \frac{1}{32}(b-a)^2$. From these equalities, we get the desired inequality. \diamond

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