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ON THE SCHWAB–BORCHARDT MEAN

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Abstract: This paper deals with the Schwab–Borchardt mean. Emphasis is on the inequalities connecting the mean in question with other means of two variables. For special values of its arguments, the Schwab–Borchardt mean simplifies to known ones. Particular means reached that way include a logarithmic mean and two means introduced recently by H.-J. Seiffert. The Ky Fan inequalities for the logarithmic mean, Seiffert means and other means are obtained. A sequential method of Sándor [10] is generalized to obtain bounds for the mean under discussion. Inequalities involving the Schwab–Borchardt mean and the Gauss arithmetic-geometric mean are also included.

1. Introduction

The Schwab–Borchardt mean of two numbers $x \geq 0$ and $y > 0$, denoted by $SB(x, y) \equiv SB$, is defined as

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$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & 0 \le x < y \\ \dfrac{\sqrt{x^2-y^2}}{\operatorname{arccosh}(x/y)}, & y < x \\ x, & x = y \end{cases} \tag{1.1}$$

(see [1, Th. 8.4], [3, (2.3)]). It follows from (1.1) that $SB(x,y)$ is not symmetric in its arguments and is a homogeneous function of degree 1 in x and y. Using elementary identities for the inverse circular function, and the inverse hyperbolic function, one can write the first two parts of formula (1.1) as

$$SB(x,y) = \frac{\sqrt{y^2-x^2}}{\arcsin\left(\sqrt{1-(x/y)^2}\right)} = \frac{\sqrt{y^2-x^2}}{\arctan\left(\sqrt{(y/x)^2-1}\right)}, \quad 0 \le x < y \tag{1.2}$$

and

$$\begin{aligned} SB(x,y) &= \frac{\sqrt{x^2-y^2}}{\operatorname{arcsinh}\left(\sqrt{(x/y)^2-1}\right)} = \frac{\sqrt{x^2-y^2}}{\operatorname{arctanh}\left(\sqrt{1-(y/x)^2}\right)} = \\ &= \frac{\sqrt{x^2-y^2}}{\ln\left(x+\sqrt{x^2-y^2}\right)-\ln y}, \qquad y < x, \end{aligned} \tag{1.3}$$

respectively.

The Schwab–Borchardt mean is the iterative mean i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n, \tag{1.4}$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = (x_n + y_n)/2, \quad y_{n+1} = \sqrt{x_{n+1} y_n}, \tag{1.5}$$

$n = 0, 1, \ldots$ (see [3, (2.3)], [2]). It follows from (1.5) that the members of two infinite sequences $\{x_n\}$ and $\{y_n\}$ satisfy the following inequalities

$$x_0 < x_1 < \cdots < x_n < \cdots < SB < \cdots < y_n < \cdots < y_1 < y_0 \quad (x < y) \tag{1.6}$$

and

$$y_0 < y_1 < \cdots < y_n < \cdots < SB < \cdots < x_n < \cdots < x_1 < x_0 \quad (y < x). \tag{1.7}$$

For later use, let us record the invariance formula for the Schwab–Borchardt mean

$$SB(x,y) = SB\left(\frac{x+y}{2}, \sqrt{\frac{x+y}{2}y}\right) \tag{1.8}$$

which follows from (1.5).

This paper deals mostly with the inequalities involving the mean under discussion and is organized as follows. Particular cases of the Schwab–Borchardt mean are studied in Section 2. They include two means introduced recently by H.-J. Seiffert, the logarithmic mean and a possible new mean of two variables. The Ky Fan inequalities for these means are also included. The main results of this paper are contained in Section 3. Lower and upper bounds for SB, that are stronger than those in (1.6)–(1.7) are contained in Th. 3.3. Inequalities involving the Schwab–Borchardt mean and the Gauss arithmetic-geometric mean are also obtained. Additional bounds for the mean under discussion are presented in Appendix 1. Inequalities involving numbers x_n and y_n and those used in Th. 3.3 are presented in Appendix 2.

2. Inequalities for the particular means

Before we state and prove the main results of this section let us introduce more notation. Let $x \geq 0$ and $y > 0$. The following function

$$R_C(x,y) = \frac{1}{2}\int_0^{\infty}(t+x)^{-1/2}(t+y)^{-1}\,dt \tag{2.1}$$

plays an important role in the theory of special functions (see [5], [7]). B. C. Carlson [3] has shown that

$$SB(x,y) = [R_C(x^2,y^2)]^{-1} \tag{2.2}$$

(see also [2, (3.21)]). It follows from (2.2) and (2.1) that the mean $SB(x,y)$ increases with an increase in either x or y.

To this end we will assume that the numbers x and y are positive and distinct. The symbols A, L, G and H will stand for the arithmetic, logarithmic, geometric, and harmonic mean of x and y, respectively. Recall that

$$L(x,y) = \frac{x-y}{\ln x - \ln y} = \frac{x-y}{2\,\mathrm{arctanh}\left(\dfrac{x-y}{x+y}\right)} \tag{2.3}$$

(see, e.g., [4]–[5]). Other means used in the paper include two means introduced recently by H.-J. Seiffert

$$P(x,y)=\frac{x-y}{2\arcsin\left(\dfrac{x-y}{x+y}\right)} \tag{2.4}$$

(see [12]) and

$$T(x,y)=\frac{x-y}{2\arctan\left(\dfrac{x-y}{x+y}\right)} \tag{2.5}$$

(see [13]). For the last two means we have used notation introduced in [10] and [11]. Several inequalities for the Seiffert means are obtained in [8], [10]–[11]). Also, we define a possibly new mean

$$M(x,y)=\frac{x-y}{2\,\mathrm{arcsinh}\left(\dfrac{x-y}{x+y}\right)}. \tag{2.6}$$

In what follows we will write $Q(x,y)\equiv Q$ for the power mean of order two of x and y

$$Q(x,y)=\sqrt{\frac{x^2+y^2}{2}}. \tag{2.7}$$

It is easy to see that the means, L, P, T, and M are the Schwab–Borchardt means. Use of (1.2) and (1.3) gives

$$\begin{aligned} &L=SB(A,G), \qquad P=SB(G,A), \qquad T=SB(A,Q),\\ &M=SB(Q,A). \end{aligned} \tag{2.8}$$

A comparison result for $SB(\cdot,\cdot)$ is contained in the following:

Proposition 2.1. *Let $x>y$. Then*

$$SB(x,y)<SB(y,x). \tag{2.9}$$

Proof. Using the invariance formula (1.8) together with the monotonicity property of the mean SB in its arguments, we obtain

$$SB(x,y)=SB(A,\sqrt{Ay})<SB(A,\sqrt{Ax})=SB(y,x).\ \diamondsuit$$

Inequalities connecting means L, P, M, and T with underlying means G, A, and Q can be established easily using (2.9). We have

$$G < L < P < A < M < T < Q. \tag{2.10}$$

For the proof of (2.10) we use monotonicity of the Schwab–Borchardt mean in its arguments, inequalities $G < A < Q$, and (2.8) to obtain

$$G = SB(G,G) < SB(A,G) < SB(G,A) < SB(A,A) =$$
$$= A < SB(Q,A) < SB(A,Q) < SB(Q,Q) = Q.$$

The first three inequalities in (2.10) are known (see [4]–[5], [12], [14]) and the sixth one appears in [13]. (See also [11] for the proof of the last inequality in (2.10) and its refinements.)

We shall establish now the Ky Fan inequalities involving the first six means that appear in (2.10). For $0 < x,\ y \le \frac{1}{2}$, let $x' = 1 - x$ and $y' = 1 - y$. In what follows we will write G' for $G(x', y')$, L' for $L(x', y')$, etc.

Proposition 2.2. *Let* $0 < x,\ y \le \frac{1}{2}$. *The following inequalities*

$$\frac{G}{G'} < \frac{L}{L'} < \frac{P}{P'} < \frac{A}{A'} < \frac{M}{M'} < \frac{T}{T'} \tag{2.11}$$

hold true.

Proof. The first inequality in (2.11) is established in [9]. For the proof of the second one we use (2.3) and (2.4) to obtain

$$\frac{L}{P} = \frac{\arcsin z}{\operatorname{arctanh}\, z}, \tag{2.12}$$

where $z = (x - y)/(x + y)$. Let $z' = (x' - y')/(x' + y')$. One can easily verify that z and z' satisfy the following inequalities

$$0 < |z'| < |z| < 1, \qquad zz' < 0. \tag{2.13}$$

Let $f(z)$ stand for the function on the right side of (2.12). The following properties of $f(z)$ will be used in the proof of (2.11). We have: $f(z) = f(-z)$, $f(z)$ is strictly increasing on $(-1, 0)$ and strictly decreasing on $(0, 1)$, $\max\{f(z) : |z| \le 1\} = f(0) = 1$. Assume that $y < x \le \frac{1}{2}$. It follows from (2.13) that $0 < -z' < z < 1$. This in turn implies that $f(-z') > f(z)$ or what is the same, $L/P < L'/P'$. One can show that the last inequality is also valid if $x < y \le \frac{1}{2}$. This completes the proof of the second inequality in (2.11). The remaining three inequalities in (2.11) can be established in the analogous manner using the formulas

$$(2.14)\qquad \frac{P}{A}=\frac{z}{\arcsin z}, \qquad \frac{A}{M}=\frac{\operatorname{arcsinh} z}{z}, \qquad \frac{M}{T}=\frac{\arctan z}{\operatorname{arcsinh} z}.$$

They follow from (2.4), (2.6), and (2.5). ◊

We close this section giving the companion inequalities to the inequalities three through five in (2.10). We have

$$(2.15)\qquad \frac{\pi}{2}P > A > \operatorname{arcsinh}(1)M > \frac{\pi}{4}T.$$

For the proof of (2.15) let us note that the functions on the right sides of (2.14) share the properties of the function $f(z)$, used above. In particular, they attain the global minima at $z = \pm 1$. This in turn implies that

$$\frac{P}{A} > \frac{2}{\pi}, \qquad \frac{A}{M} > \operatorname{arcsinh}(1), \qquad \frac{M}{T} > \frac{\pi}{4\operatorname{arcsinh}(1)}.$$

The assertion (2.15) now follows. The first inequality in (2.15) is also established in [14] by use of different means.

3. Main results

We are in position to present the main results of this paper. Several inequalities for the mean under discussion are obtained. New inequalities for the particular means discussed in the previous section are also included.

Our first result reads as follows:

Theorem 3.1. *Let x and y be positive and distinct numbers. If $x < y$, then*

$$(3.1)\qquad T(x,y) < SB(x,y)$$

and if $x > y$, then

$$(3.2)\qquad SB(x,y) < L(x,y).$$

The following inequalities

$$(3.3)\qquad SB(y,G) < SB(x,y) < SB(y,A)$$

and

$$(3.4)\qquad SB(x,y) > H(SB(y,x),y)$$

are valid.

Proof. Let $x < y$. For the proof of (3.1) we use (1.8), the inequality $xy > x^2$ and (2.8) to obtain

$$SB(x,y) = SB\left(A, \sqrt{\frac{x+y}{2}y}\right) > SB(A,Q) = T(x,y).$$

Assume now that $x > y$. Making use of (1.8) and (2.8) together with the application of the inequality $A < x$ gives

$$SB(x,y) = SB(A, \sqrt{Ay}) < SB(A,G) = L(x,y).$$

In order to establish the first inequality in (3.3) we need the following one

$$[(t+x^2)(t+y^2)]^{-1/2} \le (t+G^2)^{-1}$$

(see [4]). Multiplying both sides by $(1/2)(t+y^2)^{-1/2}$ and next integrating from 0 to infinity we obtain, using (2.1),

$$R_C(x^2,y^2) < R_C(y^2,G^2).$$

Application of (2.2) to the last inequality gives the desired result. The second inequality in (3.3) follows from the first one. Substitution $y := A$ together with (1.8) give

$$SB(A, \sqrt{Ax}) = SB(y,x) < SB(x,A).$$

Interchanging x with y in the last inequality we obtain the asserted result. For the proof of (3.4) we apply the arithmetic mean-geometric mean inequality to $[(t+x^2)(t+y^2)]^{-1/2}$ to obtain

$$[(t+x^2)(t+y^2)]^{-1/2} < (1/2)[(t+x^2)^{-1} + (t+y^2)^{-1}].$$

Multiplying both sides by $(1/2)(t+y^2)^{-1/2}$ and next integrating from 0 to infinity, we obtain

$$R_C(x^2,y^2) < \frac{1}{2}\left[R_C(y^2,x^2) + \frac{1}{y}\right].$$

Here we have used the identity $R_C(y^2,y^2) = 1/y$. Application of (2.2) to the last inequality gives

$$\frac{1}{SB(x,y)} < \frac{1}{2}\left[\frac{1}{SB(y,x)} + \frac{1}{y}\right] = \frac{1}{H\big(SB(y,x),y\big)}.$$

This completes the proof. ◊

Corollary 3.2. *The following inequalities*

(3.5) $\quad T(A,G) < P, \qquad T(A,Q) < T,$

(3.6) $\quad L < L(A,G), \qquad M < L(A,Q),$

(3.7) $\quad L > H(P,G),\ P > H(L,A),\ M > H(T,A),\ T > H(M,Q)$

hold true.

Proof. Inequalities (3.5) follows from (3.1) and (2.8) by letting $(x, y) :=$ $:= (G, A)$ and $(x, y) := (A, Q)$. Similarly, (3.6) follows from (3.2). Putting $(x, y) := (A, G)$ and $(x, y) := (Q, A)$ we obtain the desired result. Inequalities (3.7) follow from (3.4). The substitutions $(x, y) :=$ $:= (A, G)$, $(x, y) := (G, A)$, $(x, y) := (Q, A)$, and $(x, y) := (A, Q)$ together with application of (2.8) give the desired result. ◊

The first inequality in (3.6) is also established in [8].

Before we state and prove the next result, let us introduce some notation. In what follows, the symbols α and β will stand for positive numbers such that $\alpha + \beta = 1$. The weighted arithmetic mean and the weighted geometric mean of x_n and y_n (see (1.5)) with weights α and β are defined as

$$u_n = \alpha x_n + \beta y_n, \qquad v_n = x_n^{\alpha} y_n^{\beta} \tag{3.8}$$

$(n = 0, 1, \dots)$.

Theorem 3.3. *In order for the sequence $\{u_n\}_0^\infty$ ($\{v_n\}_0^\infty$) to be strictly decreasing (increasing) it suffices that $\alpha = 1/3$ and $\beta = 2/3$. Moreover,*

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = SB(x, y) \tag{3.9}$$

and the inequalities

$$(x_n y_n^2)^{1/3} < SB(x, y) < \frac{x_n + 2y_n}{3} \tag{3.10}$$

hold true for all $n \geq 0$.

Proof. For the proof of the monotonicity property of the sequence $\{u_n\}_0^\infty$ we use (3.8), (1.5), and the arithmetic mean-geometric mean inequality to obtain

$$\begin{aligned} u_{n+1} = \alpha x_{n+1} + \beta y_{n+1} &= \alpha x_{n+1} + \beta (x_{n+1} y_n)^{1/2} < \\ &< \alpha x_{n+1} + \beta \frac{x_{n+1} + y_n}{2} = \\ &= \left(\frac{\alpha}{2} + \frac{\beta}{4}\right) x_n + \left(\frac{\alpha}{2} + \frac{3\beta}{4}\right) y_n. \end{aligned}$$

In order for the inequality $u_{n+1} < u_n$ to be satisfied it suffices that

$$\left(\frac{\alpha}{2} + \frac{\beta}{4}\right) x_n + \left(\frac{\alpha}{2} + \frac{3\beta}{4}\right) y_n = \alpha x_n + \beta y_n.$$

This implies that $\alpha = 1/3$ and $\beta = 2/3$. For the proof of the monotonicity result for the sequence $\{v_n\}_0^\infty$ we follow the lines introduced above to obtain

$$v_{n+1} = x_{n+1}^{\alpha} y_{n+1}^{\beta} = x_{n+1}^{\alpha}(x_{n+1}y_n)^{\beta/2} = \left(\frac{x_n + y_n}{2}\right)^{\alpha+\beta/2} y_n^{\beta/2} >$$
$$> x_n^{\alpha/2+\beta/4} y_n^{\alpha/2+3\beta/4} = x_n^{\alpha} y_n^{\beta},$$

where the last equality holds provided $\alpha = 1/3$ and $\beta = 2/3$. The assertion (3.9) follows from (3.8) and (1.4). Inequalities (3.10) are the obvious consequence of (3.9) and the first statement of the theorem. ◊

Inequalities (3.10) for the Seiffert mean P are obtained in [10].

Corollary 3.4. *The following inequality*

$$\frac{1}{SB(x,y)} < \frac{1}{3}\left(\frac{2}{A} + \frac{1}{y}\right) \tag{3.11}$$

holds true.

Proof. Use of the first inequality in (3.10) with $n = 1$ gives $(A^2y)^{1/3} < SB(x,y)$. Application of the arithmetic mean-harmonic mean inequality with weights leads to

$$\frac{1}{SB(x,y)} < \left(\frac{1}{A}\right)^{2/3}\left(\frac{1}{y}\right)^{1/3} < \frac{2}{3}\frac{1}{A} + \frac{1}{3}\frac{1}{y} = \frac{1}{3}\left(\frac{2}{A} + \frac{1}{y}\right). \ ◊$$

Inequalities connecting the Schwab–Borchardt mean and the celebrated Gauss arithmetic-geometric mean $AGM(x,y) \equiv AGM$ are contained in Th. 3.5. For the reader's convenience, let us recall that the Gauss mean is the iterative mean, i.e.,

$$AGM = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where the sequences $\{a_n\}_0^{\infty}$ and $\{b_n\}_0^{\infty}$ are defined as

(3.12)

$$a_0 = \max(x,y),\ b_0 = \min(x,y),\ a_{n+1} = (a_n + b_n)/2,\ b_{n+1} = \sqrt{a_n b_n}$$

$(n \ge 0)$. (See, e.g., [1], [5]). Clearly,

$$b_0 < b_1 < \cdots < b_n < \cdots < AGM < \cdots < a_n < \cdots < a_1 < a_0, \tag{3.13}$$

$AGM(\cdot,\cdot)$ is a symmetric function in its arguments and

$$AGM(x,y) = AGM(a_n, b_n) \tag{3.14}$$

for all $n \ge 0$.

For later use, let us record two inequalities. If $x > y$, then

$$SB(x,y) < AGM(x,y) \tag{3.15}$$

and

$$AGM(x,y) < SB(x,y) \tag{3.16}$$

provided $x < y$. Inequality (3.15) follows from

$$SB(x,y) < L(x,y) < AGM(x,y),$$

where the first inequality is established in Th. 3.1 and the second one is due to Carlson and Vuorinen [6]. Inequality (3.16) follows from

$$AGM(x,y) < A(x,y) < T(x,y) < SB(x,y).$$

The first inequality is a special case of (3.13) when $n = 1$, the second one appears in [13] and [11], and the last inequality is established earlier (see (3.1)).

We are in position to prove the following

Theorem 3.5. *Let $n = 0, 1, \dots$. The numbers $SB(a_n, b_n)$ form a strictly increasing sequence while $SB(b_n, a_n)$ form a strictly decreasing sequence. Moreover,*

$$SB(a_n, b_n) < AGM < SB(b_n, a_n). \tag{3.17}$$

Proof. Using (1.8), (3.13), (3.15), and (3.14) we obtain

$$\begin{aligned} SB(a_n, b_n) &= SB(a_{n+1}, \sqrt{a_{n+1} b_n}) < SB(a_{n+1}, \sqrt{a_n b_n}) = \\ &= SB(a_{n+1}, b_{n+1}) < AGM(a_{n+1}, b_{n+1}) = AGM(x,y). \end{aligned}$$

Similarly, using (1.8), (3.13), (3.16), and (3.14) one obtains

$$\begin{aligned} SB(b_n, a_n) &= SB(a_{n+1}, \sqrt{a_{n+1} a_n}) > SB(b_{n+1}, a_{n+1}) > \\ &> AGM(b_{n+1}, a_{n+1}) = AGM(x,y). \end{aligned}$$

The proof is complete. ◊

Corollary 3.6. *Let the numbers a_n and b_n $(n \ge 1)$ be the same as in (3.12). If $a_0 = A$ and $b_0 = G$, then*

$$L < L(a_n, b_n) < AGM(x,y) < P(a_n, b_n) < P \tag{3.18}$$

for all $n \ge 0$. Similarly, if $a_0 = Q$ and $b_0 = A$, then

$$M < L(a_n, b_n) < AGM(A,Q) < P(a_n, b_n) < T \tag{3.19}$$

$(n \ge 0)$.

Proof. Inequalities (3.18) follow immediately from Th. 3.5 and from the formulas $SB(a_0, b_0) = SB(A, G) = L$, $SB(a_{n+1}, b_{n+1}) = L(a_n, b_n)$, $SB(b_0, a_0) = SB(G, A) = P$ and $SB(b_{n+1}, a_{n+1}) = P(b_n, a_n) = P(a_n, b_n)$ $(n \ge 0)$. Since the proof of (3.19) goes along the lines introduced above, it is omitted. ◊

Appendix 1. Bounds for the Schwab–Borchardt mean

We shall prove the following:

Proposition A1. *If $x > y$, then*

$$\text{(A1.1)} \qquad \frac{2x^2 - y^2}{2x \ln(2x/y)} < SB(x,y) < \frac{2x^2 - y^2}{2x \ln(2x/y) - (y^2/x)\ln 2}.$$

Otherwise, if $y > x \geq 0$, then

$$\text{(A1.2)} \qquad \frac{4y^3}{\pi(x^2 + 2y^2) - 4xy} \leq SB(x,y) \leq \frac{4y^3}{\pi(x^2/2 + 2y^2) - 4xy}.$$

Equalities hold in (A1.2) *if and only if $x = 0$.*

Proof. Assume that $x > y$. The following asymptotic expansion

$$R_C(x^2, y^2) = \frac{1}{2x}\left(\ln \frac{4x^2}{y^2} + \frac{y^2}{2x^2 - y^2} \ln \frac{\theta x^2}{y^2}\right),$$

$1 < \theta < 4$, is established in [7, Eq. (23)]. Letting above $\theta = 1$ and $\theta = 4$ and next using (2.2) we obtain inequalities (A1.1). Assume now that $y > x \geq 0$. Then

$$R_C(x^2, y^2) = \frac{\pi}{2y} - \frac{x}{y^2} + \frac{\pi x^2}{4y^3}\theta,$$

where $y/(x + y) \leq \theta \leq 1$ (see [7, Eq. (22)]). This in conjunction with (2.2) gives (A1.2). $\Diamond$

It is worth mentioning that the bounds (A1.1) are sharp when $x \gg y$ while (A1.2) are sharp if $y \gg x$.

Appendix 2. Inequalities connecting sequences (1.5) and (3.8)

Let the numbers x_n and y_n ($n \geq 0$) be the same as in the Schwab–Borchardt algorithm (1.5). Further, let u_n and v_n be defined in (3.8) with $\alpha = 1/3$ and $\beta = 2/3$, i.e.,

$$u_n = \frac{x_n + 2y_n}{3}, \qquad v_n = (x_n y_n^2)^{1/3}$$

($n \geq 0$). These numbers have been used in [10, Ths. 1 and 2] to obtain several inequalities involving the Seiffert mean P and other means.

The following inequalities, which hold true for all $n \geq 0$, show that the numbers u_n and v_n provide sharper bounds for SB than those obtained from x_n and y_n. We have

$$y_n < v_n \quad \text{and} \quad u_n < x_n \qquad \text{if } y < x$$

and

$$x_n < v_n \quad \text{and} \quad u_n < y_n \qquad \text{if } x < y.$$

We shall prove that these inequalities can be improved if x and y belong to certain cones in the plane.

Proposition A2. *Let $c = \sqrt{5} - 2 = 0.236\ldots$ and assume that $x > 0$, $y > 0$ with $x \neq y$. If*

$$cx < y < x, \tag{A2.1}$$

then

$$y_{n+1} < v_n \quad \text{and} \quad u_n < x_{n+1} \tag{A2.2}$$

for all $n \geq 0$. Similarly, if

$$cy < x < y, \tag{A2.3}$$

then

$$x_{n+1} < v_n \quad \text{and} \quad u_n < y_{n+1} \tag{A2.4}$$

for $n = 0, 1, \ldots$.

Proof of inequalities (A2.2) and (A2.4) is based upon results that are contained in the following lemmas.

Lemma 1. *Let x and y be distinct positive numbers. If $cx < y < x$, then*

$$\frac{x+y}{2} < (xy^2)^{1/3}. \tag{A2.5}$$

If $x < y$, then

$$\frac{x+2y}{3} < \sqrt{\frac{x+y}{2}\, y}\,. \tag{A2.6}$$

Proof. For the proof of (A2.5) let us consider a quadratic function

$$p(y) = y^2 + 4xy - x^2 = [y - (\sqrt{5}-2)x][y + (\sqrt{5}+2)x].$$

It follows that $p(y) > 0$ if $cx < y$. Inequality $p(y) > 0$ can be written as $(x-y)^2 < 2y(x+y)$. Multiplying both sides by $x - y > 0$ we obtain the desired result. In order to establish the inequality (A2.6) let us introduce a quadratic function

$$q(y) = y^2 + xy - 2x^2 = (y - x)(y + 2x).$$

Clearly $q(y) > 0$ if $x < y$. Inequality $q(y) > 0$ is equivalent to $x^2 < \frac{1}{2}(xy + y^2)$. Adding $4xy + 4y^2$ to both sides of the last inequality we obtain

$$\left(\frac{x + 2y}{3}\right)^2 < \frac{x + y}{2} y.$$

Hence, the assertion follows. $\Diamond$

Lemma 2. *If $cx < y < x$, then the following inequalities*

$$cx_n < y_n < x_n \tag{A2.7}$$

hold true for all $n \geq 0$. Similarly, if $cy < x < y$, then

$$cy_n < x_n < y_n \tag{A2.8}$$

for all $n \geq 0$.

Proof. The second inequalities in (A2.7) and (A2.8) follow from (1.7) and (1.6), respectively. For the proof of the first inequalities in (A2.7) and (A2.8) we will use the mathematical induction on n. There is nothing to prove when $n = 0$. Assume that $cx_n < y_n$, for some $n > 0$. Using (1.5), the inductive assumption and (1.7) we obtain

$$cx_{n+1} = c\frac{x_n + y_n}{2} < \frac{y_n + cy_n}{2} = \frac{\sqrt{5} - 1}{2} y_n < y_n < y_{n+1}.$$

Now let $cy < x < y$. Assume that $cy_n < x_n$ for some $n > 0$. Using (1.5), the arithmetic mean-geometric mean inequality and the inductive assumption we obtain

$$cy_{n+1} = c\sqrt{x_{n+1} y_n} < c\frac{x_{n+1} + y_n}{2} < \frac{1}{2}(cx_{n+1} + x_n) < \frac{1}{2}(cy_n + x_n) <$$
$$< \frac{1}{2}(y_n + x_n) = x_{n+1}. \ \Diamond$$

Proof of Prop. A2. For the proof of the first inequality in (A2.2) we use (A2.7) and (A2.5) to obtain

$$\frac{x_n + y_n}{2} < (x_n y_n^2)^{1/3} \tag{A2.9}$$

($n \geq 0$). Making use of (1.5) and (A2.9) we obtain

$$y_{n+1} = (x_{n+1} y_n)^{1/2} = \left(\frac{x_n + y_n}{2}\right)^{1/2} y_n^{1/2} < (x_n y_n^2)^{1/3} = v_n.$$

The second inequality in (A2.2) can be established as follows. We add to both sides of $y_n < x_n$ (see (1.7)) $2x_n + 3y_n$ and next divide the resulting inequality by 6 to obtain the desired result. For the proof of the first inequality in (A2.4) we use (A2.5) with x replaced by y and y replaced by x, the inequalities (A2.8) and $x_n < y_n$ (see (1.6)) to obtain

$$x_{n+1} = \frac{x_n + y_n}{2} < (x_n^2 y_n)^{1/3} < (x_n y_n^2)^{1/3} = v_n.$$

The second inequality in (A2.4) is obtained with the aid of (A2.8), (A2.6), and (1.5). We have

$$u_n = \frac{x_n + 2y_n}{3} < \sqrt{\frac{x_n + y_n}{2} y_n} = y_{n+1}. \ \Diamond$$

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