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ON EXPANDING ITERATION SEMIGROUPS OF SET-VALUED FUNCTIONS

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Abstract: We present some conditions under which a certain family of set-valued functions is an expanding iteration semigroup, that is, in fact, it is an iteration semigroup.

Introduction

It is well-known (cf. [2; Chap. IX, Sect. 1], [8; Ths. 5.1–8.1], [7; p. 98–99], [3; Chap. I, Sect. 1.7], also [1; Th. 1]) that the formula

$$f^t(x) = \alpha^{-1}\big(\alpha(x) + t\big)$$

yields the fundamental form of iteration semigroup which means that the translation equation

$$f^{s+t}(x) = f^t\big(f^s(x)\big)$$

is satisfied. In the present paper we consider a set-valued counterpart (A) of this formula (see Sect. 1) and continue our study made in [4]. The main aim is to give sufficient conditions for a function $F : (0, \infty) \times \times X \to 2^X$ of form (A) to be the so called expanding iteration semigroup,

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i.e.

$$F^t\big(F^s(x)\big) \subset F^{s+t}(x) \tag{E}$$

for every $x \in X$ and $s, t \in (0, \infty)$ (instead of $F(t, x)$ we write $F^t(x)$). In [5] we proved the following result showing that for multifunctions of form (A) condition (E) forces much more.

Theorem 1 ([5; Th.]). *Let $F : (0, \infty) \times X \to 2^X$ be given by* (A). *If F is an expanding iteration semigroup then F is an iteration semigroup:*

$$F^t\big(F^s(x)\big) = F^{s+t}(x)$$

for every $x \in X$ and $s, t \in (0, \infty)$.

Notice also that in [4] functions of form (A) satisfying the inclusion opposite to (E), i.e. collapsing iteration semigroups, were studied.

1. Preliminaries

Fix a set X and a set-valued function $A : X \to 2^{\mathbb{R}}$ with non-empty values. Put

$$S := A(X) \quad \text{and} \quad q := \sup S.$$

Throughout this paper we will always assume that

(**H**) for every $s, t \in (0, \infty)$ and $x, z \in X$ such that $\big[A(x)+s+t\big] \cap \cap A(z) \neq \emptyset$ there exists a $y \in X$ satisfying the conditions

$$[A(x) + s] \cap A(y) \neq \emptyset \tag{1}$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset. \tag{2}$$

It is known (see [4; Prop. 1 and Cor. 1]) that:

– if S is an interval then (H) holds;

– if all values of A are open and $(\inf S, \sup S) \subset \operatorname{cl} S$ then (H) holds;

– if all values of A are intervals and (H) holds then $(\inf S, \sup S) \subset \subset \operatorname{cl} S$;

– for a single-valued A condition (H) holds if and only if S is an interval;

– if all values of A are open intervals then (H) is equivalent to $(\inf S, \sup S) \subset \operatorname{cl} S$.

For every $x \in X$ define

$$\tau(x) := \sup\big\{t \in [0,\infty)\colon\ [A(x)+t] \cap S \neq \emptyset\big\}.$$

Fact 1 (see [4; Th. 1]). *Let $x \in X$. If $t < \tau(x)$ then $[A(x)+t] \cap S \neq \emptyset$ and if $t > \tau(x)$ then $[A(x)+t] \cap S = \emptyset$ for every $t \in (0,\infty)$.*

Fact 2 (see [4; Lemma 1 and Cor. 2]). *For every $x \in X$ we have*

$$\tau(x) = q - \inf A(x) \quad \text{and} \quad \big[A(x)+\tau(x)\big] \cap S \subset \{q\}.$$

Let $e : (0,\infty) \times X \to [0,\infty)$ be defined by

$$e(t,x) := \sup\big\{s \in [0,t]\colon\ [A(x)+s] \cap S \neq \emptyset\big\}.$$

Fact 3 (see [4; Lemma 2]). *For every $t \in (0,\infty)$ and $x \in X$*

$$e(t,x) = \min\{t, \tau(x)\}.$$

Now put

$$F^t(x) := A^{-1}\big(A(x) + e(t,x)\big), \tag{A}$$

where

$$A^{-1}(V) := \{x \in X\colon\ A(x) \cap V \neq \emptyset\}$$

for every $V \subset \mathbb{R}$.

Fact 4 (see [4; Lemma 3]). *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A) *and let $t \in (0,\infty)$ and $x \in X$. If $t < \tau(x)$ then*

$$F^t(x) = A^{-1}\big(A(x)+t\big) \neq \emptyset$$

and if $t \geqslant \tau(x)$ then

$$F^t(x) = \begin{cases} A^{-1}(\{q\}), & \text{if } q \in S \text{ and } \inf A(x) \in A(x); \\ \emptyset & \text{otherwise.} \end{cases}$$

2. Expanding iteration semigroups

Consider the following condition.

(**H1**) for every $x, z \in X$ and $s, t \in (0,\infty)$ with $s + t \leqslant \tau(x)$ if (1) and (2) hold for a $y \in X$ then

$$[A(x)+s+t] \cap [A(z)] \neq \emptyset. \tag{3}$$

Remark 1. If A is single-valued then (H1) holds.

Proof. It is enough to observe that if $x, y, z \in X$ and $s, t \in (0,\infty)$ satisfy $s + t \leqslant \tau(x)$ and conditions (1), (2) then

$$A(x) + s = A(y) = A(z) - t. \ \Diamond$$

Proposition 1. *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *If F is an expanding iteration semigroup then* (H1) *holds.*

Proof. Fix $x, z \in X$ and $s, t \in (0, \infty)$ with

$$s + t \leqslant \tau(x) \tag{4}$$

and let $y \in X$ satisfy (1) and (2). By (4) we have $s < \tau(x)$, whence, according to (1) and Fact 4,

$$y \in A^{-1}\big(A(x) + s\big) = F^s(x).$$

Similarly, by (2) and Fact 1, we have $t \leqslant \tau(y)$ and, so, using (A) and Fact 3, we get

$$z \in A^{-1}\big(A(y) + t\big) = F^t(y).$$

Thus $z \in F^t\big(F^s(x)\big)$, whence $z \in F^{s+t}(x)$ and, by (4) and Fact 3, we come to (3) which completes the proof. ◊

Ths. 2–4 below provide conditions which together with (H1) are sufficient for F of form (A) to be an expanding iteration semigroup, i.e., according to Th. 1, an iteration semigroup.

Theorem 2. *Assume* (H1) *and let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *If*

$$\tau(x) = \infty \quad \textit{for} \quad x \in X$$

then F *is an iteration semigroup.*

Proof. Fix $x \in X$ and $s, t \in (0, \infty)$. Let $z \in F^t\big(F^s(x)\big)$ and choose a $y \in F^s(x)$ such that $z \in F^t(y)$. By Fact 4,

$$y \in A^{-1}\big(A(x) + s\big) \quad \text{and} \quad z \in A^{-1}\big(A(y) + t\big),$$

that is (1) and (2) hold. Therefore, by (H1), we get (3), whence, according to Fact 4,

$$z \in F^{s+t}(x).$$

Thus F is an expanding iteration semigroup which, by Th. 1, completes the proof. ◊

Th. 2 and Fact 2 imply the following corollary.

Corollary 1. *Assume* (H1) *and let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *If either*

(i) $q = \infty$,

or

(ii) $\inf A(x) = -\infty$ *for every* $x \in X$

then F *is an iteration semigroup.*

Theorem 3. *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *If* A *is single-valued then* F *is an iteration semigroup.*

In the proof we will need the following auxiliary fact.

Lemma 1. *Assume* (H1) *and let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *If* $x \in X$ *and* card $A(x) = 1$ *then* (E) *holds for every* $s, t \in (0, \infty)$ *such that* $s + t \leqslant \tau(x)$.

Proof. Fix $s, t \in (0, \infty)$ with $s + t \leqslant \tau(x)$ and a $z \in F^t(F^s(x))$. Let $y \in F^s(x)$ be such that $z \in F^t(y)$. We will show that

$$t \leqslant \tau(y). \tag{5}$$

Since $s < \tau(x)$, by Fact 4 we have

$$F^s(x) = A^{-1}(A(x) + s)$$

whence we get (1) and, consequently,

$$A(x) + s \subset A(y). \tag{6}$$

If $s + t < \tau(x)$ then, according to Fact 1,

$$[A(x) + s + t] \cap S \neq \emptyset.$$

Thus, by (6), we get $[A(y) + t] \cap S \neq \emptyset$ and, on account of Fact 1, (5) follows. Now assume that $s + t = \tau(x)$. Then $\tau(x) < \infty$. Suppose that $t > \tau(y)$. Therefore, by Fact 4, we have $q \in S$ and $\inf A(x) \in A(x)$. Observe that, according to Fact 2, $q \in A(x) + \tau(x)$. Then

$$A(x) + s + t = A(x) + \tau(x) = \{q\}$$

whence

$$[A(x) + s + t] \cap S \neq \emptyset$$

and, by (6),

$$[A(y) + t] \cap S \neq \emptyset$$

which contradicts Fact 1 and completes the proof of inequality (5) in this case, too.

Using (5), Fact 3 and (A) we get

$$F^t(y) = A^{-1}(A(y) + t)$$

whence (2) follows. Therefore, by (1) and (H1), we obtain (3), i.e., due to the inequality $s + t \leqslant \tau(x)$,

$$z \in A^{-1}(A(x) + s + t) = F^{s+t}(x)$$

which completes the proof. ◊

Proof of Theorem 3. On account of Remark 1 condition (H1) is satisfied. Fix $x \in X$ and $s, t \in (0, \infty)$. By Th. 1 and Lemma 1 it is suffices to show (E) assuming $s + t > \tau(x)$. Take an arbitrary $z \in$ $\in F^t(F^s(x))$ and choose a $y \in F^s(x)$ such that $z \in F^t(y)$. We will prove that

$$t > \tau(y). \tag{7}$$

If $s < \tau(x)$ then, by Fact 4,

$$y \in F^s(x) = A^{-1}\big(A(x) + s\big)$$

whence $A(y) = A(x) + s$ and, by Fact 2,

$\tau(y) = q - \inf A(y) = q - \inf\big(A(x) + s)\big) = q - \inf A(x) - s = \tau(x) - s,$

which, by inequality $s+t > \tau(x)$, gives (7). If $s \geqslant \tau(x)$ then, by Fact 4,

$$y \in F^s(x) = A^{-1}(\{q\}),$$

that is $A(y) = \{q\}$ whence, by Fact 2,

$$\tau(y) = q - \inf A(y) = 0$$

which again yields (7).

Condition (7) and Fact 4 give

$$z \in F^t(y) = A^{-1}(\{q\}).$$

On the other hand, by Fact 4, we have

$$F^{s+t}(x) = A^{-1}(\{q\}).$$

Thus $z \in F^{s+t}(x)$ which completes the proof. ◊

Th. 3 shows that a set-valued settings of families having form (A) is of interest even when A is single-valued. In the classical case (cf. [2; Chap. IX, Sect. 1], [8; Ths. 5.1–8.1], [7; p. 98–99], [3; Chap. I, Sect. 1.7] (cf. also [1; Th. 1]) the generator A is always assumed to be invertible to ensure that F is single-valued. Now accepting multifunctions we may resign that very restrictive condition.

The next theorem as well as Cor. 2 deal with a mixed situation where each value of A is either a singleton, or is unbounded from below.

Theorem 4. *Assume that* (H1) *holds,* $q \notin S$ *and*

$$\operatorname{card} A(x) = 1 \quad \text{or} \quad \inf A(x) = -\infty \qquad \text{for} \quad x \in X. \tag{8}$$

Assume also that for every $x, y \in X$ *if*

$$\operatorname{card} A(x) = 1 \quad \text{and} \quad \inf A(y) = -\infty$$

then

$$\sup A(y) \leqslant u \qquad \text{where} \quad \{u\} = A(x).$$

Then (A) *defines an iteration semigroup* $F : (0, \infty) \times X \to 2^X$.

Proof. By Cor. 1(i), we can assume that $q \neq \infty$. According to Th. 1 it is sufficient to show that F is an expanding iteration semigroup. Fix $x \in X$, $s, t \in (0, \infty)$ and at first assume that $\operatorname{card} A(x) = 1$. Due to Lemma 1 we can assume that

$$(9) \qquad s+t > \tau(x).$$

Therefore, by Fact 4, $F^{s+t}(x) = \emptyset$. If $s \geqslant \tau(x)$ then, again by Fact 4, $F^s(x) = \emptyset$ and (E) follows. Now consider the case $s < \tau(x)$. Then, by Fact 4,

$$F^s(x) = A^{-1}\big(A(x)+s\big) \neq \emptyset.$$

Taking any $y \in F^s(x)$ we have (1), i.e. $A(x)+s \subset A(y)$ whence, by our assumption,

$$\operatorname{card} A(y) = 1 \quad \text{and} \quad A(y) = A(x)+s$$

and, according to Fact 2, $\tau(y) = \tau(x) - s$. Therefore, by (9), we get $t > \tau(y)$ whence, due to Fact 4, we infer that $F^t(y) = \emptyset$. Consequently, $F^t\big(F^s(x)\big) = \emptyset$ and (E) follows again.

Now consider the case $\inf A(x) = -\infty$. Hence $\tau(x) = \infty$ and $s + t < \tau(x)$. Let $z \in F^t\big(F^s(x)\big)$. Then there exists a $y \in F^s(x)$ such that $z \in F^t(y)$. Since $q \notin S$ it follows from Fact 4 that

$$z \in F^t(y) = A^{-1}\big(A(y)+t\big) \quad \text{and} \quad t < \tau(y).$$

In particular, (2) is satisfied. Since $s < \tau(x)$ also (1) holds. Thus, according to (H1), we obtain (3) and, consequently,

$$z \in A^{-1}\big(A(x)+s+t\big) = F^{s+t}(x)$$

which completes the proof. ◊

Proposition 2. *Let* $F : (0,\infty)\times X \to 2^X$ *be given by* (A). *Assume that* F *is an iteration semigroup,* $q \notin S$ *and* $q \neq \infty$. *If* $x, y \in X$ *and* $\inf A(y) = -\infty < \inf A(x)$ *then* $\sup A(y) \leqslant \inf A(x)$.

Proof. Suppose that $\sup A(y) > \inf A(x)$. Then (1) holds with an $s \in (0,\infty)$, and, according to the assumptions on q and Facts 1 and 2, we get $s < \tau(x) < \infty$. Hence, by Fact 4, $y \in F^s(x)$ and we can also find a $t \in (0,\infty)$ with $s+t > \tau(x)$. On the other hand $\tau(y) = \infty$, which gives $t < \tau(y)$. Therefore, on account of Fact 4,

$$F^{s+t}(x) = \emptyset \neq F^t(y) \subset F^t\big(F^s(x)\big)$$

which contradicts the assumption that F is an iteration semigroup. ◊

The next result follows immediately from Th. 4 and Prop. 2. It turns out that if $q \neq \infty$ and (8) holds then the condition provided by Th. 4, sufficient for F of form (A) to be an iteration semigroup, is, in fact, also necessary.

Corollary 2. *Let* $F : (0,\infty)\times X \to 2^X$ *be given by* (A). *Assume that* (H1) *holds,* $q \notin S$, $q \neq \infty$ *and* (8) *is fulfilled. Then the following conditions are equivalent:*

(i) *F is an iteration semigroup;*

(ii) *for every* $x, y \in X$ *if*

$$\operatorname{card} A(x) = 1 \quad \textit{and} \quad \inf A(y) = -\infty$$

then

$$\sup A(y) \leqslant u \qquad \text{where} \quad \{u\} = A(x).$$

The rest of the paper contains a discussion of the condition (H2) below which is a strength of (H1) and formally arises from it by deleting the requirement that $s + t \leqslant \tau(x)$ occurring there.

(**H2**) For every $x, z \in X$ and $s, t \in (0, \infty)$ if (1) and (2) hold for a $y \in X$, then (3) is fulfilled.

Remark 2. (H2) implies (H1). If $\tau(x) = \infty$ for every $x \in X$ then the conditions (H2) and (H1) are equivalent.

Remark 3. Assume (H2). If $x \in X$ and $\tau(x) < \infty$ then $\operatorname{card} A(x) = 1$.

Proof. In view of Fact 2 $\inf A(x) > -\infty$ and $q < \infty$ whence $\sup A(x) < \infty$. Suppose that $\operatorname{card} A(x) > 1$. Let $u, w \in A(x)$ and assume that $u < w$. Then we can find an $s \in (0, \infty)$ with

$$w \in A(x) + s \quad \text{and} \quad \frac{u + w}{2} < \inf\, [A(x) + s].$$

Similarly there exists a $t \in (0, \infty)$ such that

$$u \in A(x) - t \quad \text{and} \quad \sup\, [A(x) - t] < \frac{u + w}{2}.$$

Then, of course,

$$[A(x) + s] \cap A(x) \neq \emptyset \quad \text{and} \quad [A(x) - t] \cap A(x) \neq \emptyset$$

but from the inequality $\sup\, [A(x) - t] < \inf\, [A(x) + s]$ we have

$$[A(x) + s] \cap [A(x) - t] = \emptyset$$

which contradicts (H2). $\Diamond$

Example. Let $A : X \to 2^{\mathbb{R}}$ be given by $A(x) = [0, 1]$. Then $\tau \equiv 1$ and for every $x, y \in X$ and $s \in (0, \infty)$ condition (1) holds if and only if $s \leqslant 1$. Therefore A satisfies (H) and (H1) but, by Remark 3, not (H2).

For every $x, y \in X$ put

$$G(x, y) := \big\{s \in (0, \infty) \colon \ (1) \text{ holds}\big\}.$$

The following remark can be proved by a standard calculation and making use of Th. 1 whereas Remark 5 is obvious.

Remark 4. Let $F : (0, \infty) \times X \to 2^X$ be given by (A). Then F is an iteration semigroup if and only if

$$e(s+t,x) \in G(x,z)$$

for every $x, z \in X$ and $s, t \in (0, \infty)$ satisfying

$$(10) \qquad e(s,x) \in G(x,y) \quad \text{and} \quad e(t,y) \in G(y,z)$$

with a $y \in X$.

Remark 5. (H2) holds if and only if

$$s+t \in G(x,z)$$

for every $x, z \in X$ and $s, t \in (0, \infty)$ satisfying

$$(11) \qquad s \in G(x,y) \quad \text{and} \quad t \in G(y,z)$$

with a $y \in X$.

Theorem 5. *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A) *and assume that*

$$(12) \qquad \tau(x) \notin G(x,y) \quad \textit{for } x, y \in X.$$

Then F *is an iteration semigroup if and only if* (H2) *holds.*

Proof. Assume that F is an iteration semigroup. Take $x, z \in X$, $s, t \in$ $\in (0, \infty)$ and $y \in X$ such that conditions (11) are satisfied. Therefore (1) and (2) hold whence, by Fact 1 and condition (12), we have $s < \tau(x)$ and $t < \tau(y)$. Then, by Fact 3,

$$(13) \qquad e(s,x) = s \in G(x,y) \quad \text{and} \quad e(t,y) = t \in G(y,z).$$

Notice that if $s + t \geqslant \tau(x)$ then, by Fact 3, the assumption on F and Remark 4, we would have

$$\tau(x) = e(s+t,x) \in G(x,z),$$

which contradicts (12). Thus $s + t < \tau(x)$. Hence, by Fact 3 and Remark 4,

$$s + t = e(s+t,x) \in G(x,z),$$

which, by Remark 5, gives (H2).

Now assume that (H2) holds. Fix $x, z \in X$, $s, t \in (0, \infty)$ and $y \in X$ satisfying (10). We will prove that $e(s+t,x) \in G(x,z)$, which, by Remark 4, will complete the proof. By (10), (12) and Fact 3

$$e(s,x) = s < \tau(x) \quad \text{and} \quad e(t,y) = t < \tau(y).$$

Thus, according to Remark 5, $s + t \in G(x,z)$. Hence, by the definition of $G(x,z)$ and Fact 1, $s + t \leqslant \tau(x)$. Thus, on account of Fact 3,

$$e(s+t,x) = s+t \in G(x,z). \ \diamondsuit$$

Before formulating the final result we will prove the following simple fact.

Lemma 2. *Let $x, y \in X$. Then*

$$\tau(x) \in G(x,y)$$

if and only if

$$q \in A(y) \quad \text{and} \quad \inf A(x) \in A(x).$$

Proof. If $\tau(x) \in G(x,y)$ then $\big[A(x) + \tau(x)\big] \cap A(y) \neq \emptyset$ whence, by Fact 2,

$$q \in A(y) \quad \text{and} \quad q \in A(x) + \tau(x) = A(x) + q - \inf A(x),$$

that is $\inf A(x) \in A(x)$.

Conversely, due to Fact 2,

$$q \in A(x) + q - \inf A(x) = A(x) + \tau(x)$$

whence $q \in \big[A(x) + \tau(x)\big] \cap A(y)$ which means that $\tau(x) \in G(x,y)$. ◊

Theorem 6. *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A)*. If any of the following conditions holds:*

(i) $q \notin S$,

(ii) $\inf A(x) \notin A(x)$ *for every* $x \in X$

then F is an iteration semigroup if and only if (H2) *holds.*

Proof. It is enough to observe that, by Lemma 2, each of the above conditions (i), (ii) implies (12), and to apply Th. 5. ◊

References

[1] GUZIK, G., JARCZYK, W. and MATKOWSKI, J.: Cocycles of continuous itreation semigroups, *Bull. Polish Acad. Sci. Math.* **51** (2003), 187–197.

[2] KUCZMA, M.: Functional equations in a single variable, Monografie Mat. **46**, PWN, Warszawa, 1968.

[3] KUCZMA, M., CHOCZEWSKI, B. and GER, R.: Iterative functional equations, Encyclopedia of Math. and Its Applications vol. 32, Cambridge University Press, 1990.

[4] ŁYDZIŃSKA, G.: On collapsing iteration semigroups of set-valued functions, *Publ. Math. Debrecen* (in print).

[5] ŁYDZIŃSKA, G.: Iteration semigroups for which expansion implies collapse (submitted).

[6] SMAJDOR, A.: Iterations of multi-valued functions, Prace Naukowe Uniwersytetu Śląskiego w Katowicach **759** (1985).

[7] TARGONSKI, Gy.: Topics in iteration theory, Vandenhoeck and Ruprecht, Göttingen, 1981.

[8] ZDUN, M. C.: Continuous and differentiable iteration semigroups, *Prace Naukowe Uniwersytetu Ślaskiego w Katowicach* **308** (1979).