Mathematica Pannonica
14/2 (2003), 237–252

SOME CONTINUITY AND APPROXIMATION PROPERTIES OF A COUNTABLE ITERATED FUNCTION SYSTEM

Nicolae-Adrian **Secelean**

Department of Mathematics, "Lucian Blaga" University of Sibiu, Romania

Received: March 2002

MSC 2000: 28 A 80; 37 C 40, 37 C 70

Keywords: Hausdorff metric, iterated function system, countable iterated function system, countable iterated function system with probabilities, Hutchinson metric, invariant measure.

Abstract: This paper considers the problem of extending the notion of an IFS, respectively IFS with probabilities, to the case of countable iterated function system (abbreviated CIFS), respectively with probabilities (CIFSp). We prove that, in the case of CIFS, the attractor and the invariant measure are continuous with respect to a parameter, the proof being a variant of that presented in [5]. Furthermore, we show that, if a CIFS is approximated by a sequence of CIFS then the attractor will be respectively approximated. Finally, we show that if the system of probabilities of an CIFSp is the limit of a sequence of systems of probabilities, then the invariant measure is the limit of corresponding invariant measures of these CIFSp.

1. Preliminaries

We shall present some notions and results used in the sequel (more complete and rigorous treatments may be found in [1], [4], [6], [7]).

E-mail address: secelean@mail.ulbsibiu.ro

1.1. Hausdorff metric. Let (X, d) be a complete metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X.

The function $\delta : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$,

$$\delta(A, B) = \max\{\mathrm{d}(A, B),\ \mathrm{d}(B, A)\},$$
$$\text{where } \mathrm{d}(A, B) = \sup_{x\in A}(\inf_{y\in B} \mathrm{d}(x, y)),\ \forall\, A, B \in \mathcal{K}(X)$$

is called the Hausdorff metric.

The set $\mathcal{K}(X)$ is a complete metric space with respect to this metric δ. The following obvious lemma will be necessary in the sequel:

Lemma 1. *If $(E_n)_n$, $(F_n)_n$ are two sequences of sets in $\mathcal{K}(X)$, then*

$$\delta\left(\overline{\bigcup_{n\geq 1} E_n}, \overline{\bigcup_{n\geq 1} F_n}\right) \leq \sup_n \delta(E_n, F_n).$$

Proposition 1. [6, Th.1.1] *Let $(E_n)_{n\geq 1}$ be a sequence of sets in $\mathcal{K}(X)$. If $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}^*$ and the set $\bigcup_{n\geq 1} E_n$ is relatively compact, then*

$$\lim_n E_n = \overline{\bigcup_{n\geq 1} E_n},$$

the limit is taken with respect to the Hausdorff metric and the bar means the closure.

1.2. Iterated Function Systems (see [4], [1], [2]). Let (X, d) be a complete metric space. A set of contractions $(\omega_n)_{n=1}^N$, $N \geq 1$, is called according to M. Barnsley ([1]) an iterated function system (IFS). Such a system of maps induces a set function $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$,

$$\mathcal{S}(E) = \bigcup_{n=1}^{N} \omega_n(E)$$

which is a contraction on $\mathcal{K}(X)$ with contraction ratio $r \leq \max_{1\leq n\leq N} r_n$, r_n being the contraction ratio of ω_n, $n = 1, \ldots, N$. According to the Banach contraction principle, there is a unique set $A \in \mathcal{K}(X)$ which is invariant with respect to $\mathcal{S}$, that is

$$A = \mathcal{S}(A) = \bigcup_{n=1}^{N} \omega_n(A).$$

We say the set $A \in \mathcal{K}(X)$ is the attractor of IFS $(\omega_n)_{n=1}^N$.

1.3. The invariant measure of an IFS with probabilities (see [1], [4]). Let (X, d) be a compact metric space and $(\omega_n)_{n=1}^N$ an IFS of X.

Let $p_1, \ldots, p_N \in (0,1)$ such that $\sum\limits_{n=1}^{N} p_n = 1$. Then $((\omega_n)_{n=1}^N, (p_n)_{n=1}^N)$ is called iterated function system with probabilities (IFSp). We define the support of a measure μ on X to be the closed set

$$\text{supp}\mu = X \setminus \cup\{V : \ V \text{ open}, \ \mu(V) = 0\}.$$

Let μ be a Borel measure on X. If $\mu(X) = 1$ then μ is said to be normalized. Let $\mathcal{B}(X)$ denote the family of Borel subsets of X and $\mathcal{B}$ the set of normalized Borel measures on X. The map $\mathrm{d}_H : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$,

$$\mathrm{d}_H(\mu, \nu) = \sup\left\{ \int_X f d\mu - \int_X f d\nu : f : X \longrightarrow \mathbb{R}, \ |f(x) - f(y)| \leq \mathrm{d}(x,y), \forall x, y \in X \right\}$$

for all $\mu, \nu \in \mathcal{B}$, is a metric, namely the Hutchinson metric (or the Monge–Kantorovich metric). $(\mathcal{B}, \mathrm{d}_H)$ is a compact metric space ([1, ch. IX, Th. 5.1]).

The Markov operator associated with IFSp is the function $\mathrm{M} : : \mathcal{B} \longrightarrow \mathcal{B}$ defined by

$$\mathrm{M}(\nu) = p_1 \nu \circ \omega_1^{-1} + p_2 \nu \circ \omega_2^{-1} + \cdots + p_N \nu \circ \omega_N^{-1}, \ \forall \nu \in \mathcal{B}.$$

Definition 1. We reserve the notation χ_A for the characteristic function of a set $A \subset X$. It is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \in X \setminus A. \end{cases}$$

A function $f : X \rightarrow \mathbb{R}$ is called simple if can be written in the form

$$f(x) = \sum_{i=1}^{N} y_i \chi_{A_i}$$

where N is a positive integer, $A_i \in \mathcal{B}(X)$ and $y_i \in \mathbb{R}$ for $i = 1, \ldots, N$, $\bigcup\limits_{i=1}^{N} A_i = X$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Lemma 2. [1, ch. IX, L.6.1] *Let $f : X \longrightarrow \mathbb{R}$ be either a simple function or a continuous function. Choose $\nu \in \mathcal{B}$. Then $\int_X f d(\mathrm{M}(\nu)) = = \sum\limits_{n=1}^{N} p_n \int_X f \circ \omega_n d\nu$.*

The associated Markov operator with the IFSp is a contraction mapping with respect to the Hutchinson metric on $\mathcal{B}$. In particular, there is a unique measure $\mu \in \mathcal{B}$ such that $\mathrm{M}(\mu) = \mu$. μ is called the invariant measure or the Hutchinson measure of the IFSp.

Moreover, the support of μ is the attractor of the IFS $(\omega_n)_{n=1}^N$.

We consider further a metric space (T, d_T). For each $n = 1, \ldots, N$, we define $\omega_n : T \times X \longrightarrow X$, $r_n : T \longrightarrow [0,1)$ such that $\sup\limits_{t\in T} r_n(t) < 1$ and

$$\mathrm{d}(\omega_n(t,x), \omega_n(t,y)) \leq r_n(t)\mathrm{d}(x,y),$$

for all $t \in T$ and $x, y \in X$.

For every $t \in T$, we denote μ_t the Hutchinson measure associated with the IFSp $((\omega_n(t,\cdot))_{n=1}^N, (p_n)_{n=1}^N)$.

Theorem 1. [2, Th. 3.4] *We assume that, in the conditions above, for each $n \in \{1, \ldots, N\}$ and for all $x \in X$, the maps $t \mapsto \omega_n(t,x)$ are continuous. Then the function*

$$t \mapsto \mu_t$$

is continuous as a map from (T, d_T) to $(\mathcal{B}, \mathrm{d}_H)$.

1.4. Countable Iterated Function Systems (more details for this section may be found in [6]). Suppose that (X, d) is a compact metric space.

A sequence of contractions $(\omega_n)_{n\geq 1}$ on X whose contraction ratios are, respectively r_n, $r_n \geq 0$, such that $\sup\limits_n r_n < 1$ is called a countable iterated function system, for simplicity CIFS.

Let $(\omega_n)_{n\geq 1}$ be a CIFS.

We define the set function $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$, by

$$\mathcal{S}(E) = \overline{\bigcup_{n\geq 1} \omega_n(E)},$$

where the bar means the closure of the corresponding set. Then $\mathcal{S}$ is a contraction map on $(\mathcal{K}(X), \delta)$ with contraction ratio $r \leq \sup\limits_n r_n$. According to the Banach contraction principle, there exists a unique non-empty compact set $A \subset X$ which is invariant for the family $(\omega_n)_{n\geq 1}$, that is

$$A = \mathcal{S}(A) = \overline{\bigcup_{n\geq 1} \omega_n(A)}.$$

The set A is called the attractor of CIFS $(\omega_n)_{n\geq 1}$.

We denote by A_k and, respectively, by $\mathcal{S}_k$ the attractor and the contraction associated to the partial IFS $(\omega_n)_{n=1}^k$, for $k \geq 1$.

Theorem 2. [6, Cor. 2.2] *The attractor of* CIFS $(\omega_n)_{n\geq 1}$ *is*

$$A = \overline{\bigcup_{k\geq 1} A_k} = \lim_k A_k,$$

the limit being taken in $(\mathcal{K}(X), \delta)$.

Hence, the attractor of CIFS $(\omega_n)_{n\geq 1}$ *is approximated by the attractors of partial* IFS $(\omega_n)_{n=1}^k$, $k \geq 1$.

We note that each IFS $(\omega_n)_{n=1}^k$ can be considered like an CIFS according to

Proposition 2. [6, Prop. 2.2] *The set* A_k, $k \in \mathbb{N}^*$, *is the attractor of* IFS $(\omega_n)_{n=1}^k$ *if and only if* A_k *is the attractor of* CIFS $(\omega_n)_{n\geq 1}$, *where* $\omega_n \equiv e_1$ *(the fixed point of* ω_1*), for all* $n > k$.

1.5. The associated invariant measure of an CIFS with probabilities (see [7]). Let (X, d) be a compact metric space and $(\omega_n)_{n\geq 1}$ a CIFS on X. We consider a sequence of probabilities $(p_n)_{n\geq 1}$ with $0 < p_n < 1$, $\sum\limits_{n=1}^{\infty} p_n = 1$.

The pair $((\omega_n)_{n\geq 1},(p_n)_{n\geq 1})$ is called countable iterated function system with probabilities and we will denote it by CIFSp. We define the map $\mathrm{M} : \mathcal{B} \longrightarrow \mathcal{B}$,

$$\mathrm{M}(\nu) = \sum_{n=1}^{\infty} p_n \nu \circ \omega_n^{-1}, \text{ for all } \nu \in \mathcal{B}.$$

M is called the Markov operator associated with CIFSp $((\omega_n)_{n\geq 1}, (p_n)_{n\geq 1})$.

Lemma 3. [7, Lemma 3] *Let* $f : X \longrightarrow \mathbb{R}$ *be a continuous function and* $\nu \in \mathcal{B}$. *Then*

$$\int_X f d(\mathrm{M}(\nu)) = \sum_{n=1}^{\infty} p_n \int_X (f \circ \omega_n) d\nu.$$

Theorem 3. [7, Th. 2] *With the above notations,* M *is a contraction map with the contraction ratio not greater than* r *with respect to the Hutchinson metric on* $\mathcal{B}$. *That is*

$$\mathrm{d}_H(\mathrm{M}(\nu), \mathrm{M}(\mu)) \leq r \mathrm{d}_H(\nu, \mu), \quad \forall \nu, \mu \in \mathcal{B}.$$

In particular, there is a unique measure $\mu \in \mathcal{B}$ *which is invariant for* M, $\mathrm{M}(\mu) = \mu$.

The unique normalized Borel measure which exists according to the above theorem is called the Hutchinson measure associated with CIFSp.

Now, we consider for every $k \geq 2$, the partial iterated function systems $(\omega_n)_{n=1}^{k}$ with the probabilities $p_1, p_2, \ldots, p_{k-1}, \sum_{n=k}^{\infty} p_n$. The associate Markov operator is

$$\mathrm{M}_k(\nu) = \sum_{n=1}^{k-1} p_n \cdot \nu \circ \omega_n^{-1} + \Big(\sum_{n=k}^{\infty} p_n\Big) \cdot \nu \circ \omega_k^{-1}, \ \ \nu \in \mathcal{B}.$$

By **1.3** it follows that, for every $k \geq 2$, there exists uniquely $\mu_k \in \mathcal{B}$ such that $\mathrm{M}_k(\mu_k) = \mu_k$.

Theorem 4. [7, Th.3] *With the above notations, on has $\mu_k \xrightarrow{k} \mu$ with respect to the Hutchinson metric d_H.*

2. Continuity of attractors for CIFS

In this section (T, d_T), (X, d) are two metric spaces, the second being compact. We consider further the sequences of functions

$$\omega_n : T \times X \longrightarrow X, \text{ respectively, } r_n : T \longrightarrow [0,1), \ \ n \in \mathbb{N}^*,$$

with the following property: for each $t \in T$, one has

a) $\mathrm{d}(\omega_n(t,x), \omega_n(t,y)) \leq r_n(t)\mathrm{d}(x,y) \ \forall x, y \in X$;

b) $\sup_n r_n(t) < 1$.

We define $\mathcal{S} : T \times \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$,

$$\mathcal{S}(t, B) = \overline{\bigcup_{n\geq 1} \omega_n(t, B)}, \ \forall t \in T, \ B \in \mathcal{K}(X).$$

By **1.3** it follows that, for each $t \in T$, $\mathcal{S}(t, \cdot)$ is a contraction map on $\mathcal{K}(X)$ with the contraction ratio $r(t) \leq \sup_n r_n(t) < 1$.

Theorem 5. *Assume that there is a constant $C > 0$ so that*

$$\mathrm{d}(\omega_n(t,x), \omega_n(s,x)) \leq C\mathrm{d}_T(t,s), \tag{1}$$

for all $x \in X$, $t, s \in T$, $n \in \mathbb{N}^*$.

Then, for every $B \in \mathcal{K}(X)$, one has $\delta(\mathcal{S}(t,B), \mathcal{S}(s,B)) \leq C\mathrm{d}_T(t,s)$, and hence $\mathcal{S}(\cdot, B)$ is uniformly continuous on T.

Proof. Choose $t, s \in T$ and $B \in \mathcal{K}(X)$. Then, for each $n \in \mathbb{N}^*$, one has

$$\delta(\omega_n(t,B), \omega_n(s,B)) \leq C\mathrm{d}_T(t,s).$$

Indeed, by symmetry, it is sufficient to prove that

$$\mathrm{d}(\omega_n(t,B),\omega_n(s,B)) \le C\mathrm{d}_T(t,s).$$

If $y \in \omega_n(t,B)$, then there exists $x \in B$ such that $y = \omega_n(t,x)$. Put $z = \omega_n(s,x)$. It follows

$$\mathrm{d}(y,z) = \mathrm{d}(\omega_n(t,x),\omega_n(s,x)) \le C\mathrm{d}_T(t,s)$$

hence $\sup\limits_{y\in\omega_n(t,B)} \inf\limits_{z\in\omega_n(s,B)} \mathrm{d}(x,y) \le C\mathrm{d}_T(t,s)$.

Now, using Lemma 1, we deduce

$$\delta(\mathcal{S}(t,B),\mathcal{S}(s,B)) = \delta\left(\overline{\bigcup_{n\ge1}\omega_n(t,B)}, \overline{\bigcup_{n\ge1}\omega_n(s,B)}\right) \le$$

$$\le \sup_n \delta(\omega_n(t,B),\omega_n(s,B)) \le C\mathrm{d}_T(t,s). \ \Diamond$$

Remarks. 1° If we assume, like in the case of IFS, only the condition that the maps $t \mapsto \omega_n(t,B)$, $n \ge 1$, are continuous for every $B \in \mathcal{K}(X)$, it did not follow that the function $t \mapsto \mathcal{S}(t,B)$ is continuous, as it follows by the following counter-example:

Let us consider $T = [0,1]$, $X = [-2,2]$ and the contraction maps

$$\omega_n(t,x) = \frac{1}{2}x + \sin\frac{nt\pi}{2}, \ n \ge 1.$$

It is clear that the conditions a) and b) hold with $r_n \equiv \frac{1}{2}$ and that $\omega_n(\cdot,x)$ is continuous for all $x \in X$, $n \ge 1$.

Choosing $x \in X$ and $B = \{x\}$, we will show that $\mathcal{S}(\cdot,B)$ is not continuous in $t_0 = 0$.

Thus, we consider the sequence $t_k = \frac{1}{k} \to 0$. We have

$$\mathcal{S}(t_k,\{x\}) = \overline{\bigcup_{n=1}^{\infty}\left\{\frac{1}{2}x + \sin\frac{n\pi}{2k}\right\}} = \left[\frac{1}{2}x - 1, \frac{1}{2}x + 1\right], \ \forall k \in \mathbb{N},$$

$$\mathcal{S}(t_0,\{x\}) = \overline{\bigcup_{n=1}^{\infty}\left\{\frac{1}{2}x\right\}} = \left\{\frac{1}{2}x\right\},$$

but $\delta(\mathcal{S}(t_k,\{x\}),\mathcal{S}(t_0,\{x\})) = 1, \ \forall k \in \mathbb{N}^*$.

2° Since, in $(\mathcal{K}(X),\delta)$, we have that (see Prop. 1) for each $t \in T$,

$$\mathcal{S}_k(t,B) := \bigcup_{n=1}^{k}\omega_n(t,B) \xrightarrow{k} \overline{\bigcup_{n\ge1}\omega_n(t,B)} = \mathcal{S}(t,B), \ \forall B \in \mathcal{K}(X),$$

it follows that, if the maps $\omega_n(\cdot,x)$, $n \ge 1$ are only continuous but they do not verify condition (1), the convergence of the sequences of functions $(\mathcal{S}_k(t,\cdot))_k$ is not uniform.

We will use the following elementary lemma:

Lemma 4. *Let (Y,ρ) be a complete metric space and $(f_k)_{k\geq 1}$ a sequence of contractions on Y with the contraction ratios, respectively $r_k \in [0,1)$, such that $r = \sup_k r_k < 1$. We assume that $(f_k)_{k\geq 1}$ is pointwise convergent to $f : Y \longrightarrow Y$. Then*

a) *f is a contraction with ratio less than or equal to r;*

b) $\lim_k \xi_k = \xi$ *(ξ_k, resp. ξ are the fixed points of f_k, resp. f).*

Theorem 6. *Assume that the condition (1) is fulfilled. For each $t \in T$ we denote $A(t)$ the attractor of CIFS $(\omega_n(t,\cdot))_{n\geq 1}$. Then the function*

$$t \mapsto A(t)$$

is continuous from T to $\mathcal{K}(X)$.

Proof. Let $t_0 \in T$ and $(t_k)_{k\geq 1} \subset T$, $t_k \longrightarrow t_0$. Then, by Th. 5, we deduce that the sequence of contraction mappings $(\mathcal{S}(t_k,\cdot))_k$ having the ratios, respectively $r(t_k) \leq \sup_n r_n(t_k)$ is pointwise convergent to $\mathcal{S}(t_0,\cdot)$ on the complete metric space $(\mathcal{K}(X),\delta)$.

On the other hand, for each $k \in \mathbb{N}^*$, $A(t_k)$ is the fixed point of contraction map $\mathcal{S}(t_k,\cdot)$, respectively $A(t_0)$ is the fixed point of $\mathcal{S}(t_0,\cdot)$.

By applying Lemma 4 and using the fact that $\sup_k r(t_k) < 1$, it follows that

$$A(t_k) \xrightarrow{k} A(t_0). \ \Diamond$$

In the following example one can see the continuous dependence of the attractor of an CIFS.

Example 1. (The CIFS of Sierpinski-infinite type [6].) We denote

$$X = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\}$$

the plane surface of the closed triangle having its vertices in the points $(0,0)$, $(0,1)$, $(1,0)$.

Let $p \in \mathbb{N}$, $p \geq 2$, and consider the maps

$$\omega_{ij}(t,(x,y)) = \left(\left(\frac{1}{p^i} - \frac{t}{10}\right)x + \frac{t}{5}y + (j-1)\cdot\frac{1}{p^i} + t, -\frac{3}{10}tx + \right.$$
$$\left. + \left(\frac{1}{p^i} + \frac{3}{10}\right)y + \left(\frac{p^i-1}{p-1} - j\right)\cdot\frac{1}{p^i} - \frac{t}{10}\right),$$

$i = 1,2,\ldots,$ $j = 1,2,\ldots,\dfrac{p^i-1}{p-1}$, $t \in [0,1]$. In Fig. 1 are presented the images for $p = 2$, $t = 1$, $t = 3^{-1}$, $t = 10^{-1}$, $t = 10^{-5}$.

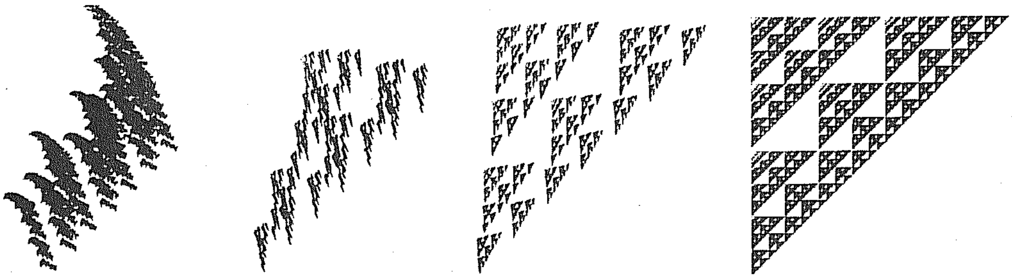

Fig. 1. The evolution of attractors of CIFS Sierpinski-infinite type for different values of parameter

Theorem 7. *Suppose that $T = X$ and that the sequences of maps $(\omega_n)_{n\geq 1}$, $(r_n)_{n\geq 1}$ satisfy the condition (1) for $C \in (0,1)$.*

Then, for each $B \in \mathcal{K}(X)$, there exists a point $x_B \in X$ such that $x_B \in \mathcal{S}(x_B, B)$.

In particular, if $A(x)$ is the attractor of CIFS $(\omega_n(x,\cdot))_n$, *there exists a point $x_0 \in X$ with $x_0 \in A(x_0)$.*

Proof. Choose $B \in \mathcal{K}(X)$. By Th. 5 it follows that

$$\delta(\mathcal{S}(x,B), \mathcal{S}(y,B)) \leq C\mathrm{d}(x,y), \quad \forall x, y \in X. \tag{2}$$

Let $p_0 \in X$ be a fixed point and $p_1 \in \mathcal{S}(p_0, B)$. Then, from (2),

$$\delta(\mathcal{S}(p_0,B), \mathcal{S}(p_1,B)) \leq C\mathrm{d}(p_0,p_1),$$

and hence $\sup\limits_{p\in\mathcal{S}(p_0,B)} \inf\limits_{q\in\mathcal{S}(p_1,B)} \mathrm{d}(p,q) \leq C\mathrm{d}(p_0,p_1)$.

Thus, for $p_1 \in \mathcal{S}(p_0, B)$, there is $p_2 \in \mathcal{S}(p_1, B)$ such that

$$\mathrm{d}(p_1,p_2) \leq C\mathrm{d}(p_0,p_1).$$

When proceeding in this way, we obtain a sequence $(p_i)_{i\geq 0} \subset X$ which has the following properties:

α) $p_{i+1} \in \mathcal{S}(p_i, B)$;

β) $\mathrm{d}(p_i, p_{i+1}) \leq C\mathrm{d}(p_{i-1}, p_i)$

for $i = 1, 2, \ldots$.

We deduce that $\mathrm{d}(p_i,p_{i+1}) \leq C^i\mathrm{d}(p_0,p_1)$, $i = 1, 2, \ldots$, hence

$$\mathrm{d}(p_i, p_{i+j}) \leq (c^i + c^{i+1} + \cdots + c^{i+j-1})\mathrm{d}(p_0,p_1) =$$

$$= c^i \cdot \frac{1-c^j}{1-c}\mathrm{d}(p_0,p_1), \ \forall i,j \in \mathbb{N}^*.$$

It follows that the sequence $(p_i)_i$ is a Cauchy sequence. Put $x_B :=$ $:= \lim_i p_i$.

By (2) we deduce that $(\mathcal{S}(p_i,B))_i$ converges to $\mathcal{S}(x_B,B)$ and, since $p_i \in \mathcal{S}(p_{i-1},B)$, it follows that $x_B \in \mathcal{S}(x_B,B)$.

The second assertion is obvious by taking into account that for each $x \in X$, $A(x) = \mathcal{S}(x, A(x))$. $\Diamond$

Proposition 3. *Let $\omega_k^n : X \longrightarrow X$, $k,n \in \mathbb{N}^*$ be contraction maps with contraction ratios $r_k^n \in [0,1)$, $r := \sup\limits_{n,k} r_k^n < 1$ which constitutes a sequence of* CIFS, *the system $(\omega_k^n)_{n\geq 1}$ having the attractor A_k for every $k = 1,2,\dots$.*

We accept that there is a sequence of functions $(\omega^n)_{n\geq 1}$, where $\omega^n : X \longrightarrow X$, $n \in \mathbb{N}^*$ are such that for each $x \in X$,

$$\sup_n \mathrm{d}(\omega_k^n(x), \omega^n(x)) \xrightarrow{k} 0. \tag{3}$$

Then $(\omega^n)_n$ is an CIFS, *whose attractor A is approximated by $(A_k)_k$. That is*

$$A_k \xrightarrow{k} A$$

in the Hausdorff metric.

Proof. By (3) it follows immediately that, for each $n \in \mathbb{N}$, one has $\omega_k^n \to \omega^n$ (pointwise) and hence, using Lemma 4, ω^n is a contraction map with contraction ratio not greater than r.

We will prove that $\delta(A_k, A) \xrightarrow{k} 0$.

First we check that

$$\sup_n \delta(\omega_k^n(A), \omega^n(A)) \xrightarrow{k} 0. \tag{4}$$

Suppose that the relation (4) did not hold and let $\varepsilon > 0$ such that

$$\sup_n \delta(\omega_k^n(A), \omega^n(A)) > \varepsilon, \quad \text{for any } k \geq 1.$$

Then, for each $k \geq 1$, there is a $n_k \geq 1$ so that $\delta(\omega_k^{n_k}(A), \omega^{n_k}(A)) > \varepsilon$.

Taking, eventually, a subsequence, we distinguish two cases:

A. $\mathrm{d}(\omega_k^{n_k}(A), \omega^{n_k}(A)) = \sup\limits_{x\in\omega_k^{n_k}(A)} \inf\limits_{y\in\omega^{n_k}(A)} \mathrm{d}(x,y) > \varepsilon$.

It follows that there exists a point $x_k' \in A$ such that for every $y' \in A$, we have

$$\mathrm{d}(\omega_k^{n_k}(x_k'), \omega^{n_k}(y')) > \varepsilon, \ \forall\, k \in \mathbb{N}. \tag{5}$$

Since the sequence $(x_k')_k$ is contained in the compact set A, we deduce that it contains a convergent subsequence which, for simplicity, will be denoted in the same way. Thus $x_k' \to x' \in A$.

Then, by taking $y' = x'$ in (5), we obtain

$$\varepsilon < \mathrm{d}(\omega_k^{n_k}(x'_k), \omega^{n_k}(x')) \leq \mathrm{d}(\omega_k^{n_k}(x'_k), \omega_k^{n_k}(x')) + \mathrm{d}(\omega_k^{n_k}(x'), \omega^{n_k}(x')) \leq$$

$$\leq r\mathrm{d}(x'_k, x') + \sup_n \mathrm{d}(\omega_k^n(x'), \omega^n(x')), \ \forall k \in \mathbb{N}^*.$$

This inequality contradicts (3) and the fact that $x'_k \to x'$.

B. The case $\mathrm{d}(\omega^{n_k}(A), \omega_k^{n_k}(A)) > \varepsilon$ may be treated in an analogous way.

Now we can write, using Lemma 1,

$$\delta(A_k, A) = \delta\left(\overline{\bigcup_{n\geq 1} \omega_k^n(A_k)}, \overline{\bigcup_{n\geq 1} \omega^n(A)}\right) \leq \sup_n \delta(\omega_k^n(A_k), \omega^n(A)) \leq$$

$$\leq \sup_n \delta(\omega_k^n(A_k), \omega_k^n(A)) + \sup_n \delta(\omega_k^n(A), \omega_n(A)) \leq$$

$$\leq r\delta(A_k, A) + \sup_n \delta(\omega_k^n(A), \omega_n(A)).$$

It follows, using (4),

$$\delta(A_k, A) \leq \frac{1}{1-r} \sup_n \delta(\omega_k^n(A), \omega_n(A)) \xrightarrow{k} 0. \ \Diamond$$

We will show that the condition (3) from the above proposition is not necessary.

Thus, we consider a CIFS $(\omega^n)_{n\geq 1}$ on X, the contraction maps having the contraction ratios, respectively r_n, $n \geq 1$. We denote e_1 the unique fixed point of ω^1. Assume that there are $C > 0$ and $N \in \mathbb{N}^*$ so that

$$\mathrm{d}(e_1, \omega^n(x)) > C, \tag{6}$$

for all $n \geq N$ and all $x \in X$.

The following example shows that there exists an CIFS as above.

Example 2. (The CIFS of Cantor-infinite type [6].) We work in the compact metric space $X = [0, 1]$ with the Euclidean metric. Let $q \in \left(0, \frac{1}{2}\right]$.

We define, for each $n \in \mathbb{N}^*$, the sequence of contractions $\omega_n : [0,1] \longrightarrow [0,1]$,

$$\omega_n(x) = q^n x + \alpha_n,$$

where $\alpha_1 = 0$, $\alpha_n = q^{n-1} + \left(\frac{1-2q}{2-3q}\right)^{n-1} + \alpha_{n-1}$, $n \geq 2$.

Thus, for this CIFS, we have $C \in \left(0, \frac{1}{3}\right)$, $N \geq 2$, $e_1 = 0$.

Now we define a sequence of CIFS $((\omega_k^n)_n)_k$ as follows:

$$\omega_k^n := \omega^n, \text{ if } n \le k \text{ and } \omega_k^n \equiv e_1 \text{ for } n > k.$$

That is $(\omega_k^n)_n = (\omega^1, \omega^2, \ldots, \omega^k, e_1, e_1, \ldots)$, $k = 1, 2, \ldots$.

Proposition 4. *In the above conditions, we have*

a) $\forall n \ge 1,\ \forall x \in X,\ \omega_k^n(x) \xrightarrow{k} \omega^n(x)$;

b) $\sup\limits_n \mathrm{d}(\omega_k^n(x), \omega^n(x)) \overset{k}{\nrightarrow} 0,\ \forall x \in X$;

c) $A_k \xrightarrow{k} A$,

where, for each $k \ge 1$, we have denoted by A_k the attractor of CIFS $(\omega_k^n)_n$ *and A means the attractor of* CIFS $(\omega^n)_n$, *the convergence being taken in the space $(\mathcal{K}(X), \delta)$.*

Proof. a) If $x \in X$ and $n \ge 1$, then for each $k \ge n$, we have by definition

$$\omega_k^n(x) = \omega^n(x)$$

and hence the convergence is trivially.

b) The assertion results taking into account that for each $k \in \mathbb{N}^*$, there is a number $n_k > \max\{k, N\}$ such that

$$\mathrm{d}(\omega_k^{n_k}(x), \omega^{n_k}(x)) = \mathrm{d}(e_1, \omega^{n_k}(x)) > C,\ \forall x \in X,$$

by the hypothesis (6).

c) By using Prop. 2 we deduce that, for each $k \ge 1$, A_k is the attractor of IFS $(\omega^n)_{n=1}^k$.

Now, the conclusion follows from Th. 2. ◊

3. The dependence on parameter of the invariant measure of a CIFS

In this section we will assume the same context like in the above section. Let $(p_n)_n$ be a sequence of probabilities $p_n \in (0,1)$, $n \in \mathbb{N}^*$, $\sum\limits_{n=1}^{\infty} p_n = 1$.

For some $k \in \mathbb{N}$, $k \ge 2$, we denote $q_1 = p_1, \ldots, q_{k-1} = p_{k-1}, q_k = \sum\limits_{n=k}^{\infty} p_n$, k probabilities and, for each $t \in T$, we will denote M_t^k and, respectively μ_t^k the Markov operator, respectively the Hutchinson measure associated of the countable iterated function system with probabilities $((\omega_n(t,\cdot))_{n=1}^k, (q_n)_{n=1}^k)$. We also denote for every $t \in T$ by μ_t the

Hutchinson measure associated of CIFSp $((\omega_n(t,\cdot)_{n=1}^{\infty}, (p_n)_{n=1}^{\infty})$ and we will suppose further that $r := \sup_{t\in T}\sup_{n\in\mathbb{N}} r_n(t) < 1$.

Theorem 8. *Suppose that for each $x \in X$ and $n \in \mathbb{N}^*$ the function $t \mapsto \omega_n(t,x)$ is continuous. Then the map*

$$t \mapsto \mu_t$$

from T to $(\mathcal{B}, \mathrm{d}_H)$ is continuous.

Proof. By Th. 4 it follows that for all $t \in T$,

$$\mu_t^k \xrightarrow{k} \mu_t \tag{7}$$

with respect to the metric d_H.

Next, from Th. 1 it results that the map $t \mapsto \mu_t^k$ is continuous.

We will prove that the convergence in (7) is uniformly (with respect to t).

Choose $\varepsilon > 0$ and $K_\varepsilon \in \mathbb{N}$ such that, for every $k \geq K_\varepsilon$,

$$\frac{1}{1-r}\cdot \mathrm{diam}(X)\cdot \sum_{n=k+1}^{\infty} p_n < \varepsilon, \tag{8}$$

where $\mathrm{diam}(X)$ is the diameter of the set X.

Let $k \geq K_\varepsilon$ and $f : X \longrightarrow \mathbb{R}$ be continuous with $\mathrm{Lip} f \leq 1$ ($\mathrm{Lip} f = \sup_{x\neq y} \frac{|f(x)-f(y)|}{\mathrm{d}(x,y)}$ being the Lipschitz constant of f).

Denoting by M_t the Markov operator associated of CIFSp $((\omega_n(t,\cdot))_n, (p_n)_n)$, and using lemmas 2 and 3, we have for all $t \in T$,

$$\begin{aligned}
&\int_X f d\mathrm{M}_t^k(\mu_t^k) - \int_X f d\mathrm{M}_t(\mu_t^k) = \\
&= \sum_{n=1}^{k-1} p_n \int_X f\circ\omega_n(t,\cdot)d\mu_t^k + \sum_{n=k}^{\infty} p_n \int_X f\circ\omega_k(t,\cdot)d\mu_t^k - \\
&\quad - \sum_{n=1}^{k-1} p_n \int_X f\circ\omega_n(t,\cdot)d\mu_t^k - \sum_{n=k}^{\infty} p_n \int_X f\circ\omega_n(t,\cdot)d\mu_t^k = \\
&= \sum_{n=k+1}^{\infty} p_n \int_X \left(f\circ\omega_k(t,\cdot) - f\circ\omega_n(t,\cdot)\right) d\mu_t^k \leq \\
&\leq \mathrm{d}(X)\cdot \underbrace{\mu_t^k(X)}_{=1} \sum_{n=k+1}^{\infty} p_n < \varepsilon(1-r)
\end{aligned}$$

according to (8) and using the inequalities

$$| f \circ \omega_k(t,x) - f \circ \omega_n(t,x) | \leq \mathrm{d}(\omega_k(t,x), \omega_n(t,x)) \leq \mathrm{diam}(X),$$

for all $t \in T$, $x \in X$ (we have use also the inequality $\mathrm{Lip} f \leq 1$).

Thus one obtains

$$\text{(9)} \qquad \mathrm{d}_H(\mathrm{M}_t^k(\mu_t^k), \mathrm{M}_t(\mu_t^k)) < \varepsilon(1-r), \ \forall t \in T \text{ and } k \geq K_\varepsilon.$$

Next, for $k \geq K_\varepsilon$ and $t \in T$ one has

$$\begin{aligned}\mathrm{d}_H(\mu_t^k, \mu_t) &= \mathrm{d}_H(\mathrm{M}_t^k(\mu_t^k), \mathrm{M}_t(\mu_t)) \leq \\ &\leq \mathrm{d}_H(\mathrm{M}_t^k(\mu_t^k), \mathrm{M}_t(\mu_t^k)) + \mathrm{d}_H(\mathrm{M}_t(\mu_t^k), \mathrm{M}_t(\mu_t)) \leq \\ &\leq \mathrm{d}_H(\mathrm{M}_t^k(\mu_t^k), \mathrm{M}_t(\mu_t^k)) + r\mathrm{d}_H(\mu_t^k, \mu_t).\end{aligned}$$

Hence, using (9), we have

$$\mathrm{d}_H(\mu_t^k, \mu_t) \leq \frac{1}{1-r} \cdot \varepsilon(1-r) = \varepsilon, \ \forall t \in T, \ \forall k \geq K_\varepsilon,$$

and consequently $\mu_t^k \xrightarrow{k} \mu_t$ uniformly. Thus $t \mapsto \mu_t$ is a continuous map. ◊

4. A new approximation for Hutchinson measure

We will study the dependence of Hutchinson measure associated to a CIFSp in the case when the system of probabilities is the limit of a sequence of systems of probabilities.

Write $\mathcal{C}(X) = \{f : X \longrightarrow \mathbb{R} : \ f \text{ continuous}\}$ and let $\mathcal{M}$ be the family of Borel measures on X. The convergence in the weak topology on $\mathcal{M}$ will be, for $(\mu_k)_k \subset \mathcal{M}$, $\mu \in \mathcal{M}$,

$$\mu_k \xrightarrow{weak} \mu \iff \int_X f d\mu_k - \int_X f d\mu \xrightarrow{k} 0, \ \forall f \in \mathcal{C}(X).$$

Clearly, X being compact, all measures in $\mathcal{M}$ have a compact support.

It is a standard fact that the d_H topology and the weak topology coincide on the space of Borel normalized measures with compact support.

Let us consider a CIFS $(\omega_n)_{n\geq 1}$ with ratios r_n, $n \geq 1$, with $r = \sup_n r_n < 1$ and for each $k = 1, 2, \ldots$ the system of probabilities $(p_n^k)_{n\geq 1}$ which has the following properties:

a) $p_n^k \in (0,1)$, $\forall n, k \geq 1$;

b) $\sum_{n=1}^{\infty} p_n^k = 1,\ \forall k = 1, 2, \ldots;$

c) there exists a sequence $(p_n)_n \subset (0,1)$ such that $|p_n^k - p_n| \leq$ $\leq \frac{1}{k \cdot 2^n}$, for all $k = 1, 2, \ldots,\ n = 1, 2, \ldots$.

Lemma 5. *In the above conditions,*

$$p_n^k \xrightarrow{k} p_n,\ \forall n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} p_n = 1.$$

Thus $(p_n)_n$ is a system of probabilities which is approximated by $((p_n^k)_k)_n$.

Proof. We have: $| p_n^k - p_n | \leq \frac{1}{k \cdot 2^n}$ implies $p_n^k \xrightarrow{k} p_n,\ \forall n.$

Next, for any $k, n \in \mathbb{N}^*$ we have

$$-\frac{1}{k \cdot 2^n} \leq p_n^k - p_n \leq \frac{1}{k \cdot 2^n}$$

and, summing with respect to n,

$$-\sum_{n=1}^{\infty} \frac{1}{k \cdot 2^n} \leq \sum_{n=1}^{\infty} p_n^k - \sum_{n=1}^{\infty} p_n \leq \sum_{n=1}^{\infty} \frac{1}{k \cdot 2^n} \iff -\frac{1}{k} \leq 1 - \sum_{n=1}^{\infty} p_n \leq \frac{1}{k}, \forall k.$$

Hence $\sum_{n=1}^{\infty} p_n = 1.$ $\Diamond$

For each $k \geq 1$ we denote μ^k, respectively M^k the Hutchinson measure, respectively the Markov operator associated to CIFSp $((\omega_n)_{n\geq 1}, (p_n^k)_{n\geq 1})$. We denote further μ, respectively M the Hutchinson measure, respectively the Markov operator associated of the system $((\omega_n)_{n\geq 1}, (p_n)_{n\geq 1})$.

Theorem 9. *Under the above conditions and using the same notations, we have*

$$\mu^k \xrightarrow{k} \mu$$

with respect to the Hutchinson metric d_H.

Proof. For an arbitrary $k \geq 1$, we have

$$\begin{aligned} \mathrm{d}_H(\mu^k, \mu) &= \mathrm{d}_H(\mathrm{M}^k(\mu^k), \mathrm{M}(\mu)) \leq \\ &\leq \mathrm{d}_H(\mathrm{M}^k(\mu^k), \mathrm{M}^k(\mu)) + \mathrm{d}_H(\mathrm{M}^k(\mu), \mathrm{M}(\mu)) \leq \\ &\leq r\mathrm{d}_H(\mu^k, \mu) + \mathrm{d}_H(\mathrm{M}^k(\mu), \mathrm{M}(\mu)). \end{aligned}$$

It follows

$$d_H(\mu^k, \mu) \leq \frac{1}{1-r} d_H(M^k(\mu), M(\mu)).$$

To establish that $d_H(M^k(\mu), M(\mu)) \xrightarrow{k} 0$, it is sufficient to prove that $M^k(\mu) \xrightarrow{k} M(\mu)$ with respect to the weak topology.

Take some $f \in \mathcal{C}(X)$. Then, using Lemma 3 and the condition c),

$$\left| \int_X f dM^k(\mu) - \int_X f dM(\mu) \right| =$$

$$= \left| \sum_{n=1}^{\infty} p_n^k \int_X f \circ \omega_n d\mu - \sum_{n=1}^{\infty} p_n \int_X f \circ \omega_n d\mu \right| \leq$$

$$\leq \sum_{n=1}^{\infty} | p_n^k - p_n | \cdot \int_X | f \circ \omega_n | d\mu \leq$$

$$\leq \sup_{x \in X} | f(x) | \cdot \mu(X) \cdot \sum_{n=1}^{\infty} \frac{1}{k \cdot 2^n} = \sup_{x \in X} | f(x) | \cdot \frac{1}{k} \xrightarrow{k} 0.$$

Consequently it follows $M^k(\mu)(f) \xrightarrow{k} M^k(\mu)(f)$, $\forall f \in \mathcal{C}(X)$. ◊

References

[1] BARNSLEY, M. F.: Fractals everywhere, Academic Press, Harcourt Brace Janovitch, 1988

[2] CENTORE, P. M. and VRSCAY, E. R.: Continuity of attractors and invariant measures for Iterated Function Systems, *Canad. Math. Bull.* **37** (3) (1994), 315–329.

[3] FERNAU, H.: Iterierte Funktionen, Sprachen und Fraktale, Wissenschaftsverlag, 1994.

[4] HUTCHINSON, J.: Fractals and self-similarity, *Indiana Univ. J. Math.* **30** (1981), 713–747.

[5] MENDIVIL, F.: A generalization of IFS with probabilities to infinitely many maps, *Rocky Mountain Journal of Mathematics* **28** (3) (1998), 1043–1051.

[6] SECELEAN, N. A.: Countable Iterated Function Systems, *Far East Journal of Dynamical Systems*, Pushpa Publishing House, **3** (2) (2001), 149–167.

[7] SECELEAN, N. A.: The invariant measure of an countable iterated function system, *Seminarberichte aus dem Fachbereich Mathematik de Fernuniversität Hagen* **73** (2002), 3–10.