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ITERATIONS OVER QUATERNIONS

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Abstract: In the center of this work stands the function $f_c(q) = q^n + c$, $n \in \mathbb{N} \setminus \{1\}$ over the quaternions $\mathbb{H}$. We start with $q = 0$ and then we iterate again and again. In this way a Mandelbrot set with its main body is born. All this is done exactly as we did already over complex numbers [6]. A corresponding paper due to J. Kosi-Ulbl [4] is essentially extended. Some ideas of H. Schwarzmeier are used. Everywhere in this paper we are confronted with two difficulties. In respect of algebra: The non-commutativity of multiplication in $\mathbb{H}$. In respect of geometry: The impossibility to imagine figures in a four-dimensional space.

1. Some fundamental properties of complex numbers and quaternions

Let $\mathbb{R}, \mathbb{C} = \{x_0 + ix_1 \mid x_0, x_1 \in \mathbb{R}\}$ and $\mathbb{H} = \{x_0 + ix_1 + jx_2 + kx_3 \mid \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ be the sets of real numbers, complex numbers and quaternions, respectively. We mention, just as a warning, some well-known properties of complex numbers and quaternions, used in this article.

We can calculate in $\mathbb{C}$ and $\mathbb{H}$ in the same way as in $\mathbb{R}$, but we have to use $i^2 = -1$ in $\mathbb{C}$ and the multiplication table

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

in $\mathbb{H}$. With respect to the operations of addition and multiplication, $\mathbb{C}$ is a field and $\mathbb{H}$ is a skew field (as the multiplication is not commutative).

For the absolute values $|z| = \sqrt{x_0^2 + x_1^2}$ of $z = x_0 + ix_1 \in \mathbb{C}$ and $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$of $q = x_0+ix_1+jx_2+kx_3 \in \mathbb{H}$ the "additional inequality" holds:

$|z_1 + z_2| \leq |z_1| + |z_2|$ $(z_1, z_2 \in \mathbb{C})$ and $|q_1 + q_2| \leq |q_1| + |q_2|$ $(q_1, q_2 \in \mathbb{H})$.

As in $\mathbb{C}$, the square A^2 of the imaginary part $A = ix_0 + jx_2 + kx_3$ of a quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is a real number:

$$A^2 = -(x_1^2 + x_2^2 + x_3^2) \in \mathbb{R}.$$

In this paper a "polynomial representation" is important:

$$q^2 = (x_0 + A)^2 = x_0^2 + 2Ax_0 + A^2 = (x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_0A$$

and analogously

$$q^n = G + AF$$

where

$$G = x_0^n + \binom{n}{2} x_0^{n-2} A^2 + \binom{n}{4} x_0^{n-y} A^4 + \ldots \in \mathbb{R}$$

$$F = \binom{n}{1} x_0^{n-1} + \binom{n}{3} x_0^{n-3} A^2 + \ldots \in \mathbb{R}.$$

There is a bijective mapping of $\mathbb{C}$ into the Euclidean plane. This geometric representation of $\mathbb{C}$ is called the Gaussian plane. However, the geometric representation of $\mathbb{H}$ makes some difficulties. It is possible to use the four-dimensional Euclidean space, spanned by $1, i, j, k$.

2. The Mandelbrot set and its main body

2.1. Over $\mathbb{C}$

Let $f_c(z) = z^n + c$, $n \in \mathbb{N} \setminus \{1\}$ be a function of a complex variable z.

The *Mandelbrot set* in respect of f_c is defined as the set of all $c \in \mathbb{C}$ so that the sequence $\{f_c^{(s)}(0)\}$ does not tend to ∞ as s tends to ∞:

$$M_{\mathbb{C}} = \left\{ c \in \mathbb{C} \mid f_c^{(s)}(0) \not\to \infty \text{ if } s \to \infty \right\}.$$

We observe that $z_0 = 0$ is the only critical point of $f_c(z)$. Because of a theorem due to Fatou [2] therefore this special point is chosen as a starting point of the iteration.

A fixed point p of f_c is called attractive (stable) if there exists a neighborhood $U(p)$ such that for all $z \in U(p)$ we have $\lim_{s\to\infty} f_c^{(s)}(z) = p$. This case happens if and only if $|f_c'(p)| < 1$. We are speaking about the so-called stability criterion [2], [7].

The *main body of* $M_{\mathbb{C}}$ is the following subset of $M_{\mathbb{C}}$:

$$H_{\mathbb{C}} = \{c \in M_{\mathbb{C}} \mid f_c(z) \text{ has exactly one attractive fixed point within the } z\text{-plane}\}$$

2.2. Over $\mathbb{H}$

Let $f_c(q) = q^n + c$, $n \in \mathbb{N} \setminus \{1\}$ be a function of a quaternion variable q.

The *Mandelbrot set* in respect of f_c is defined as

$$M_{\mathbb{H}} = \{c \in \mathbb{H} \mid f_c^{(s)}(0) \not\to \infty \text{ if } s \to \infty\}.$$

In $\mathbb{H}$ a derivative of $f_c(q)$ is not defined and therefore we have no critical points. The theorem of Fatou cannot work here. In spite of these facts we chose the seed $q_0 = 0$, hoping that by iteration really all cycles are reached.

The *main body of* $M_{\mathbb{H}}$ is defined as the following subset of $M_{\mathbb{H}}$:

$$H_{\mathbb{H}} = \{c \in M_{\mathbb{H}} \mid f_c(q) \text{ has exactly one fixed point } p \text{ within the } q\text{-plane such that } |np^{n-1}| < 1\}.$$

We observe that exactly as in $\mathbb{C}$ a fixed point is called attractive (stable) if there exists a neighborhood $U(p)$ such that for all $q \in U(p)$ we have $\lim_{s\to\infty} f_c^{(s)}(q) = p$.

The stability criterion now is a little bit more difficult as over $\mathbb{C}$. We calculate at first the functional matrix (Jacobian matrix), then the characteristic equation and finally the eigenvalues λ_i. The spectral radius ϱ is defined as follows $\varrho = \max |\lambda_i|$. Then we have the criterion:

The fixed point p is attractive if and only if $\varrho < 1$.

In the case $n = 2$ we succeeded to prove $\varrho = 2|q|$. We suppose that for all $n \in \mathbb{N} \setminus \{1\}$ it follows $\varrho = n|q^{n-1}|$. That would be analogous to the criterion over $\mathbb{C}$. With this supposition we defined $H_{\mathbb{H}}$ – without using any criterion.

3. Mandelbrot set and main body over $\mathbb{C}$

3.1. Theorem. *The main body $H_\mathbb{C}$ in respect of $f_c(z) = z^n + c$ is the set of all points within an epicycloid with $(n-1)$ cusps* [5].

Proof. (a) Fixed points. If $p \in \mathbb{C}$ is a fixed point of $f_c(z)$ then we have $p^n + c = p$. There exist solutions of this equation (fundamental theorem of algebra). We write $p = \lambda w$ with $\lambda \in \mathbb{R}^+$. The exact value of λ is given later. So we obtain $c = \lambda w - \lambda^n w^n$.

(b) Attractivity. Using the stability criterion – given above – we now obtain $|n\lambda^{n-1}w^{n-1}| < 1$. We choose λ such that $n\lambda^{n-1} = 1$ or $\lambda = \frac{1}{\sqrt[n-1]{n}}$. Because of $n > 1$ this means $|w| < 1$.

(c) w-plane or (x_0, x_1)-plane. With $w = x_0 + ix_1$ we obtain in the w-plane the set of all points within the circle $x_0^2 + x_1^2 = 1$. Now we use polar coordinates $x_0 = \cos t$, $x_1 = \sin t$. Then the boundary circle has the equation $w = \cos t + i \sin t = e^{it}$. The parameter t is running from 0 to 2π.

(d) c-plane or (y_0, y_1)-plane. Now we switch to the c-plane

$$c = \lambda w - \lambda^n w^n = \lambda(x_0 + ix_1) - \lambda^n (x_0 + ix_1)^n.$$

Using polar coordinates we obtain

$$c = \lambda(\cos t + i \sin t) - \lambda^n (\cos t + i \sin t)^n = \lambda e^{it} - \lambda^n e^{itn} = y_0 + iy_1,$$

with $y_0 = \lambda \cos t - \lambda^n \cos tn$, $y_1 = \lambda \sin t - \lambda^n \sin tn$. Due to [6] this is the equation of an epicycloid with $(n-1)$ cusps. In case $n = 2$ we have a cardioid.

We delete here the proof that exactly one fixed point exists. This proof works exactly as for $\mathbb{H}$ in Section 4.1. ◊

3.2. Mandelbrot set. In the case $n = 2$ the Mandelbrot set is well known as the so-called "apple manikin".

In all cases n a lot of decorations are growing out of the main body: buds, antennas, filaments and satellites. We do not investigate all these nice and difficult things here. But we refer to the extensive literature [2] and we give some pictures for the cases $n = 2, 3, 4$ (Figures 1, 2, 3). The main body is very clear to make out.

4. Main body over $\mathbb{H}$

In the following text we delete the index $\mathbb{H}$. So we denote the Mandelbrot set and the main body over $\mathbb{H}$ simply by M and H.

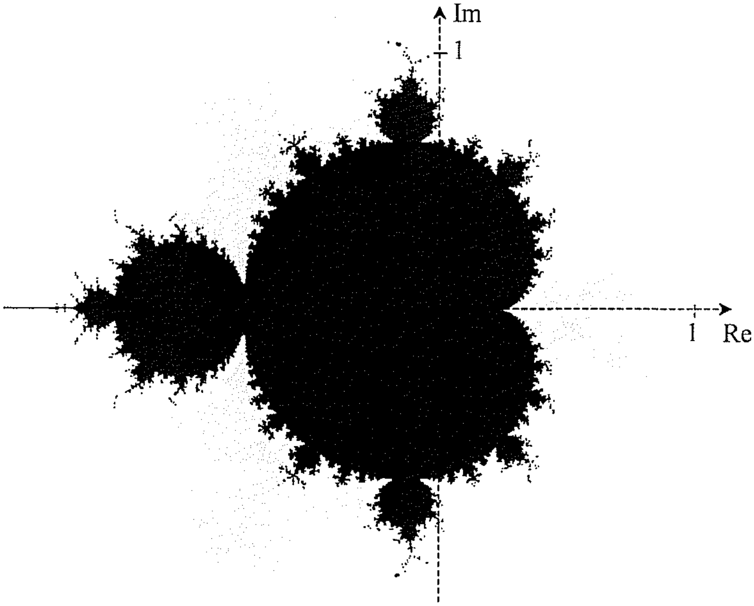


Fig. 1. The so-called "apple manikin"

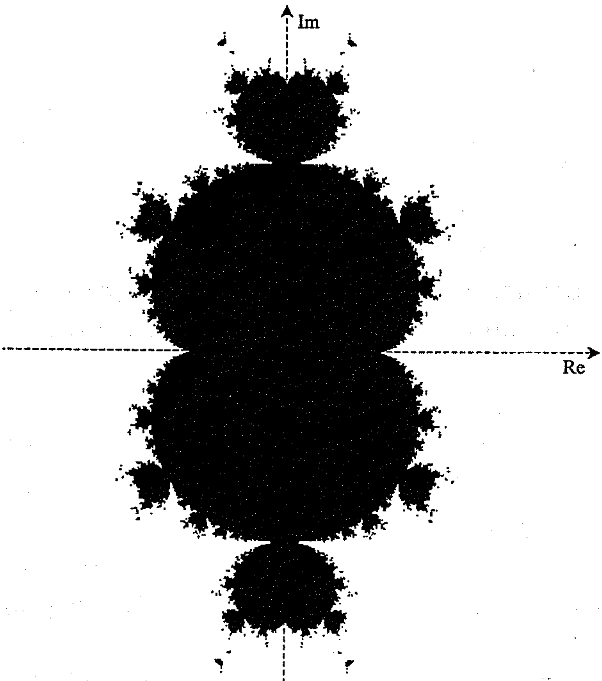


Fig. 2. Mandelbrot set in case $n = 3$

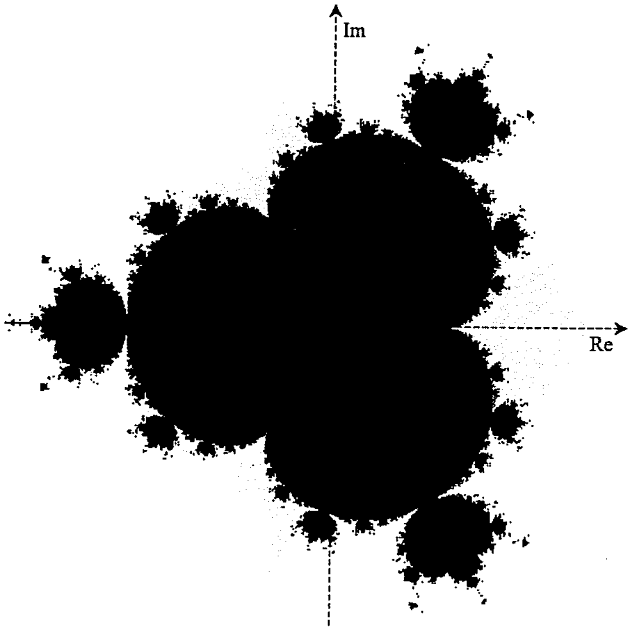


Fig. 3. Mandelbrot set in case $n = 4$

4.1. Theorem. *The main body H in respect of $f_c(q) = q^n + c$ is the set of all points inside the boundary*

$$c = \lambda(x_0 + ix_1 + jx_2 + kx_3) - \lambda^n(x_0 + ix_1 + jx_2 + kx_3) \tag{1}$$
$$\text{with } \lambda = \frac{1}{\sqrt[n-1]{n}} \in \mathbb{R}^+ \text{ and } x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

Proof. The proof is running similar as in 3.1.

(a) Fixed points. If p is a fixed point of $f_c(q)$ then we have $p^n + c = p$. The fundamental theorem of algebra does not work in $\mathbb{H}$. Nevertheless there exist solutions of this equation. We write $p = \lambda w$ with $\lambda \in \mathbb{R}^+$. So it follows $c = \lambda w - \lambda^n w^n$.

(b) Attractivity. Using the unproved criterion given above we obtain $|n\lambda^{n-1}w^{n-1}| < 1$. Now we choose $n\lambda^{n-1} = 1$, $\lambda = \frac{1}{\sqrt[n-1]{n}}$. Because of $n > 1$ it turns out that $|w| < 1$.

(c) w-space or (x_0, x_1, x_2, x_3)-space. With $w = x_0 + ix_1 + jx_2 + kx_3$ we obtain in the (x_0, x_1, x_2, x_3)-space the set of all points within the sphere $S^3 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$.

(d) c-space or (y_0, y_1, y_2, y_3)-space. Now we switch to the c-space. This yields

$$c = \lambda w - \lambda^n w^n = \lambda(x_0 + ix_1 + jx_2 + kx_3) - \lambda^n(x_0 + ix_1 + jx_2 + kx_3)^n$$
$$= y_0 + iy_1 + jy_2 + ky_3 \text{ with } y_0, y_1, y_2, y_3 \in \mathbb{R}.$$

(e) Uniqueness. In the definition of the main body (Chapter 2) the existence of exactly one fixed point with special qualities was required. We had deleted the corresponding proof for complex numbers. But now the missing proof is given (it holds for $\mathbb{H}$ as well as for $\mathbb{C}$). So a gap is closed.

We assume that we have two fixed points p, q with $p \neq q$. We write again $q = \lambda w$, $p = \lambda v$ and choose λ such that $n\lambda^{n-1} = 1$ or $\lambda^n = = \frac{\lambda}{n}$. Exactly as in 3.1 it turns out that $|w| < 1$ and $|v| < 1$. Further we obtain

$$v^n - w^n = \frac{1}{\lambda^n}(p^n - q^n) = \frac{n}{\lambda}(p^n - q^n) = \frac{n}{\lambda}(p - q) =$$
$$= \frac{n}{\lambda}(\lambda v - \lambda w) = n(v - w)$$

and finally

$$|v^n - w^n| = n|v - w|.$$

But this is impossible, because now we show that $|v^n - w^n| < n|v - w|$. (With this we have a contradiction to our assumption $p \neq q$, $v \neq w$.) We work with complete induction.

$\boxed{n = 2}$

$$|v^2 - w^2| = |v^2 - vw + vw - w^2| = |v(v - w) + (v - w)w| \leq$$
$$\leq |v(v - w)| + |(v - w)w|.$$

But $|v| < 1$, $|w| < 1$, therefore $|v^2 - w^2| < 2|v - w|$.

$\boxed{n - 1 \to n}$

The inequality shall be proved up to $n - 1$. Then it holds

$$|v^{n-1} - w^{n-1}| < (n - 1)|v - w|.$$

Now we go from $n - 1$ to n.

$$|v^n - w^n| = |v^n - vw^{n-1} + vw^{n-1} - w^n| =$$
$$= |v(v^{n-1} - w^{n-1}) + (v - w)w^{n-1}| \leq$$
$$\leq |v(v^{n-1} - w^{n-1})| + |(v - w)w^{n-1}|.$$

But $|v| < 1$, $|w| < 1$, therefore – using also the assumption of induction, we have

$$|v^n - w^n| < |v^{n-1} - w^{n-1}| + |v - w| <$$
$$< (n-1)|v - w| + |v - w| = n|v - w|. \quad \Diamond$$

4.2. Other formulations. Using qualities of the skewfield sketched in Chapter 1, Th. 4.1 may be formulated in different ways. Using (1) one can write

$$c = \lambda[x_0 + A] - \lambda^n[x_0 + A]^n = \left[\lambda x_0 - \lambda^n x_0^n - \lambda^n \binom{n}{2} x_0^{n-2} A^2 - \dots\right] +$$
$$+ A\left[\lambda - \lambda^n \binom{n}{1} x_0^{n-1} - \lambda^n \binom{n}{3} x_0^{n-3} A^2 - \dots\right] = G + AF$$

with $G, F \in \mathbb{R}$, and thus

$$(2) \qquad c = G + iFx_1 + jFx_2 + kFx_3 = y_0 + iy_1 + jy_2 + ky_3$$

with $y_0, y_1, y_2, y_3 \in \mathbb{R}$.

With this we went from the w-space to the c-space, from the (x_0, x_1, x_2, x_3)-space to the (y_0, y_1, y_2, y_3)-space.

5. Visualization

Unfortunately we do not have four-dimensional eyes. That is why we cannot see the object described with our nice equation (2). Therefore we use an old trick and investigate some sections. So we obtain shapes which we can really see and which we can grasp with our hands.

5.1. A first section. We cut the main body described by (2) with a three-dimensional (x_0, x_1, x_2)-space, with a hyperplane. Then the coefficient of k has to disappear. This is done by setting $x_3 = 0$, respectively $y_3 = 0$. In this way we find the boundary surface of our cutting object.

$$(3) \qquad \begin{aligned} c &= \lambda(x_0 + ix_1 + jx_2) - \lambda^n(x_0 + ix_1 + jx_2)^n = \\ &= G + iFx_1 + jFx_2 = y_0 + iy_1 + jy_2. \end{aligned}$$

How does this point set look like?

5.2. Polar coordinates. As we did in the c-plane we now introduce polar coordinates.

In our (x_0, x_1, x_2)-space we have the boundary surface $x_0^2 + x_1^2 + x_2^2 = 1$. This is a sphere S^2. Let P be a point of this sphere ($\overline{OP} = 1$). Then we take from Fig. 4 – with all notions used there – the polar coordinates of P:

$$x_1 = \cos\varphi \sin\psi, \quad x_2 = \sin\varphi \sin\psi, \quad x_0 = \cos\psi.$$

Substitution in (3) yields

$$\begin{aligned} c = {} & \lambda(\cos\psi + i\cos\varphi\sin\psi + j\sin\varphi\sin\psi) - \\ & - \lambda^n(\cos\psi + i\cos\varphi\sin\psi + j\sin\varphi\sin\psi)^n = \\ = {} & G + i\cos\varphi\sin\psi F + j\sin\varphi\sin\psi F = y_0 + iy_1 + jy_2. \end{aligned} \tag{4}$$

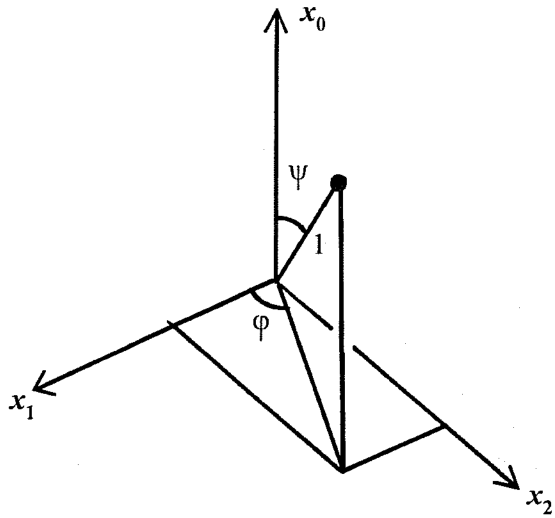


Fig. 4. Polar coordinates in the (x_0, x_1, x_2)-space

Using polar coordinates yields once more beautiful equations. But we still have no visual idea of our cutting object in c-space.

5.3. A second section. We cut our three-dimensional object once more using a plane. This is reached by setting $x_2 = y_2 = 0$ in (3) or with $\varphi = 0$ in (4). It follows

$$\begin{aligned} c &= \lambda(\cos\psi + i\sin\psi) - \lambda^n(\cos\psi + i\sin\psi)^n = \\ &= \lambda e^{i\psi} - \lambda^n e^{i\psi n} = y_0 + iy_1. \end{aligned} \tag{5}$$

This is an old friend, namely the epicycloid from 3.1.

In the case $n = 2$ we have $\lambda = \frac{1}{2}$ and with $\psi = 0$, $\psi = \pi$ two special points $\left(y_0 = \frac{1}{4}, y_1 = 0\right)$, $\left(y_0 = -\frac{3}{4}, y_1 = 0\right)$ are found. In Fig. 5 this situation is "freehand" sketched.

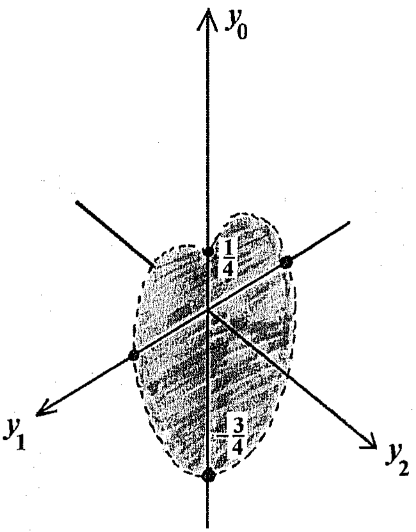


Fig. 5. Section with (y_0, y_1)-plane in case $n = 2$

In the same way with $x_1 = y_1 = 0$ respectively $\varphi = \frac{1}{2}\pi$ we obtain a second section – namely by cutting with the (y_0, y_2)-plane. This yields an epicycloid congruent to the last one. Step by step a visual idea of our three-dimensional section develops. We suppose an object arising by rotation of our epicycloid around the y_0-axis. But this assumption needs a proof.

5.4. The apple theorem. *Within the (y_0, y_1, y_2)-space the boundary of the main body consists in a surface which is created by rotation of our epicycloid (equation (5)) around the y_0-axis.*

All points within this surface are the elements of the "filled" main body. In case $n = 2$ this object looks like an apple.

Proof. Let $Q(x_0, x_1, x_2)$ be a point on the boundary sphere $x_1 = \cos\varphi \sin\psi$, $x_2 = \sin\varphi \sin\psi$. Now ψ and with this $x_0 = \cos\psi$ are constant. Switching to the (y_0, y_1, y_2)-space the point Q is (due to (4)) mapped on the point

$$P(y_0 = G, y_1 = \cos\varphi \sin\psi F, y_2 = \sin\varphi \sin\psi F).$$

Naturally P is a point of the boundary surface in (y_0, y_1, y_2)-space. Further the point P is element of the plane $\mathcal{E}$: $y_0 = G$ (Fig. 6). Then G is a function of $x_0 = \cos\psi$ and $A^2 = -x_1^2 - x_2^2 - x_3^2 = -\cos^2\varphi \sin^2\psi - \sin^2\varphi \sin^2\psi = -\sin^2\psi$, totally of $\cos\psi$. So if $x_0 = \cos\psi$ is constant, then G is constant, too.

Now we have $y_1^2 + y_2^2 = F^2 \cos^2\varphi \sin^2\psi + F^2 \sin^2\varphi \sin^2\psi = F^2 \sin^2\psi$. F is also a function of $x_0 = \cos\psi$ and $A^2 = -\sin^2\psi$.

It follows $y_1^2 + y_2^2 = R^2$, where R^2 is non-negative and constant. Therefore P is element of a circle in $\mathcal{E}$.

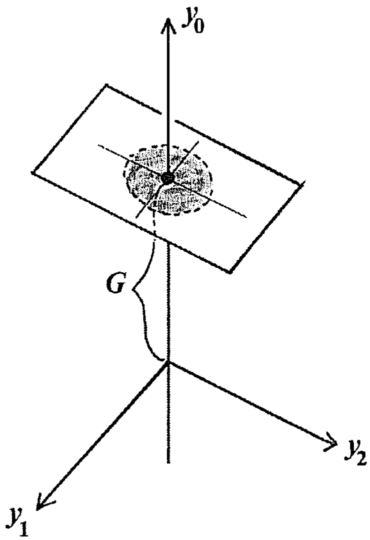


Fig. 6. Section with the plane $y_0 = G$

Every point on the boundary surface and $\mathcal{E}$ is element of such a circle. And vice versa every point of this circle is contained in the boundary set. Naturally every point of our epicycloid in the (y_0, y_1)-plane determines a plane $\mathcal{E}$. Now we see that G must be bounded such that the corresponding plane really intersects the epicycloid.

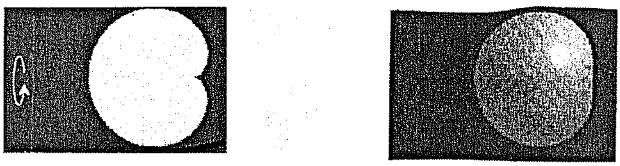
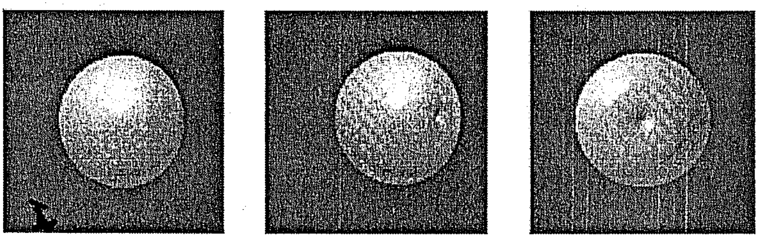


Fig. 7. The apple



Views from different directions

Summarizing we can say that every plane parallel to the (y_1, y_2)-

plane intersects the boundary of the main body in circles or in a point or the intersection is empty. With this the proof is done. ◊

Fig. 7 shows different views from the apple.

5.5. A very special example. I always preach that mathematics really must be done. Therefore now I give a detailed investigation in case $n = 2$. The results are formulated in a corollary.

Corollary. *The intersections with planes $y = G$ yield the following different results.*

$G < -\frac{3}{4}$	*the empty set*
$G = -\frac{3}{4}$	*one point*
$-\frac{3}{4} < G < \frac{1}{4}$	*one circle*
$G = \frac{1}{4}$	*one circle and one point*
$\frac{1}{4} < G < \frac{3}{8}$	*two circles*
$G = \frac{3}{8}$	*one circle*
$G > \frac{3}{8}$	*the empty set.*

Proof. What is already known?

(a) $n = 2 \Longrightarrow \lambda = \frac{1}{2}$.

(b) From (5) we take

$$G = \frac{1}{2}\cos\psi - \frac{1}{4}\cos 2\psi = \frac{1}{2}\cos\psi - \frac{1}{2}\cos^2\psi + \frac{1}{4}$$

and with $x_0 = \cos\psi$ we obtain

$$G = \frac{1}{2}x_0 - \frac{1}{2}x_0^2 + \frac{1}{4} \text{ or } x_0 = \frac{1}{2}\left(1 \pm \sqrt{3 - 8G}\right).$$

(c) $R^2 = F^2 \sin\psi = \frac{1}{4}\sin^2\psi(1 - \cos\psi)^2$. We know

$$c = \frac{1}{2}(x_0 + A) - \frac{1}{4}(x_0 + A)^2 =$$
$$= \left(\frac{1}{2}x_0 - \frac{1}{2}x_0^2 - \frac{1}{4}A^2\right) + i\frac{1}{2}x_1(1 - x_0) + j\frac{1}{2}x_2(1 - x_0).$$

With this

$$y_1 = \frac{1}{2}x_1(1 - x_0) = \frac{1}{2}\cos\varphi\sin\psi(1 - \cos\psi).$$

Comparison with (4) yields

$$y_1 = \frac{1}{2}\cos\varphi\sin\psi(1 - \cos\psi) = \cos\varphi\sin\psi F$$

and finally $F = 1 - \cos\psi$.

(d) Fig. 8 shows our cardioid in the (y_0, y_1)-plane. The distance between the tangent t and the y_1-axes is $\frac{3}{8}$. This follows from the cardioid equation.

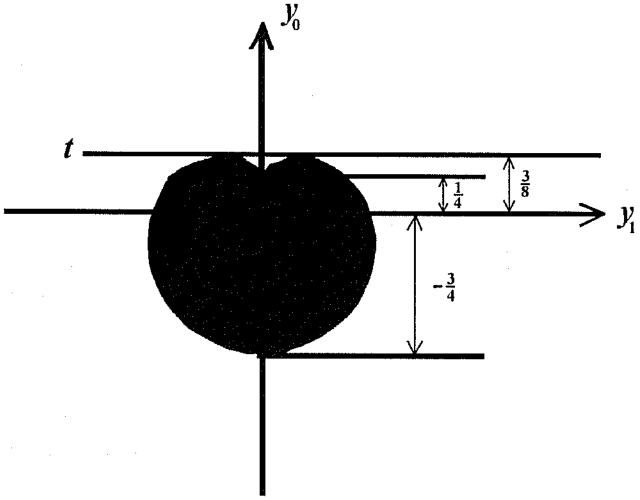


Fig. 8. The cardioid in the (y_0, y_1)-plane

Now we prove the results in different cases of the corollary.

$$G = \frac{3}{8} \qquad x_0 = \frac{1}{2} = \cos\psi \Longrightarrow R^2 = \frac{3}{64} \ldots \text{circle}$$

$$G = \frac{1}{4} \qquad \begin{cases} x_0 = \cos\psi = 1 \Longrightarrow R^2 = 0 \ldots \text{point} \\ x_0 = \cos\psi = 0 \Longrightarrow R^2 = \frac{1}{4} \ldots \text{circle} \end{cases}$$

$$G = -\frac{3}{4} \begin{cases} x_0 = \cos\psi = 2 & \text{impossible} \\ x_0 = \cos\psi = -1 \Longrightarrow R^2 = 0 \ldots \text{point.} \end{cases}$$

All these special limit cases are excluded in the following investigations.

From (b) follows immediately a necessary condition for the existence of intersections, namely $G < \frac{3}{8}$. This means that $G > \frac{3}{8}$ yields the empty set.

Now we choose in (b) the positive sign and require $\cos\psi < 1$. Then it follows $G > \frac{1}{4}$. If $\frac{1}{4} < G < \frac{3}{8}$ then $0 < \cos\psi < 1$. Therefore we have a circle in this case.

Let the sign in (b) be negative then $x_0 = \cos\psi = \frac{1}{2}(1-\sqrt{3-8G})$.
First case. $1 - \sqrt{3-8G} > 0 \Longrightarrow G > \frac{1}{4}$.
Therefore in case $\frac{1}{4} < G < \frac{3}{8}$ occurs a second circle.
Second case. $\cos\psi = 1 - \sqrt{3-8G} < 0 \Longrightarrow G < \frac{1}{4}$.
Further it turns out

$$|\cos\psi| = \frac{1}{2}\left(\sqrt{3-8G} - 1\right) < 1 \Longrightarrow G > -\frac{3}{4}.$$

In the area $-\frac{3}{4} < G < \frac{1}{4}$ we have a circle. It remains only $G < -\frac{3}{4}$. Here we have $|\cos\psi| = \frac{1}{2}\left(\sqrt{3-8G}-1\right) > 1$. Impossible – this means again the empty set. ◊

5.6. Some more intersections. In this chapter the main body H was cut with the (y_0, y_1, y_2)-space. The boundary of this section was a rotation surface. The same procedure can be performed with the (y_0, y_1, y_3)-space and with the (y_0, y_2, y_3)-space, too. By rotation of epicycloids around the y_0-axis we obtain further boundary surfaces, congruent to one another.

There are quite other possibilities for cutting our main body H within the (y_0, y_1, y_2)-space. For instance by planes $y_2 = B$ parallel to the (y_0, y_1)-plane.

All these possibilities sound indeed very nice. But how the total main body H really looks like? We do not see, we can never see.

6. Mandelbrot set over $\mathbb{H}$

6.1. Lemma. *Let $f_1(q) = q^n + c_1 = q^n + a + A$, $f_2(q) = q^n + c_2 = = q^n + b + B$ be two functions. We perform iterations in $\mathbb{H}$ with seed $q_0 = 0$. If the real parts of c_1 and c_2 are equal and the absolute values of the imaginary parts coincide then the absolute values of every iteration with number s are equal.*

In short: $a = b$ and $|A| = |B|$ then $|f_1^{(s)}(0)| = |f_2^{(s)}(0)|$.

Proof. $f_1(0) = c_1 = a + A$, $f_1^{(2)}(0) = c_1^n + c_1 = (a+A)^n + a + A$. We use the polynomial presentation as in 4.2. With this we obtain

$$f_1^{(2)}(0) = \left[a + a^n + \binom{n}{2}a^{n-2}A^2 + \binom{n}{4}a^{n-4}A^4 + \dots\right] +$$

$$+A\left[1+\binom{n}{1}a^{n-1}+\binom{n}{3}a^{n-3}A^2+\ldots\right]=$$
$$=G^{(2)}+AF^{(2)} \text{ with } G^{(2)},F^{(2)}\in\mathbb{R}.$$

$G^{(2)}$ and $F^{(2)}$ are functions of a and A^2. Therefore we write $G^{(2)}(a,A^2)$, $F^{(2)}(a,A^2)$.

Using the principle of induction we show that for each generation $s\in\mathbb{N}$ a representation of this form exists.

We know that this is true for $s=2$. The assumption shall be proved up to $s-1$. Then it holds $G^{(s-1)}(a,A^2), F^{(s-1)}(a,A^2)$.

Now we go from $s-1$ to s.

$$f_1^{(s)}(0)=\left(G^{(s-1)}+AF^{(s-1)}\right)^n+a+A=$$
$$=\left[a+\left(G^{(s-1)}\right)^n+\binom{n}{2}\left(G^{(s-1)}\right)^{n-2}A^2(F^{(s-1)})^2+\ldots\right]+$$
$$+A\left[1+\binom{n}{1}(G^{(s-1)})^{n-1}F^{(s-1)}+\binom{n}{3}(G^{(s-1)})^{n-3}A^2(F^{(s-1)})^3+\ldots\right]=$$
$$=G^{(s)}+AF^{(s)}.$$

It is immediately seen that $G^{(s)}$ and $F^{(s)}$ are again functions of a and A^2. So we can write $G^{(s)}(a,A^2)$, $F^{(s)}(a,A^2)$.

Now we consider the absolute values

$$|f_1^{(s)}(0)|=\sqrt{(G^{(s)})^2-|A|^2(F^{(s)})^2}.$$

The investigation of $f_2(q)$ is running in exactly the same way. We have only to substitute a by b and A^2 by B^2. In this way we obtain

$$|f_2^{(s)}(0)|=\sqrt{(G^{(s)})^2-|B|^2(F^{(s)})^2}.$$

$G^{(s)}$ and $F^{(s)}$ now are functions of b and B^2. We write $G^{(s)}(b,B^2)$, $F^{(s)}(b,B^2)$. Because of the assumptions $a=b$ and $|A|=|B|$ it follows $|f_2^{(s)}(0)|=|f_1^{(s)}(0)|$ and our lemma is proved. ◊

(The spelling technique is a little bit confusing. It may be that there arise difficulties in understanding.)

6.2. The extended apple theorem. *Let the Mandelbrot set M intersect the (y_0,y_1,y_2)-space. The boundary of this section consists in a surface which is created by rotation of the corresponding boundary line in the (y_0,y_1)-plane around the y_0-axis.*

Proof. Due to 5.3 we know already the section of our main body H with the (y_0, y_1)-plane. This was exactly the "filled" epicycloid from 3.1. This point set was found by iteration with

$$f_1(q) = q^n + c_1 = q^n + a + A = q^n + a + ia_1.$$

Using the same function we obtain the Mandelbrot set in the (y_0, y_1)-plane following Def. 2.1. In this way the main body is extended to the Mandelbrot set. So Figures 1, 2, 3 were produced.

Now we perform a rotation (as in 5.4) of the (y_0, y_1)-plane around the y_0-axis to the plane γ. Doing so the element $c_1 = a + ia_1$ is mapped into the point $c_2 = b + ib_1 + jb_2 + kb_3$ in γ. From Fig. 9 we take out $a = b$, $b_3 = 0$ and $b_1^2 + b_2^2 = a_1^2$.

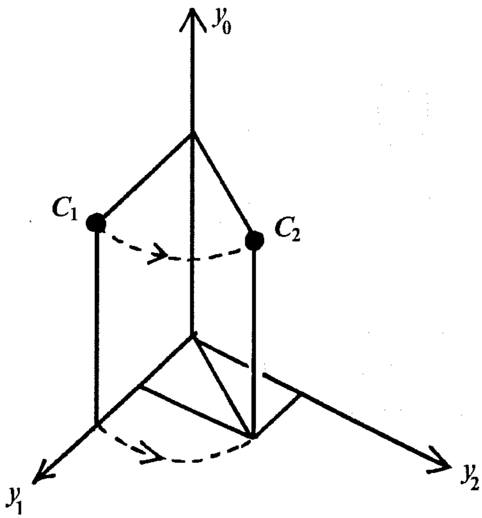


Fig. 9. Rotation around the y_0-axis

Two functions are considered:
$f_1(q) = q^n+a+A = q^n+a+ia_1$ and $f_2(q) = q^n+b+B = q^n+b+ib_1+jb_2$. The first working over the (y_0, y_1)-plane and the second over γ. With $a = b$, $|A| = \sqrt{a_1^2} = \sqrt{b_1^2 + b_2^2} = |B|$ the conditions of our lemma are fulfilled. Because of $|f_1^{(s)}(0)| = |f_2^{(s)}(0)|$ the points are running in both planes exactly in the same way – performing iterations. If the points do not "escape" in one case, then they do not also in the other. We obtain in both planes point sets congruent to one another.

The filled rotation surface is the intersection of M with the (y_0, y_1, y_2)-space. Figures 10, 11, 12 show in a very impressive way such rotation solids for $n = 2, 3, 4$.

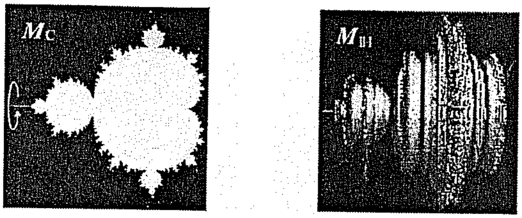
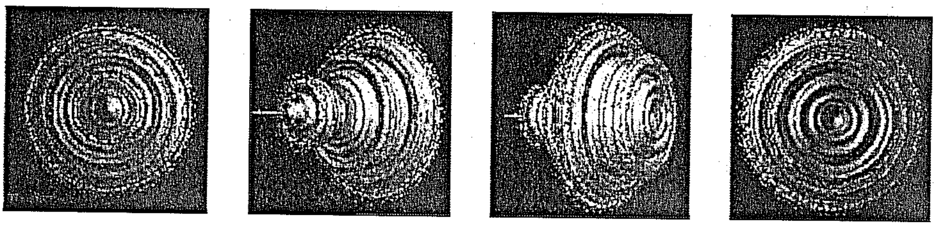
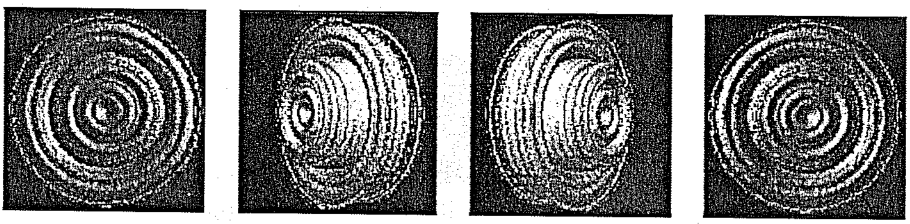


Fig. 10. The extended apple theorem in case $n = 2$



Views from different directions

Fig. 11. The extended apple theorem in case $n = 3$



Views from different directions

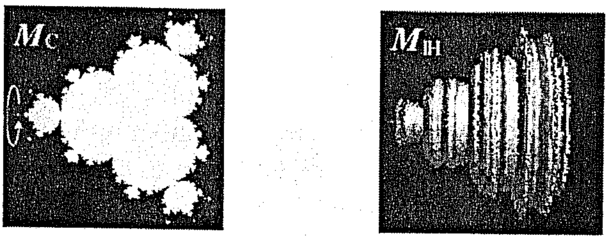
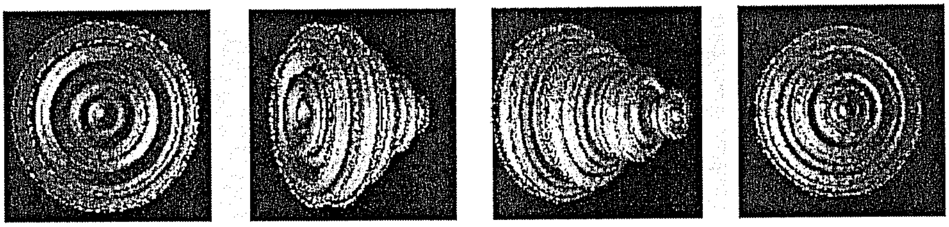


Fig. 12. The extended apple theorem in case $n = 4$



Views from different directions

All our investigation can also be done in respect of the (y_0, y_1, y_3) or the (y_0, y_2, y_3)-space. In any of the three cases we obtain solids congruent to one another. ◊

Exactly as with H the true shape of M remains hidden within the darkness of the fourth dimension.

7. Conclusion

A lot of interesting questions concerning iterations in quaternions remain unanswered. We quote some of them.

7.1. In Chapter 2 two problems were already mentioned.

Is there any motivation to choose the starting point $q_0 = 0$?

What happens if the seed is not zero?

Is the stability criterion using the spectral radius ϱ equivalent to our assumption $|np^{n-1}| < 1$ for all functions $f_c(q) = q^n + c$ with $n \in \mathbb{N} \setminus \{1\}$?

7.2. Is it possible to obtain more visual clarity about M and H using totally other cross-sections? In [1] some interesting pictures (with Cassini curves) are given.

7.3. Not every quadratic function over $\mathbb{H}$ can be reduced to the case $f_c(q) = q^2 + c$ (in contrast to $\mathbb{C}$). Is it possible to classify these functions in respect of iterations?

7.4. Investigate other functions over $\mathbb{H}$ performing iteration – for instance rational or transcendental ones!

7.5. The Mandelbrot set over $\mathbb{C}$ may also be defined as the set of all points whose corresponding Julia sets are connected. This definition is equivalent to the definition given in Chapter 2. Is it possible to define the Mandelbrot set over $\mathbb{H}$ in a similar way?

Summarizing we can say that we have much more questions than satisfying answers. And the more mathematicians are doing research in this field, the more questions arise.

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