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EXPECTED UTILITY WITH PSEUDOTRANSITIVE PREFERENCES

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Abstract: Given a separable metric space Y, and a σ-algebra $\mathcal{B}(Y)$ of subsets of Y, consider the space $\mathcal{M}(Y)$ of all (countably additive) probability measures on the measurable space $(Y, \mathcal{B}(Y))$, endowed with the topology of weak convergence. Further, denote by $\prec$ a preference relation on a σ-convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. Necessary and sufficient conditions are presented for the existence of a pair of real continuous bounded functions u, v on Y, such that, for every $p, q \in \mathcal{P}$, $[p \prec q$ if and only if $\int_Y udp < \int_Y vdq]$, where the real functionals $p \to \int_Y udp$ and $p \to \int_Y vdp$ are utility functionals for two weak orders naturally associated to $\prec$.

1. Introduction

Grandmont [8, Th. 3] proved a classical theorem in expected utility theory. Given a separable metric space Y, a σ-algebra $\mathcal{B}(Y)$ of subsets of Y, and a weak order (i.e., an asymmetric and negatively transitive binary relation) $\prec$ on a σ-convex subspace $\mathcal{P}$ of the space $\mathcal{M}(Y)$ of all (countably additive) probability measures on the measurable space $(Y, \mathcal{B}(Y))$, Grandmont presented necessary and sufficient

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conditions for the existence of a continuous bounded real function u on Y, such that, for every $p, q \in \mathcal{P}$,

$$p \prec q \quad \text{if and only if} \quad \int_Y u dp < \int_Y u dq.$$

In this case, u is said to be a (continuous) von Neumann–Morgenstern utility function for the weak order $\prec$.

Several authors pointed out that indifference relations should not be transitive (see e.g. Armstrong [1], Bridges [4], Chateauneuf [5], Chipman [6], Fishburn [7], Luce [10]). While (semi)continuous representations of preferences with intransitive indifference seem to have received a considerable attention in literature (see e.g. Bridges [4], Chateauneuf [5] and Bosi et al. [3]), only a few authors were concerned with linear representations of preferences of this kind (see e.g., Fishburn [7], Vincke [13], and Nakamura [11]).

In this paper, given a preference relation $\prec$ on a σ-convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$, we are concerned with the existence of a pair of continuous bounded real functions u, v on the consequence space Y, such that, for every $p, q \in \mathcal{P}$,

$$p \prec q \quad \text{if and only if} \quad \int_Y u dp < \int_Y v dq.$$

In such a representation, u and v are von Neumann–Morgenstern utility functions for two weak orders naturally associated to $\prec$.

2. Notation and preliminaries

Denote by Y the set of all consequences, and let $\mathcal{B}(Y)$ be a σ-algebra of subsets of Y. It is assumed that Y is a separable metric space. Moreover, let $\mathcal{M}(Y)$ be the space of all (countably additive) probability measures (lotteries) on the measurable space $(Y, \mathcal{B}(Y))$, endowed with the topology of weak convergence. We recall that a sequence $\{p_n, n \geq \geq 1\}$ of probability measures in $\mathcal{M}(Y)$ converges weakly to a probability measure p if

$$\lim \int_Y f dp_n = \int_Y f dp$$

for every real bounded continuous function f on Y (see Parthasarathy [12]).

A subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ is said to be

(i) convex if $\lambda p_1 + (1-\lambda)p_2$ belongs to $\mathcal{P}$ for any p_1, p_2 in $\mathcal{P}$, and for any real number λ in $[0,1]$,

(ii) σ-convex if $p_0 = \sum_1^\infty \lambda_n p_n$ belongs to $\mathcal{P}$ for any sequence $\{p_n, n \geq 1\}$ of elements of $\mathcal{P}$, and for any sequence $\{\lambda_n, n \geq \geq 1\}$ of nonnegative real numbers such that $\sum_1^\infty \lambda_n = 1$.

A real functional f on a convex (σ-convex) subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ is linear (σ-linear) if, for every p, q in $\mathcal{P}$, and any real number λ in $[0,1]$, it is $f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda) f(q)$ (respectively, for any sequence $\{p_n, n \geq 1\}$ of elements of $\mathcal{P}$, and for any sequence $\{\lambda_n, n \geq 1\}$ of nonnegative real numbers such that $\sum_1^\infty \lambda_n = 1$, it is $f(\sum_1^\infty \lambda_n p_n) = = \sum_1^\infty \lambda_n f(p_n)$).

Let $\prec$ be a preference relation (i.e. an asymmetric binary relation) on a subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. Denote by $\preceq$ and $\sim$ the preference-indifference relation, and respectively the indifference relation associated with $\prec$, namely, for $p, q \in \mathcal{P}$,

$$p \preceq q \Leftrightarrow \text{ not } (q \prec p),$$

$$p \sim q \Leftrightarrow (p \preceq q) \text{ and } (q \preceq p).$$

A preference relation $\prec$ on $\mathcal{P}$ is said to be a weak order if $\prec$ is negatively transitive. If $\prec$ is a weak order, then the associated preference-indifference relation $\preceq$ is a complete preorder (i.e., $\preceq$ is transitive and complete).

The preference-indifference relation $\preceq$ associated with a preference relation $\prec$ on $\mathcal{P}$ is said to be pseudotransitive if, for every $p, p', q, q' \in \mathcal{P}$,

$$p \prec p' \preceq q' \prec q \Rightarrow p \prec q.$$

We say that a preference relation $\overset{c}{\prec}$ on Y is induced by a preference relation $\prec$ on $\mathcal{M}(Y)$ if, for every $y, z \in Y$,

$$y \overset{c}{\prec} z \Leftrightarrow p_y \prec p_z,$$

where, for every $y \in Y$, p_y is the probability distribution concentrated at the point $y \in Y$. Denote by D the subspace of $\mathcal{M}(Y)$ whose elements are the probability distributions which are concentrated, namely

$$D = \{p \in \mathcal{M}(Y) : \exists y \in Y, p = p_y\}.$$

A preference relation $\prec$ is represented by a utility functional U on $\mathcal{P}$ if, for every $p, q \in \mathcal{P}$,

$$p \prec q \Leftrightarrow U(p) < U(q).$$

If such a representation exists, then $\prec$ is a weak order. Grandmont [8] found necessary and sufficient conditions for the existence of a continuous von Neumann–Morgenstern utility function representing a weak order $\prec$ on a σ-convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ containing D. We recall that u is said to be a von Neumann–Morgenstern utility function for a preference relation $\prec$ on $\mathcal{P}$ if u is a real function on Y representing the preference relation $\overset{c}{\prec}$ among sure consequences and, for every $p, q \in \mathcal{P}$,

$$p \prec q \Leftrightarrow \int_Y udp < \int_Y udq.$$

It is clear that, if $\overset{c}{\prec}$ is induced by $\prec$, and there exists a real function u on Y such that

$$p \rightarrow U(p) = \int_Y udp$$

is a utility functional for $\prec$, then u is a von Neumann–Morgenstern utility function for $\prec$.

A preference relation $\prec$ is represented by a pair of real functionals U, V on $\mathcal{P}$ if, for every $p, q \in \mathcal{P}$,

$$p \prec q \Leftrightarrow U(p) < V(q).$$

If such a representation exists, then $\preceq$ is pseudotransitive.

A preference relation $\prec$ on $\mathcal{P}$ is continuous if $\{q \in P : p \prec q\}$ and $\{q \in P : q \prec p\}$ are open sets in $\mathcal{P}$ for every $p \in \mathcal{P}$.

To each preference relation $\prec$ on $\mathcal{P}$ we may associate the binary relations $\prec^*$ and $\prec^{**}$ defined as follows:

$$p \prec^* q \Leftrightarrow \exists r \in \mathcal{P} : p \prec r \preceq q,$$

$$p \prec^{**} q \Leftrightarrow \exists s \in \mathcal{P} : p \preceq s \prec q.$$

Fishburn [7] proved that, if $\prec$ is a preference relation with pseudotransitive preference-indifference, then $\prec^*$ and $\prec^{**}$ are both weak orders. The indifference relations associated to $\prec^*$ and $\prec^{**}$ are denoted by $\sim^*$ and $\sim^{**}$, respectively.

3. Expected utility with pseudotransitive preferences

In the following lemma, we present a necessary and sufficient condition for the continuity of a linear utility functional on a convex subspace of $\mathcal{M}(Y)$.

Lemma 1. *Let Y be a separable metric space, and let $\prec$ be a preference relation on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. Assume that there exists a* *linear utility functional U for $\prec$. Then U is continuous if and only if $\prec$ is continuous.*

Proof. Let U be a linear utility functional for a preference relation $\prec$ on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. It is clear that, if U is continuous, then $\prec$ is continuous. Conversely, assume that $\prec$ is continuous. In order to show that U is upper semicontinuous, consider $p \in \mathcal{P}$, and $\beta \in \mathbb{R}$, such that $U(p) < \beta$. If p is a maximal element relative to $\prec$, then $U(q) \leq \leq U(p)$ for every $q \in \mathcal{P}$, and therefore $\mathcal{P}$ is an open set containing p such that $U(q) < \beta$ for every $q \in \mathcal{P}$. If p is not a maximal element relative to $\prec$, then there exists $q' \in \mathcal{P}$ such that $p \prec q'$, and therefore $U(p) < < U(q')$. Since $\alpha \to (1-\alpha)U(p) + \alpha U(q')$ is a continuous function from the closed real interval $[0,1]$ onto the closed real interval $[U(p), U(q')]$, there exists $\bar{\alpha} \in [0,1]$ such that $U(p) < (1-\bar{\alpha})U(p) + \bar{\alpha}U(q') < \beta$. Define $\bar{q} = (1-\bar{\alpha})p + \bar{\alpha}q'$. Since U is a linear utility functional for $\prec$, it is $p \prec \bar{q}$, $U(\bar{q}) < \beta$. Since $\prec$ is continuous, $L(\bar{q}) = \{q \in \mathcal{P} : q \prec \bar{q}\}$ is an open set containing p, such that $U(q) < \beta$ for every $q \in L(\bar{q})$. Analogously, it can be shown that U is lower semicontinuous. $\Diamond$

In the following proposition, necessary and sufficient conditions are given for the existence of an integral representation of a linear utility functional on a σ-convex subspace of $\mathcal{M}(Y)$.

Proposition 1. *Let $\overset{c}{\prec}$ be a preference relation on a separable metric space Y, and let $\prec$ be a preference relation on a σ-convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ containing D. Assume that there is a linear utility functional U for $\prec$. Then there exists a real bounded continuous function u on Y, which is a utility function for $\overset{c}{\prec}$, such that, for every $p \in \mathcal{P}$, $U(p) = = \int_Y u dp$, if and only if $\prec$ is continuous and $\overset{c}{\prec}$ is induced by $\prec$.*

Proof. It is easily seen that, if u is a real bounded continuous function on Y, and $p \to U(p) = \int_Y u dp$ is a utility functional for $\prec$, then $\prec$ is continuous and $\overset{c}{\prec}$ is induced by $\prec$. Let us show that, if U is a linear utility functional for $\prec$, $\prec$ is continuous and $\overset{c}{\prec}$ is induced by $\prec$, then there exists a real bounded continuous function u on Y, which is a utility function for $\overset{c}{\prec}$, such that, for every $p \in \mathcal{P}$, $U(p) = \int_Y u dp$. First observe that U is continuous by Lemma 1. Define, for every $y \in Y$, $u(y) = = U(p_y)$. Since U is linear and continuous, and $\mathcal{P}$ is σ-convex, we have that u is bounded (see Grandmont [8, Lemma 2]). From Parthasarathy [12, Chap. 2, Lemma 6.1], u is continuous. Since $\mathcal{P}$ is convex and contains p_y for every $y \in Y$, any finite support probability distribution

p in $\mathcal{M}(Y)$ belongs to $\mathcal{P}$. From Parthasarathy [12, Ths. 6.2 and 6.3], each element p of $\mathcal{P}$ is the limit in the topology of weak convergence of a sequence $\{p_n, n \geq 1\} \subseteq \mathcal{P}$ of finite support probability measures. Since U is linear, it is easily seen that $U(p_n) = \int_Y u dp_n$ for every $n \geq 1$. By continuity of U, $\lim U(p_n) = U(p)$, and therefore, using the fact that u is continuous and bounded, $U(p) = \lim \int_Y u dp_n = \int_Y u dp$. $\Diamond$

Let us consider necessary and sufficient conditions for the existence of a pair U, V of linear functionals representing a preference relation $\prec$ with pseudotransitive preference-indifference on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. In this axiomatization, U and V are utility functionals for the associated weak orders $\prec^*$ and $\prec^{**}$, respectively. It is assumed that there is not a maximal element relative to $\prec$. We recall that another axiomatization was presented by Nakamura [11, Th. 1]. The following theorem allows us to recover an integral representation of both U and V, and this is the reason why we present it.

Theorem 1. *Let Y be a separable metric space, and let $\prec$ be a preference relation without a maximal element on a convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$. There exists a pair U, V of real continuous linear functionals on $\mathcal{P}$ representing $\prec$, such that U and V are utility functionals for $\prec^*$ and $\prec^{**}$, respectively, if and only if*

$$(1)\quad \begin{cases} \underline{A1}.\ \preceq \text{ is pseudotransitive}, \\ \underline{A2}.\ p \sim^{**} q \Rightarrow \lambda p + (1-\lambda) r \sim^{**} \lambda q + (1-\lambda) r \\ \qquad\qquad\qquad\qquad \forall p, q, r \in \mathcal{P},\ \lambda \in [0,1], \\ \underline{A3}.\ \prec^* \text{ and } \prec^{**} \text{ are both continuous}, \\ \underline{A4}.\ \lambda p + (1-\lambda) q \prec r \Rightarrow \exists r_1, r_2 \in \mathcal{P} : \lambda r_1 + (1-\lambda) r_2 \prec^{**} r, \\ \qquad p \prec r_1, q \prec r_2 \qquad\qquad \forall p, q, r \in \mathcal{P},\ \lambda \in [0,1], \\ \underline{A5}.\ p \prec q,\ r \prec s \Rightarrow \lambda p + (1-\lambda) r \prec \lambda q + (1-\lambda) s \\ \qquad\qquad\qquad\qquad \forall p, q, r, s \in \mathcal{P},\ \lambda \in [0,1]. \end{cases}$$

If U, V and U', V' are two pairs of such real functionals, then there exist two real numbers $a > 0$ and b, such that $U' = aU + b$ and $V' = aV + b$.

Proof. It is easily seen that conditions (1) are necessary for the existence of a pair U, V of real continuous linear functionals on $\mathcal{P}$ representing $\prec$, such that U and V are utility functionals for $\prec^*$ and $\prec^{**}$, respectively. So let us prove that axioms (1) are sufficient for the existence of such a representation. By axiom $\underline{A1}$, $\prec^*$ and $\prec^{**}$ are both weak orders. By axioms A2 and A3, for any $p, q, r \in \mathcal{P}$, the sets $\{\lambda \in \in [0,1] : p \prec^{**} \lambda q + (1-\lambda) r\}$ and $\{\lambda \in [0,1] : \lambda p + (1-\lambda) q \prec^{**} r\}$ are open (see the proof of Th. 2 in Grandmont [8]). According to Herstein and Milnor [9, Th. 8], there exists a real linear utility functional V on $\mathcal{P}$ representing $\prec^{**}$. Define, for every $p \in \mathcal{P}$,

$$U(p) = \inf\{V(q) : p \prec q, q \in \mathcal{P}\}.$$

Let us show that the pair U, V represents $\prec$. Consider $p, q \in \mathcal{P}$ such that $p \prec q$. By axioms A4 and A5, there exists $p' \in \mathcal{P}$ with $p \prec p' \prec^{**} q$. Since $V(p') < V(q)$, it is $U(p) < V(q)$ from the definition of U. Conversely, assume that $U(p) < V(q)$. Then there exists $p' \in \mathcal{P}$ such that $U(p) < V(p') < V(q)$, $p \prec p'$. Hence $p \prec p' \prec^{**} q$, and therefore $p \prec q$ by axiom A1.

Let us prove that U is a utility functional for $\prec^{*}$. If $p \prec^{*} q$, then there exists $q' \in \mathcal{P}$ such that $p \prec q' \preceq q$. Then $U(p) < V(q') \leq U(q)$, and therefore $U(p) < U(q)$. Conversely, assume that $U(p) < U(q)$. Then there exists $q' \in \mathcal{P}$ such that $U(p) < V(q') < U(q)$, $p \prec q' \preceq q$, and therefore $p \prec^{*} q$.

Now, let us show that U is linear. Assume that there exist $p, q \in \in \mathcal{P}$, and $\lambda \in [0,1]$, such that $\lambda U(p) + (1-\lambda) U(q) < U(\lambda p + (1-\lambda) q)$. From the definition of U, and from linearity of V, there exist $r_1, r_2 \in \mathcal{P}$ with $p \prec r_1$, $q \prec r_2$, $\lambda U(p) + (1-\lambda) U(q) < V(\lambda r_1 + (1-\lambda) r_2) < < U(\lambda p + (1-\lambda) q)$. By axiom A5, it is $\lambda p + (1-\lambda) q \prec \lambda r_1 + (1-\lambda) r_2$, and therefore $V(\lambda r_1 + (1-\lambda) r_2) < U(\lambda p + (1-\lambda) q)$ is contradictory. Using similar considerations, it can be shown that for no $p, q \in \mathcal{P}$, and $\lambda \in [0,1]$, it is $U(\lambda p + (1-\lambda) q) < \lambda U(p) + (1-\lambda) U(q)$.

Since U and V are real linear utility functionals for $\prec^{*}$ and $\prec^{**}$, respectively, and $\prec^{*}$ and $\prec^{**}$ are both continuous by axiom A3, then U and V are continuous by Lemma 1.

Let U, V and U', V' be two pairs of real functionals both satisfying axioms (1). From Herstein and Milnor [9, Th. 8], there exist two real numbers $a > 0$ and b, and two real numbers $a' > 0$ and b', such that $U' = aU + b$ and $V' = a'V + b'$. Assume that either $a \neq a'$ or $b \neq b'$, and consider $p, q \in \mathcal{P}$, such that $p \prec q$. Then it is both $U(p) < V(q)$ and $U(p) < 1/a\,(a'V(q) + b' - b)$. If $1/a\,(a'V(q) + b' - b) < V(q)$, then, using the fact that U is linear, it is easily seen that there exists $p' \in \mathcal{P}$ such that $1/a\,(a'V(q) + b' - b) < U(p') < V(q)$, and this is impossible since U, V and U', V' are two representations of $\prec$. Analogous considerations lead to a contradiction in the case when $V(q) \leq 1/a\,(a'V(q) + b' - b)$.

So the proof is complete. ◊

Remark 1. Observe that axiom A5 is found in the axiomatization presented by Nakamura [11, Th. 1]. Axiom A4 is a continuity axiom involving the preference relation $\prec$ and the associated weak order $\prec^{**}$. ◊

Now we are able to present the main result of this section.

Theorem 2. *Let $\overset{c}{\prec}$ be a preference relation on a separable metric space Y, and let $\prec$ be a preference relation without maximal elements on a σ-convex subspace $\mathcal{P}$ of $\mathcal{M}(Y)$ containing D. There exists a pair u, v of real continuous bounded functions on Y, which are utility functions for $\overset{c}{\prec}{}^{*}$ and $\overset{c}{\prec}{}^{**}$, respectively, such that, for every $p, q \in \mathcal{P}$,*

$$(2)\qquad \begin{cases} \underline{B1}.\ p \prec q \Leftrightarrow \int_Y u dp < \int_Y v dq, \\ \underline{B2}.\ p \prec^{*} q \Leftrightarrow \int_Y u dp < \int_Y u dq, \\ \underline{B3}.\ p \prec^{**} q \Leftrightarrow \int_Y v dp < \int_Y v dq, \end{cases}$$

if and only if axioms (1) *of* Th. 1 *hold, and*

$$(3)\qquad \overset{c}{\prec} \text{ is induced by } \prec .$$

If u, v and u', v' are two pairs of such real functions on Y, then there exist two real numbers $a > 0$ and b, such that $u' = au + b$ and $v' = av + b$.

Proof. It is easily seen that axioms (1) of Th. 1, and condition (3) are necessary for the existence of a pair u, v of real continuous bounded functions on Y satisfying conditions (2). So, let us prove the sufficiency part. From Th. 1, there exists a pair of real continuous linear functionals U, V on Y, representing $\prec^{*}$ and $\prec^{**}$, respectively. From Prop. 1, since it is easily seen that, if $\overset{c}{\prec}$ is induced by $\prec$, then $\overset{c}{\prec}{}^{*}$ and $\overset{c}{\prec}{}^{**}$ are induced by $\prec^{*}$ and $\prec^{**}$, respectively, there exists a pair u, v of real bounded continuous functions on Y, which are utility functions for $\overset{c}{\prec}{}^{*}$ and $\overset{c}{\prec}{}^{**}$, respectively, such that, for every $p \in \mathcal{P}$, $U(p) = \int_Y u dp$ and $V(p) = \int_Y v dp$.

Finally, if u, v and u', v' are two pairs of such real functions on Y, then, by Th. 1, there exist two real numbers $a > 0$ and b, such that $u' = au + b$, $v' = av + b$. So the proof is complete. ◊

References

[1] ARMSTRONG, W. E.: A note on the theory of consumers' behavior, *Oxford Economics Papers* **2** (1950), 119–122.

[2] BOSI, G.: Continuous von Neumann–Morgenstern utility representation of pseudotransitive preferences, Department of Applied Mathematics "Bruno de Finetti", Technical Report **9** (1996).

[3] BOSI, G., CANDEAL, J. C., INDURÁIN, E., OLORIZ, E. and ZUDAIRE, M.: Numerical representations of interval orders, *Order* **18** (2001), 171–190.

[4] BRIDGES, D. S.: A numerical representation of preferences with intransitive indifference, *Journal of Mathematical Economics* **11** (1983), 25–42.

[5] CHATEAUNEUF, A.: Continuous representation of a preference relation on a connected topological space, *Journal of Mathematical Economics* **16** (1987), 139–146.

[6] CHIPMAN, J. S.: Consumption theory without transitive indifference, in Preferences, utility and demand, J. S. Chipman, L. Hurwicz, M. Richter and H. Sonnenschein, eds., New York, 1971.

[7] FISHBURN, P. C.: Utility theory for decision making, Wiley, New York, 1970.

[8] GRANDMONT, J. M.: Continuity properties of a von Neumann–Morgenstern utility, *Journal of Economic Theory* **4** (1972), 45–67.

[9] HERNSTEIN, I. N. and MILNOR, J.: An axiomatic approach to measurable utility, *Econometrica* **21** (1953), 291–297.

[10] LUCE, R.: Semiorders and a theory of utility discrimination, *Econometrica* **24** (1956), 178–191.

[11] NAKAMURA, Y.: Expected utility with an interval ordered structure, *Journal of Mathematical Psychology* **32** (1988), 298–312.

[12] PARTHASARATHY, K. R.: Probability measures on metric spaces, Academic Press, 1967.

[13] VINCKE, P.: Linear utility functions on semiordered mixture spaces, *Econometrica* **48** (1980), 771–775.