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ON CERTAIN GENERALIZED CIRCULANT MATRICES

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Abstract: Let h, n be positive integers, where $1 \leq h < n$, $k = (n, h)$ and $n = kn'$. We call h-generalized circulant a matrix A of order n which can be partitioned into h-circulant submatrices of type $n' \times n$. We determine a characterization of h-generalized circulant matrices and, using this result, we prove that $A = \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^{jh}$ is permutation similar to the direct sum of k matrices coinciding with $\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_{n'}^{j}$, where P_n denote the $(0,1)$-circulant matrix of order n whose first row is null but the element in position $(1, 2)$. This implies new results on the values of the permanent and also on the determination of the eigenvalues of $(0,1)$-circulant matrices. A partial proof of a conjecture on the maximum value of permanents is achieved.

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1. Introduction

Recall that a matrix A of type $m \times n$ $(m \leq n)$ is said h-circulant when each row other than the first one is obtained from the preceding row by shifting the elements cyclically h columns to the right. In the case of $h = 1$ A is said circulant.

Let P_n denote the $(0,1)$-circulant matrix of type $n \times n$ with first row $(010\dots0)$. If there is not possibility of ambiguity we often drop the subscript n and simply write P_n as P.

If $(a_0, a_1, \dots, a_{n-1})$ is the first row of a circulant matrix A of order n, then $A = \sum_{i=0}^{n-1} a_i P^i$.

It is easy to see that a matrix A of type $m \times n$ is h-circulant if and only if it satisfies the relation $AP_n^h = P_m A$.

For $i = 1, 2, \dots, k$, let A_i be a square matrix of order n_i. The block diagonal square matrix

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_k \end{bmatrix}$$

of order $n_1 + n_2 + \dots n_k$ is called the direct sum of the matrices $A_1, \dots, A_k$. It is denoted as $A = \operatorname{diag}\{A_1, A_2, \dots, A_k\}$.

Recall that the permanent of a $n \times n$ matrix $A = [a_{i,j}]$, denoted by per$A$, is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the sum extends over all permutations σ of the symmetric group of all permutations of the first n integers.

For every $1 \leq r \leq n$, we denote by (r) and $[r]$ the r-row and the r-column respectively of a matrix of order n.

Definition 1. Let h, n be positive integers, where $1 \leq h < n$, $k = (n, h)$ and $n = kn'$. A matrix A of order n is said *h-generalized circulant* when it is partitioned into k submatrices of type $n' \times n$, which are h-circulant.

In other words a matrix A of order n is h-generalized circulant when it can be partitioned in the form

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_k \end{bmatrix} \tag{1}$$

where A_i, $1 \le i \le k$ are h-circulant $n' \times n$-submatrices, i.e. they satisfy $A_j P_n^h = P_n A_j$.

The main result of this paper is proving a characterization of the h-generalized circulant matrices (Th. 1). By using this result we are able to prove that the matrix $A = \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^{jh}$ is permutation similar to the matrix $B = \text{diag}\left\{ \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^j, \dots, \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^j \right\}$, the direct sum of k matrices coinciding with $\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^j$. As these matrices have the same permanent we obtain new values for the permanent of $(0,1)$ circulant matrices. In particular we obtain $\text{per}\left(\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} P_n^{jh} \right) = \left(\text{per}\left(\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} P_{n'}^j \right) \right)^k$; in the particular case of three ones for row

$$\text{per}\,(I + P_n^h + P_n^{2h}) = \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n'} \left(\frac{1-\sqrt{5}}{2} \right)^{n'} + 2 \right]^k .$$

A partial proof of a conjecture by Codenotti, Crespi and Resta [1] on the maximum value for permanents of very sparse matrices is achieved; the computation of the permanent of this class of matrices was extensively studied also in [2], [3] and [4]. Results are also obtained in relation to the characteristic polynomials of A and B.

2. Characterization

Let us consider a matrix A of order n; we denote by A_j, $1 \le j \le k$, the submatrix of A of type $n' \times n$ formed by the rows of A

$$(1 + (j-1)n'), (2 + (j-1)n'), \dots, (jn').$$

Theorem 1. *A matrix A of order n is h-generalized circulant, where $(n,h) = k$ and $n = kn'$, if and only if it satisfies the relation*

(2) $$AP^h = P'A$$

where P' *is direct sum of* k *matrices coinciding with* $P_{n'}$, *i.e.* $P' = \mathrm{diag}\{P_{n'}, \dots, P_{n'}\}$.

Proof. Let us assume that a matrix $A = [a_{i,j}]$ of order n, where $1 \leq i, j \leq n$ satisfies (2). The matrix AP^h is obtained by shifting cyclically the columns of A of h positions to the right. Taking into account the partitioned form of A we have

$$AP^h = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} P^h = \begin{bmatrix} A_1P^h \\ A_2P^h \\ \vdots \\ A_kP^h \end{bmatrix}.$$

Hence $(AP^h)_j = A_jP^h$ for $1 \leq j \leq k$.

Now consider the product $P'A$. From the definition of P' and the partitioned form of A we have

$$\begin{bmatrix} P_{n'} & & & \\ & P_{n'} & & \\ & & \ddots & \\ & & & P_{n'} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} = \begin{bmatrix} P_{n'}A_1 \\ P_{n'}A_2 \\ \vdots \\ P_{n'}A_k \end{bmatrix},$$

which shows that $(P'A)_j = P_{n'}A_j$ for $1 \leq j \leq k$.

The equality (2) implies the equality of the submatrices $(AP^h)_j$ and $(P'A)_j$, where $1 \leq j \leq k$. But we have seen that $(AP^h)_j = A_jP^h$. Hence from the assumption (2) it follows $A_jP^h = P_{n'}A_j$ $(1 \leq j \leq k)$, i.e. the submatrices A_j $(1 \leq j \leq k)$ are h-circulant $n' \times n$ matrices and therefore A is h-generalized circulant.

Conversely, assume that every A_j, $1 \leq j \leq k$, is h-circulant, i.e. $P_{n'}A_j = A_jP^h$ $(1 \leq j \leq k)$. Then $(P'A)_j = P_{n'}A_j$ and $(AP^h)_j = A_jP^h$ for $1 \leq j \leq k$.

Denote $Q = [q_{ij}]$ the permutation matrix of order n, that rearranges the columns of an arbitrary $m \times n$ matrix M by postmultiplication MQ according to the permutation α. Using Kronecker symbols δ the entries q_{ij} can be written in the form

$$q_{ij} = \delta_{\alpha(i),j} = \begin{cases} 1 & \text{if } j = \alpha(i) \\ 0 & \text{otherwise.} \end{cases}$$

By the assumption that A_j is h-circulant it follows that the rows of $(P'A)_j$ and $(AP^h)_j$ coincide. Then $(P'A)_j = (AP^h)_j$ and A satisfies (2). ◊

When $k = 1$ a matrix A which satisfies (2) turns out to be a h-circulant matrix; thus this definition turns out to be a generalization of the notion of h-circulant matrix.

Now we will consider the particular case of h-generalized circulant permutation matrices.

Proposition 1. *Let n and h be positive integers, where $1 \leq h \leq n$, $k = (n, h)$ and $n = kn'$. The function $\alpha : i \mapsto 1 + (i-1)h + t$, where $1 + tn' \leq i \leq (t+1)n'$, $0 \leq t \leq k-1$ and the integers are taken* $\bmod n$ *is a permutation, whose representing matrix satisfies* (2).

Proof. In order to prove that α is a permutation it is sufficient to prove it is injective. Let $i, j \in [1, n]$, $i < j$ and $h = kh'$, where $(h', n') = 1$. Assume that $\alpha(i) = \alpha(j)$, that is

$$1 + (i-1)h + t = 1 + (j-1)h + t'$$

where $0 \leq t, t' \leq k-1$. Let us distinguish the cases of $t = t'$ or $t \neq$ $\neq t'$. The condition of $t = t'$ implies the relation $(j-i)h \equiv 0 \pmod n$, which is impossible because $j - i < n'$. In the case of t and t' distinct, without loss of generality we may assume $t' > t$ and represent $t' =$ $= t + r$, where $0 < r < k$. Thus we obtain $(i-j)h \equiv r (\bmod n)$. It implies that for a suitable integer m we obtain $k((i-j)h' - mn') = r$, which is impossible by the assumption on r. Then α is a permutation. Denote by Q the matrix which represents such a permutation. Then by the construction we have that the submatrices Q_t formed by the rows $(1 + (t-1)n'), \ldots, (tn')$, where $1 \leq t \leq k$ are h-circulant. Then Q is h-generalized circulant. $\Diamond$

Also the permutation α is said *h-generalized circulant.*

Corollary 1. *Let α an h-generalized circulant permutation; α is uniquely determined when*

$$\alpha(1), \alpha(1 + n'), \ldots, \alpha(1 + (k-1)n')$$

are assigned.

Proof. When we assign $\alpha(1)$ then the first row and therefore the consecutive $k - 1$ rows of the matrix which represents α are assigned. This means that in the decomposition (1) the submatrix A_1 is given. Similar considerations hold for the remaining submatrices. $\Diamond$

Proposition 2. *The number of h-generalized circulant matrices of order n, where $k = (n, h)$ and $n = kn'$, is*

$$n(n - n')(n - 2n') \ldots n'$$

Proof. From the above considerations, α is uniquely determined when we assign $\alpha(1+jn')$, for all $0 \le j \le k-1$. We see that $\alpha(1)$ may assume n values. When $\alpha(1)$ is assigned, also the following $n'-1$ rows are assigned; then $\alpha(n'+1)$ may assume $n-n'$ values. By continuing in this way the result follows. $\Diamond$

We call *regular* an h-generalized circulant permutation matrix $Q = [q_{i,j}]$ of order $n = k.n'$, when

$$q_{1,1} = q_{1+n',2} = \cdots = q_{1+(k-1)n',k} = 1.$$

In other words a h-generalized matrix A, representing the permutation α, is regular when α satisfies the conditions

$$\alpha(1) = 1, \alpha(n'+1) = 2, \ldots, \alpha((k-1)n'+1) = k.$$

When we need to remember the parameter h in relation to an h-generalized circulant permutation matrix Q, we write $Q(h)$.

Theorem 2. *Let $A = a_0 I + a_1 P^h + \ldots + a_t P^{th}$ be a matrix of order n, where $1 < h < n$, $(n,h) = k$, $n = kn'$, $t = \lfloor \frac{n}{h} \rfloor$ and $a_i, 1 \le i \le n'-1$, real numbers; moreover let Q be the h-generalized regular permutation matrix of order n. Then the matrix $B = QAQ^T$ is direct sum of k matrices coinciding with $\sum_{i=0}^{t} a_i P_{n'}^i$.*

Proof. By (2) $QP^hQ^T = P'$; then $B = QAQ^T = a_o I + a_1 P' + \cdots + a_t (P')^t$. $\Diamond$

An immediate consequence is that the circulant matrix $A = I + P^h + \cdots + P^{sh}$ where $s \le \lfloor \frac{n}{h} \rfloor$, is permutation similar to the circulant matrix $B = I + P + \cdots + P^s$. Another consequence is the following

Corollary 2. *Let $A = \sum_{j=0}^{r} P_n^{jh}$ be a square matrix of order n, where $1 < h < n$, $(n,h) = k$, $n = kn'$, $t = \lfloor \frac{n}{h} \rfloor$ and $1 \le r \le t$. Then we have*

$$\operatorname{per}\left(\sum_{j=0}^{t} P_n^{jh}\right) = \left(\operatorname{per}\left(\sum_{j=0}^{t} P_{n'}^{j}\right)\right)^k.$$

Proof. As the permanent is invariant with respect to permutation of rows or columns, the result follows from Th. 1. $\Diamond$

As example of regular 3-generalized permutation matrix we may consider the following matrix of order 9:

$$Q(3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then in relation to the matrix of order 9 $A = I + P^3 + P^6$, we obtain that

$$(3) \qquad Q(3)AQ(3)^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

3. Very sparse matrices

In this section we consider the case of $(0, 1)$ circulant matrices with 3 ones in each row. First consider the matrix $I + P^h + P^{2h}$, where $(n, h) = 1$ and P denotes the permutation matrix P_n. By using a Minc's formula for the permanent of $I + P + P^2$ [8] we have:

Corollary 3. *Let P be of order n, where $1 < h < n$, $(n, h) = k$, $n = kn'$; then*

$$\text{per}\left(I + P^h + P^{2h}\right) = \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n'} + \left(\frac{1-\sqrt{5}}{2}\right)^{n'} + 2\right]^k.$$

Now, consider the matrix $D = I + P^m + P^{2m}$ of order $n = 3m$. From Th. 2 we have that A is permutation similar to the direct sum of m submatrices coinciding with J_3, the matrix of all ones of order 3, then per $mA = (3!)^m$. It is known that in the class of $vk \times vk$ $(0, 1)$-matrices with row sums and column sums equal to k the permanent function takes its maximum on the direct sum of $k \times k$ matrices of 1's. Thus the matrix D satisfies partially the conjecture by Codenotti, Crespa and Resta [1].

Now consider the case of a matrix $A = I+P^h+P^j$, where $(n,h) = 1$ and $j \neq 2h \pmod{n}$.

Proposition 3. *Let h,n be positive integers, such that $1 < h < n$, $(n,h) = 1$ and Q is the regular h-generalized permutation matrix of order n. Then $QPQ^T = P^s$ where s is the unique solution, modulo n, of the equation*

$$sh \equiv 1. \tag{4}$$

Proof. Denote by α, π and β the h-generalized regular circulant permutations represented by Q_h, P and $Q_hPQ_h^T$ respectively. Then $\beta(1) = \alpha^{-1}(\pi(\alpha(1))) = \alpha^{-1}(\pi(1)) = \alpha^{-1}(2)$. Denote by s the integer, $1 < s \leq n$, such that $1 + sh \equiv 2$. Because $(n,h) = 1$, it easy to see that the equation $sh \equiv 1$ has a unique solution. Thus, for every $1 < i \leq n$, we have

$$\beta(i) = \alpha^{-1}(\pi(\alpha(i))) = \alpha^{-1}(\pi(1+(i-1)h)) =$$
$$= \alpha^{-1}(2+(i-1)h) = \alpha^{-1}(1+(s+i-1)h) = s+i.$$

This implies that $\beta = \pi^s$. ◊

Proposition 4. *Let $A = I + P^h + P^j$ be a square matrix of order n, where $1 < h < j < n$, $j \neq 2h \pmod{n}$, $(h,n) = 1$ and Q is the regular h-generalized permutation matrix of order n. Then $QAQ^T = I+P+P^v$, where v is the unique solution of the equation $vh \equiv j \pmod{n}$.*

Proof. From Prop. 4 we have that $Q_hP^jQ_h^T = (Q_hPQ_h^T)^j = P^{sj}$. Denoted $v = sj$, from the equation $sh = 1$, we obtain $vh = j \pmod{n}$. This implies $Q_hAQ_h^T = I + P + P^v$. ◊

In the case when $(n,h) \neq 1$, but $(n,j) = 1$ or $(n,j-h) = 1$, we have a similar situation by multiplying A by a suitable power of P.

4. Eigenvalues

Recall that if A is a circulant matrix hose first row is $[a_0 a_1 \dots a_{n-1}]$, the polynomial $p(\lambda) = \sum_{i=o}^{n} a_i \lambda^i$ is said the Hall polynomial of the matrix A. If $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the eigenvalues of A are $1, p(\omega), p(\omega^2), \dots, p(\omega^{n-1})$.

Let $p(\lambda)$ and $q(\lambda)$ the Hall polynomials of the matrices $A = I + P^h + \cdots + P^{rh}$ and $B = I + P + \cdots + P^r$ respectively, where $0 < h < n, k = (n, h)$, $n = kn'$ and $1 < r \leq \lfloor \frac{n}{h} \rfloor$. Moreover let $r(\lambda) = 1 + \lambda + \cdots + \lambda^{n'-1}$ be the Hall polynomial of the matrix $C = I + P_{n'} + \cdots + P_{n'}^r$ and $\alpha = \cos \frac{2\pi}{n'} + i \sin \frac{2\pi}{n'}$. From Th. 2 it follows the following

Proposition 5. *The sets of eigenvalues of* $A = I + P^h + \cdots + P^{rh}$ *and* $C = I + P_{n'} + \cdots + P_{n'}^r$ *coincide, when* $k = 1$. *In the case of* $k > 1$, *the set of eigenvalues of* A *is the union of* k *sets coinciding with* $\{1, r(\alpha), \ldots, r(\alpha^{n'-1})\}$.

A consequence is that, when $k > 1$, every eigenvalue of A has multiplicity at least k.

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